Chapter 3

Collisions in Plasmas

3.1 Binary collisions between charged particles

Reduced-mass for binary collisions:

Two particles interacting with each other have forces

$\mathbf{F}_{12}$ force on 1 from 2.

$\mathbf{F}_{21}$ force on 2 from 1.

By Newton's 3rd law, $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Equations of motion:

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_{12} \qquad ; \qquad m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_{21} \tag{3.1}$$

Combine to get

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \mathbf{F}_{12} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \tag{3.2}$$

which may be written

$$\frac{m_1 m_2}{m_1 + m_2} \frac{d^2}{dt^2} (\mathbf{r}_1 - \mathbf{r}_2) = \mathbf{F}_{12} \tag{3.3}$$

If F_{12} depends only on the difference vector $\mathbf{r}_1 - \mathbf{r}_2$, then this equation is identical to the equation of a particle of "Reduced Mass" $m_r \equiv \frac{m_1 m_2}{m_1 + m_2}$ moving at position $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ with respect to a fixed center of force:

$$m_r \ddot{\mathbf{r}} = \mathbf{F}_{12}(\mathbf{r}) \quad . \tag{3.4}$$

This is the equation we analyse, but actually particle 2 *does* move. And we need to recognize that when interpreting mathematics.

If $\mathbf{F}_{21}$ and $\mathbf{r}_1 - \mathbf{r}_2$ are always parallel, then a general form of the trajectory can be written as an integral. To save time we specialize immediately to the *Coulomb force*

$$\mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\mathbf{r}}{r^3} \tag{3.5}$$

Solution of this standard (Newton's) problem:

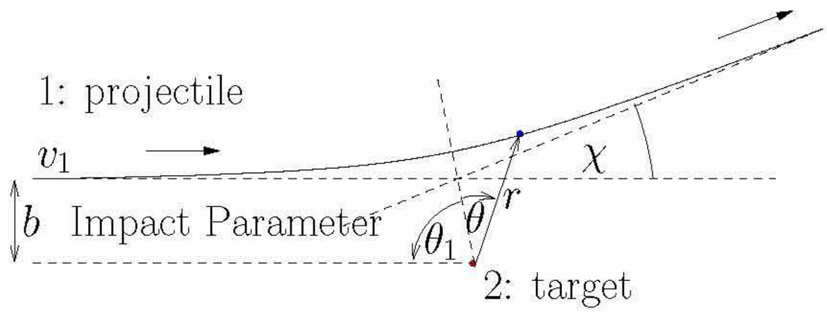

Figure 3.1: Geometry of the collision orbit

Angular momentum is conserved:

$$m_r r^2 \dot{\theta} \;=\; \text{const.} \;=\; m_r b v_1 \qquad (\theta \text{ clockwise from symmetry}) \tag{3.6}$$

Substitute $u \equiv \frac{1}{r}$ then $\dot{\theta} = \frac{bv_1}{r^2} = u^2\, bv_1$
Also

$$\dot{r} \;=\; \frac{d}{dt}\,\frac{1}{u} \;=\; -\frac{1}{u^2}\,\frac{du}{d\theta}\,\dot{\theta} \;=\; -bv_1\,\frac{du}{d\theta} \tag{3.7}$$

$$\ddot{r} \;=\; -bv_1\,\frac{d^2u}{d\theta^2}\,\dot{\theta} \;=\; -(bv_1)^2\,u^2\,\frac{d^2u}{d\theta^2} \tag{3.8}$$

Then radial acceleration is

$$\ddot{r} - r\dot{\theta}^2 = -(bv_1)^2\,u^2\left(\frac{d^2u}{d\theta^2} + u\right) \;=\; \frac{|F_{12}|}{m_r} \tag{3.9}$$

i.e.

$$\frac{d^2u}{d\theta^2} + u = -\frac{q_1q_2}{4\pi\epsilon_0}\,\frac{1}{m_r\,(bv_1)^2} \tag{3.10}$$

This orbit equation has the elementary solution

$$u \equiv \frac{1}{r} \;=\; C\cos\theta \;-\; \frac{q_1q_2}{4\pi\epsilon_0}\,\frac{1}{m_r\,(bv_1)^2} \tag{3.11}$$

The $\sin\theta$ term is absent by symmetry. The other constant of integration, C, must be determined by initial condition. At initial (far distant) angle, θ_1, $u_1 = \frac{1}{\infty} = 0$. So

$$0 = C\cos\theta_1 - \frac{q_1q_2}{4\pi\epsilon_0}\,\frac{1}{m_r\,(bv_1)^2} \tag{3.12}$$

There:

$$\dot{r}_1 = -v_1 = -bv_1\frac{du}{d\theta}|_1 = +bv_1C\sin\theta_1 \tag{3.13}$$

Hence

$$\tan\theta_1 = \frac{\sin\theta_1}{\cos\theta_1} = \frac{-1/Cb}{\frac{q_1q_2}{4\pi\epsilon_0}\frac{1}{m_r(bv_1)^2}/C} = -\frac{b}{b_{90}} \tag{3.14}$$

where

$$b_{90} \equiv \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{m_r v_1^2} \quad . \tag{3.15}$$

Notice that $\tan\theta_1 = -1$ when $b = b_{90}$. This is when $\theta_1 = -45°$ and $\chi = 90°$. So particle emerges at 90° to initial direction when

$$b \;=\; b_{90} \qquad\qquad \text{“90° impact parameter”} \tag{3.16}$$

Finally:

$$C = -\frac{1}{b}\mathrm{cosec}\theta_1 = -\frac{1}{b}\left(1 + \frac{b_{90}^2}{b^2}\right)^{\frac{1}{2}} \tag{3.17}$$

3.1.1 Frames of Reference

Key quantity we want is the scattering angle but we need to be careful about reference frames.

Most “natural” frame of ref is “Center-of-Mass” frame, in which C of M is stationary. C of M has position:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{3.18}$$

and velocity (in lab frame)

$$\mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} \tag{3.19}$$

Now

$$\mathbf{r}_1 \;=\; \mathbf{R} + \frac{m_2}{m_1 + m_2}\mathbf{r} \tag{3.20}$$

$$\mathbf{r}_2 \;=\; \mathbf{R} - \frac{m_1}{m_1 + m_2}\,\mathbf{r} \tag{3.21}$$

So motion of either particle in C of M frame is a factor times difference vector, $\mathbf{r}$.

Velocity in lab frame is obtained by adding $\mathbf{V}$ to the C of M velocity, e.g. $\frac{m_2\dot{\mathbf{r}}}{m_1+m_2} + \mathbf{V}$.

Angles of position vectors and velocity *differences* are *same* in all frames.

Angles (i.e. directions) of velocities are *not same.*

3.1.2 Scattering Angle

In *C of M frame* is just the final angle of $\mathbf{r}$.

$$-\,2\theta_1 + \chi = \pi \tag{3.22}$$

(θ_1 is negative)

$$\chi = \pi + 2\theta_1 \quad ; \quad \theta_1 = \frac{\chi - \pi}{2} \quad . \tag{3.23}$$

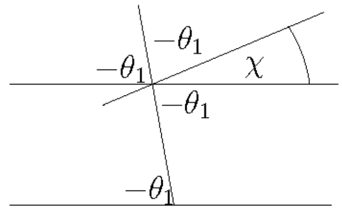

Figure 3.2: Relation between θ_1 and χ.

$$\tan\theta_1 = \tan\left(\frac{\chi}{2} = \frac{\pi}{2}\right) = -\cot\frac{\chi}{2} \tag{3.24}$$

So

$$\cot\frac{\chi}{2} = \frac{b}{b_{90}} \tag{3.25}$$

$$\tan\frac{\chi}{2} = \frac{b_{90}}{b} \tag{3.26}$$

But scattering angle (defined as exit velocity angle relative to initial velocity) in lab frame is *different.*

Final velocity in CM frame

$$\mathbf{v}'_{\rm CM} = v_{1\rm CM}\left(\cos\chi_c, \sin\chi_C\right) = \frac{m_2}{m_1+m_2}\, v_1\left(\cos\chi_c, \sin\chi_c\right) \tag{3.27}$$

[$\chi_c \equiv \chi$ and v_1 is initial relative velocity]. Final velocity in Lab frame

$$\mathbf{v}'_L = \mathbf{v}'_{\rm CM} + \mathbf{V} = \left(V + \frac{m_2 v_1}{m_1+m_2}\cos\chi_c, \frac{m_2 v_1}{m_1+m_2}\sin\chi_c\right) \tag{3.28}$$

So angle is given by

$$\cot\chi_L = \frac{V + \frac{m_2 v_1}{m_1+m_2}\cos\chi_c}{\frac{m_2 v_1}{m_1+m_2}\sin\chi_x} = \frac{V}{v_1}\frac{m_1+m_2}{m_2}{\rm cosec}\chi_c + \cot\chi_c \tag{3.29}$$

For the specific case when m_2 is initially a *stationary target* in lab frame, then

$$V = \frac{m_1 v_1}{m_1+m_2} \qquad \text{and hence} \tag{3.30}$$

$$\cot\chi_L = \frac{m_1}{m_2}{\rm cosec}\chi_c + \cot\chi_c \tag{3.31}$$

This is *exact.*

Small angle approximation ($\cot\chi \to \frac{1}{\chi}, {\rm cosec}\chi \to \frac{1}{\chi}$ gives

$$\frac{1}{\chi_L} = \frac{m_1}{m_2}\frac{1}{\chi_c} + \frac{1}{\chi_c} \Leftrightarrow \chi_L = \frac{m_2}{m_1+m_2}\chi_c \tag{3.32}$$

So small angles are proportional, with ratio set by the mass-ratio of particles.

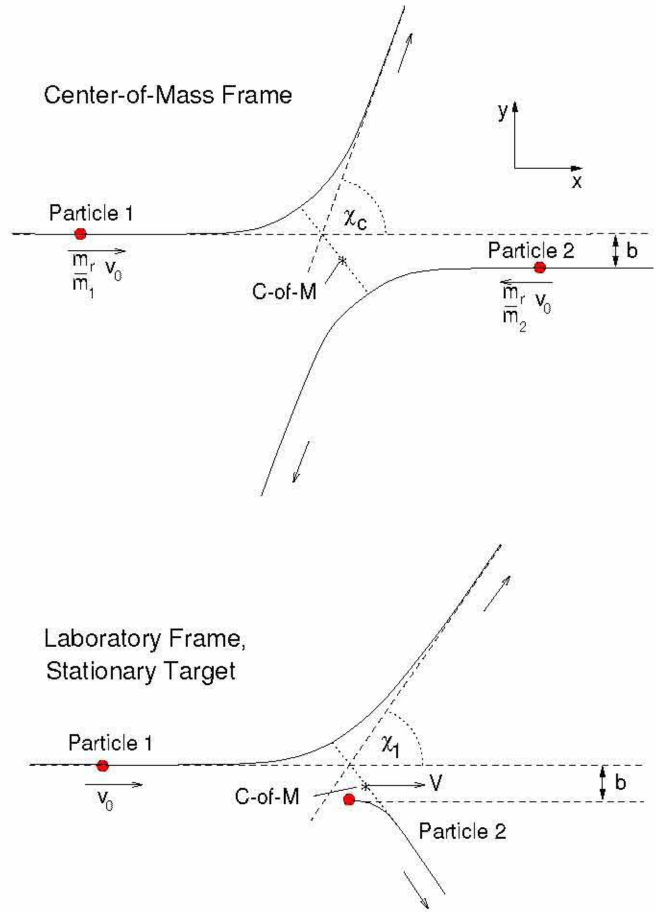

Figure 3.3: Collisions viewed in Center of Mass and Laboratory frame.

3.2 Differential Cross-Section for Scattering by Angle

Rutherford Cross-Section

By definition the cross-section, σ, for any specified collision process when a particle is passing through a density n_2 of targets is such that the number of such collisions per unit path length is $n_2\sigma$.

Sometimes a continuum of types of collision is considered, e.g. we consider collisions at different angles (χ) to be distinct. In that case we usually discuss *differential cross-sections* (e.g $\frac{d\sigma}{d\chi}$) defined such that number of collisions in an (angle) element $d\chi$ per unit path length is $n_2\frac{d\sigma}{d\chi}dx$. [Note that $\frac{d\sigma}{d\chi}$ is just notation for a number. Some authors just write $\sigma(\chi)$, but I find that less clear.]

Normally, for scattering-angle discrimination we discuss the differential cross-section per unit *solid angle*:

$$\frac{d\sigma}{d\Omega_s}. \tag{3.33}$$

This is related to scattering angle integrated over all azimuthal directions of scattering by:

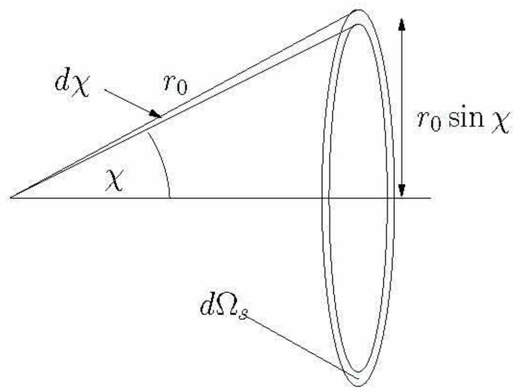

Figure 3.4: Scattering angle and solid angle relationship.

$$d\Omega_s = 2\pi \sin\chi d\chi \tag{3.34}$$

So that since

$$\frac{d\sigma}{d\Omega_s} d\Omega_s = \frac{d\sigma}{d\chi} d\chi \tag{3.35}$$

we have

$$\frac{d\sigma}{d\Omega_s} = \frac{1}{2\pi \sin\chi} \frac{d\sigma}{d\chi} \tag{3.36}$$

Now, since χ is a function (only) of the impact parameter, b, we just have to determine the number of collisions per unit length at impact parameter b.

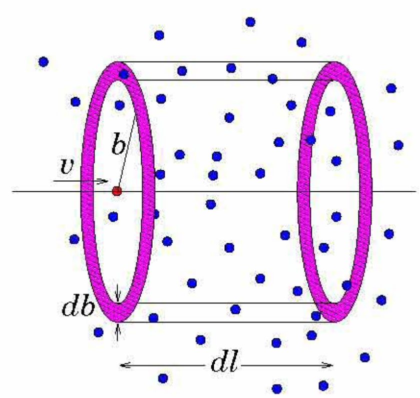

Figure 3.5: Annular volume corresponding to db.

Think of the projectile as dragging along an annulus of radius b and thickness db for an elementary distance along its path, $d\ell$. It thereby drags through a volume:

$$d\ell 2\pi b db \quad . \tag{3.37}$$

Therefore in this distance it has encountered a total number of targets

$$d\ell 2\pi b db \ . \ n_2 \tag{3.38}$$

at impact parameter $b(db)$. By definition this is equal to $d\ell\frac{d\sigma}{db}dbn_2$. Hence the differential cross-section for scattering (encounter) at impact parameter b is

$$\frac{d\sigma}{db} = 2\pi b \quad . \tag{3.39}$$

Again by definition, since χ is a function of b

$$\frac{d\sigma}{d\chi}dx = \frac{d\sigma}{db}db \Rightarrow \frac{d\sigma}{d\chi} = \frac{d\sigma}{db} = \left|\frac{db}{d\chi}\right| \quad . \tag{3.40}$$

[$db/d\chi$ is negative but differential cross-sections are positive.]

Substitute and we get

$$\frac{d\sigma}{d\Omega_s} = \frac{1}{2\pi\sin\chi}\,\frac{d\sigma}{db}\left|\frac{db}{d\chi}\right| \;=\; \frac{b}{\sin}\left|\frac{db}{d\chi}\right| \quad . \tag{3.41}$$

[This is a general result for classical collisions.]

For Coulomb collisions, in C of M frame,

$$\cot\left(\frac{\chi}{2}\right) = \frac{b}{b_{90}} \tag{3.42}$$

$$\Rightarrow \frac{db}{d\chi} \;=\; b_{90}\,\frac{d}{d\chi}\cot\frac{\chi}{2} \;=\; -\frac{b_{90}}{2}\mathrm{cosec}^2\frac{\chi}{2} \quad . \tag{3.43}$$

Hence

$$\frac{d\sigma}{d\Omega_s} \;=\; \frac{b_{90}\cot\frac{\chi}{2}}{\sin\chi}\,\frac{b_{90}}{2}\,\mathrm{cosec}^2\frac{\chi}{2} \tag{3.44}$$

$$= \frac{b_{90}^2}{2}\,\frac{\cos\frac{\chi}{2}/\sin\frac{\chi}{2}}{2\sin\frac{\chi}{2}\cos\frac{\chi}{2}}\,\frac{1}{\sin^2\frac{\chi}{2}} \tag{3.45}$$

$$= \frac{b_{90}^2}{4\sin^4\frac{\chi}{2}} \tag{3.46}$$

This is the Rutherford Cross-Section.

$$\frac{d\sigma}{d\Omega_s} = \frac{b_{90}^2}{4\sin^4\frac{\chi}{2}} \tag{3.47}$$

for scattering by Coulomb forces through an angle χ measured in C of M frame.

Notice that $\frac{d\sigma}{d\Omega_s} \to \infty$ as $\chi \to 0$.

This is because of the long-range nature of the Coulomb force. Distant collisions tend to dominate. ($\chi \to 0 \Leftrightarrow b \to \infty$).

3.3 Relaxation Processes

There are 2 (main) different types of collisional relaxation process we need to discuss for a test particle moving through a background of scatterers:

1. Energy Loss (or equilibrium)
2. Momentum Loss (or angular scattering)

The distinction may be illustrated by a large angle (90°) scatter from a heavy (stationary) target.

If the target is fixed, no energy is transferred to it. So the *energy loss* is *zero* (or small if scatterer is just 'heavy'). However, the *momentum* in the x direction is *completely 'lost'* in this 90° scatter.

This shows that the timescales for Energy loss and momentum loss may be very different.

3.3.1 Energy Loss

For an initially stationary target, the final velocity in lab frame of the projectile is

$$v_L' = \left(\frac{m_l v_1}{m_1 + m_2} + \frac{m_2 v_1}{m_1 + m_2} \cos\chi_c \ , \ \frac{m_2 v_1}{m_1 + m_2} \sin\chi_c\right) \tag{3.48}$$

So the final kinetic energy is

$$K' = \frac{1}{2} m_1 v_L'^2 = \frac{1}{2} m_1 v_1^2 \left\{ \left(\frac{m_1}{m_1 + m_2}\right)^2 + \frac{2 m_1 m_2}{(m_1 + m_2)^2} \cos\chi_c \right. \tag{3.49}$$

$$\left. + \frac{m_2^2}{(m_1 + m_2)^2} \left(\cos^2\chi_c + \sin^2\chi_c\right) \right\} \tag{3.50}$$

$$= \frac{1}{2} m_1 v_1^2 \left\{ 1 + \frac{2 m_1 m_2}{(m_1 + m_2)^2} (\cos\chi_c - 1) \right\} \tag{3.51}$$

$$= \frac{1}{2} m_1 v_1^2 \left\{ 1 + \frac{2 m_1 m_2}{(m_1 + m_2)^2} 2 \sin^2\frac{\chi_c}{2} \right\} \tag{3.52}$$

Hence the kinetic energy lost is $\Delta K = K - K'$

$$= \frac{1}{2} m_1 v_2 \frac{4 m_1 m_2}{(m_1 + m_2)^2} \sin^2\frac{\chi_c}{2} \tag{3.53}$$

$$= \frac{1}{2} m_1 v_1^2 \frac{4 m_1 m_2}{(m_1 + m_2)^2} \frac{1}{\left(\frac{b}{b_{90}}\right)^2 + 1} \qquad [\text{using } \cot\frac{\chi_c}{2} = \frac{b}{b_{90}}] \tag{3.54}$$

(exact). For small angles $\chi \ll 1$ i.e. $b/b_{90} \gg 1$ this energy lost in a single collision is approximately

$$\left(\frac{1}{2} m_1 v_1^2\right) \frac{4 m_1 m_2}{(m_1 + m_2)^2} \left(\frac{b_{90}}{b}\right)^2 \tag{3.55}$$

If what we are asking is: how fast does the projectile lose energy? Then we need add up the effects of all collisions in an elemental length $d\ell$ at all relevant impact parameters.

The contribution from impact parameter range db at b will equal the number of targets encountered times ΔK:

$$\underbrace{n_2 d\ell 2\pi b db}_{\text{encounters}} \quad \underbrace{\frac{1}{2}m_1 v_1^2 \frac{4m_1m_2}{(m_1+m_2)^2}\left(\frac{b_{90}}{b}\right)^2}_{\text{Loss per encounter }(\Delta K)} \tag{3.56}$$

This must be integrated over all b to get total energy loss.

$$dK = n_2 d\ell K \frac{4m_1m_2}{(m_1+m_2)^2} \int \left(\frac{b_{90}}{b}\right)^2 2\pi b db \tag{3.57}$$

so

$$\frac{dK}{d\ell} = K\ n_2 \frac{m_1m_2}{(m_1+m_2)^2}\ 8\pi b_{90}^2\ \left[\ln|b|\right]_{\text{min}}^{\text{max}} \tag{3.58}$$

We see there is a *problem* both limits of the integral ($b \to 0$, $b \to \infty$) diverge logarithmically. That is because the formulas we are integrating are approximate.

1. We are using small-angle approx for ΔK.
2. We are assuming the Coulomb force applies but this is a plasma so there is screening.

3.3.2 Cut-offs Estimates

1. Small-angle approx breaks down around $b = b_{90}$. Just truncate the integral there; ignore contributions from $b < b_{90}$.
2. Debye Shielding says really the potential varies as

$$\phi \propto \frac{\exp\left(\frac{-r}{\lambda_D}\right)}{r} \qquad \text{instead of } \propto \frac{1}{r} \tag{3.59}$$

so approximate this by cutting off integral at $b = \lambda_D$ equivalent to

$$b_{\text{min}} = b_{90}. \qquad b_{\text{max}} = \lambda_D. \tag{3.60}$$

$$\frac{dK}{d\ell} = K n_2 \frac{m_1m_2}{(m_1+m_2)^2}\ 8\pi b_{90}^2\ \ln|\Lambda| \tag{3.61}$$

$$\Lambda = \frac{\lambda_D}{b_{90}} = \left(\frac{\epsilon_0 T_e}{ne^2}\right)^{\frac{1}{2}} \Big/ \left(\frac{q_1q_2}{4\pi\epsilon_0 m_r v_1^2}\right) \tag{3.62}$$

So *Coulomb Logarithm* is '$\ln\Lambda$'

$$\Lambda \;=\; \frac{\lambda_D}{b_{90}} \;=\; \left(\frac{\epsilon_0 T_e}{ne^2}\right)^{\frac{1}{2}} \Bigg/ \left(\frac{q_1 q_2}{4\pi\epsilon_0 m_r v_1^2}\right) \tag{3.63}$$

Because these cut-offs are in ln term result is not sensitive to their exact values.

One commonly uses *Collision Frequency*. Energy Loss Collision Frequency is

$$\nu_K \equiv v_1 \frac{1}{K}\frac{dK}{dL} = n_2 v_1 \; \frac{m_1 m_2}{(m_1+m_2)^2} \; 8\pi b_{90}^2 \;\; \ln|\Lambda| \tag{3.64}$$

Substitute for b_{90} and m_r (in b_{90})

$$\nu_K \;=\; n_2 v_1 \; \frac{m_1 m_2}{(m_1+m_2)^2} \; 8\pi \left[\frac{q_1 q_2}{4\pi\epsilon_0 \frac{m_1 m_2}{m_1+m_2} v_1^2}\right]^2 \;\ln\Lambda \tag{3.65}$$

$$\;=\; n_2 \; \frac{q_1^2 q_2^2}{(4\pi\epsilon_0)^2} \; \frac{8\pi}{m_1 m_2 v_1^3} \;\ln\Lambda \tag{3.66}$$

Collision time $\quad \tau_K \equiv 1/\nu_K$

Effective (Energy Loss) *Cross-section* $\quad \left[\frac{1}{K}\frac{dK}{d\ell} = \sigma_K n_2\right]$

$$\sigma_K = \nu_K / n_2 v_1 = \frac{q_1^2 q_2^2}{(4\pi\epsilon_0)^2} \; \frac{8\pi}{m_1 m_2 v_1^4} \;\ln\Lambda \tag{3.67}$$

3.3.3 Momentum Loss

Loss of x-momentum in 1 collision is

$$\Delta p_x \;=\; m_1 (v_1 - v'_{Lx}) \tag{3.68}$$

$$\;=\; m_1 v_1 \left\{1 - \left(\frac{m_1}{m_1+m_2} + \frac{m_2}{m_1+m_2}\cos\chi_c\right)\right\} \tag{3.69}$$

$$\;=\; p_x \; \frac{m_2}{m_1+m_2} \; (1-\cos\chi_c) \tag{3.70}$$

$$\;\simeq\; p_x \; \frac{m_2}{m_1+m_2} \; \frac{\chi_c^2}{2} = p_x \; \frac{m_2}{m_1+m_2} \; \frac{2b_{90}^2}{b^2} \tag{3.71}$$

(small angle approx). Hence rate of momentum loss can be obtained using an integral identical to the energy loss but with the above parameters:

$$\frac{dp}{d\ell} \;=\; n_2 p \int_{b_{min}}^{b_{max}} \frac{m2}{m1+m_2} \; \frac{2b_{90}^2}{b^2} \; 2\pi b db \tag{3.72}$$

$$\;=\; n_2 p \; \frac{m_2}{m_1+m_2} \; 4\pi \; b_{90}^2 \;\; \ln\Lambda \tag{3.73}$$

Note for the future reference:

$$\frac{dp}{dt} = v\frac{dp}{d\ell} = n_2 v^2 \; \frac{m_1 m_2}{m_1+m_2} \; 4\pi b_{90}^2 \;\; \ln\Lambda. \tag{3.74}$$

Therefore *Momentum Loss.*

Collision Frequency

$$\nu_p = v_1 \frac{1}{p}\frac{dp}{d\ell} = n_2 v_1 \; \frac{m_2}{m_1+m_2} \; 4\pi b_{90}^2 \; \ln\Lambda \tag{3.75}$$

$$= n_2 v_1 \; \frac{m_2}{m_1+m_2} \; 4\pi \left[\frac{q_1 q_2}{4\pi\epsilon_0 \frac{m_1 m_2}{m_1+m_2} v_1^2}\right]^2 \; \ln\Lambda \tag{3.76}$$

$$= n_2 \; \frac{q_1^2 q_2^2}{(4\pi\epsilon_0)^2} \; \frac{4\pi\,(m_1+m_2)}{m_2 m_1^2 v_1^3} \; \ln\Lambda \tag{3.77}$$

Collision Time $\tau_p = 1/\nu_p$

Cross-Section (effective) $\sigma = \nu_p / n_2 v_1$

Notice ratio

$$\frac{\text{Energy Loss } \nu_{\rm K}}{\text{Momentum loss } \nu_{\rm p}} = \frac{2}{m_1 m_2} \Bigg/ \frac{m_1+m_2}{m_2 m_1^2} = \frac{2m_1}{m_1+m_2} \tag{3.78}$$

This is

$$\simeq 2 \quad \text{if} \quad m_1 >> m_2 \tag{3.79}$$

$$= 1 \quad \text{if} \quad m_1 = m_2 \tag{3.80}$$

$$\ll 1 \quad \text{if} \quad m_1 \ll m_2. \tag{3.81}$$

Third case, e.g. electrons → shows that mostly the *angle* of velocity scatters. Therefore Momentum 'Scattering' time is often called '90° scattering' time to 'diffuse' through 90° in angle.

3.3.4 'Random Walk' in angle

When $m_1 << m_2$ energy loss << momentum loss. Hence $|\mathbf{v}'_L \simeq v_1$. All that matters is the scattering angle: $\chi_L \simeq \chi_c \simeq 2b_{90}/b$.

Mean angle of deviation in length L is zero because all directions are equally likely.
But:
Mean *square* angle is

$$\overline{\Delta\alpha^2} = n_2 L \int_{b_{\min}}^{b_{\max}} \chi^2 \; 2\pi b db \tag{3.82}$$

$$= L n_2 \; 8\pi \; b_{90}^2 \; \ln\Lambda \tag{3.83}$$

Spread is 'all round' when $\overline{\Delta\alpha^2} \simeq 1$. This is roughly when a particle has scattered 90° on average. It requires

$$L n_2 \; 8\pi b_{90}^2 \; \ln\Lambda = 1 \quad . \tag{3.84}$$

So can think of a kind of 'cross-section' for 'σ_{90}' 90° scattering as such that

$$n_2 L`\sigma_{90}' = 1 \text{ when } Ln_2\, 8\pi\, b_{90}^2 \ln\Lambda = 1 \quad (3.85)$$
$$\text{i.e. } `\sigma_{90}' = 8\pi\, b_{90}^2 \ln\Lambda \ \ (= 2\sigma_p) \quad (3.86)$$

This is $8\ln\Lambda$ larger than cross-section for 90° scattering *in single collision.*

Be Careful! 'σ_{90}' is not a usual type of cross-section because the whole process is really diffusive in angle.

Actually all collision processes due to coulomb force are best treated (in a Mathematical way) as a diffusion in velocity space

→ *Fokker-Planck equation.*

3.3.5 Summary of different types of collision

The *Energy Loss* collision frequency is to do with slowing down to rest and exchanging energy. It is required for calculating

Equilibration Times (of Temperatures)
Energy Transfer between species.

The *Momentum Loss* frequency is to do with loss of *directed* velocity. It is required for calculating

Mobility: Conductivity/Resistivity
Viscosity
Particle Diffusion
Energy (Thermal) Diffusion

Usually we distinguish between electrons and ions because of their very different mass:

Energy Loss [Stationary Targets] *Momentum Loss*

$$\begin{aligned}
{}^K\nu_{ee} &= n_e \frac{e^4}{(4\pi\epsilon_0)^2}\frac{8\pi}{m_e^2 v_e^3}\ln\Lambda & {}^p\nu_{ee} &={}^K \nu_{ee}\times\left[\frac{m_e+m_e}{2m_e}=1\right]\\
{}^K\nu_{ei} &= n_i \frac{Z^2e^4}{(4\pi\epsilon_0)^2}\frac{8\pi}{m_e m_i v_e^3}\ln\Lambda & {}^p\nu_{ei} &={}^K \nu_{ei}\times\left[\frac{m_e+m_i}{2m_e}\simeq\frac{m_i}{2m_e}\right]\\
{}^k\nu_{ii} &= n_i \frac{Z^2e^4}{(4\pi e_o)^2}\frac{8\pi}{m_i^2 v_i^3}\ln\Lambda & {}^p\nu_{ii} &={}^K \nu_{ii}\times\left[\frac{m_i+m_i}{2m_i}=1\right]\\
{}^K\nu_{ie} &= n_e \frac{Z_e^2e^4}{(4\pi\epsilon_0)^2}\frac{8\pi}{m_i m_e v_i^3}\ln\Lambda & {}^p\nu_{ie} &={}^K \nu_{ie}=\times\left[\frac{m_e+m_i}{2mi_i}\simeq\frac{1}{2}\right]
\end{aligned} \quad (3.87)$$

Sometimes one distinguishes between 'transverse diffusion' of velocity and 'momentum loss'. The ratio of these two is

$$\frac{\overline{\Delta p_\perp^2}}{p^2\Delta L}\Bigg/\left|\frac{\Delta p_\parallel}{p\Delta L}\right| = \frac{d\chi_L^2}{dL}\Bigg/\left|\frac{1}{p}\frac{dp}{dL}\right| \quad (3.88)$$

$$= \frac{\left(\frac{m_2}{m_1+m_2}\chi_c\right)^2}{\frac{m_2}{m_1+m_2}\frac{\chi_c^2}{2}} = \frac{2m_2}{m_1+m_2}. \quad (3.89)$$

So

$$
\begin{aligned}
\frac{`\sigma_{90}\text{'}}{`\sigma_p\text{'}} = \frac{2m_2}{m_1+m_2} \qquad &= 1 \qquad \text{like particles} && (3.90)\\
&\simeq 2 \qquad m_1 << m_2 && (3.91)\\
&\simeq \frac{2m_2}{m_1} \qquad m_2 << m_1 \ . && (3.92)
\end{aligned}
$$

Hence

$$
\begin{aligned}
{}^{\perp}\nu_{ee} &= {}^{p}\nu_{ee} = {}^{K}\nu_{ee} \, (= `\nu_{ee}\text{'!!}) && (3.93)\\
{}^{\perp}\nu_{ei} &= 2{}^{p}\nu_{ei} = {}^{K}\nu_{ee}\frac{n_i}{n_e}Z^2 \, (= Z\nu_{ee}) \, (= `\nu_{ei}\text{'}) && (3.94)\\
{}^{\perp}\nu_{ii} &= {}^{p}\nu_{ii} = {}^{K}\nu_{ii} \, (= \nu_{ii}!!) \ \text{(Like Ions)} && (3.95)\\
{}^{\perp}\nu_{ie} &= \frac{2m_e}{m_i} \, {}^{p}\nu_{ie} = \frac{m_e}{m_i} \, {}^{K}\nu_{ie} = {}^{K}\nu_{ii} = \nu_{ii} && (3.96)
\end{aligned}
$$

[But note: ions are slowed down by electrons long before being angle scattered.]

3.4 Thermal Distribution Collisions

So far we have calculated collision frequencies with stationary targets and single-velocity projectiles but generally we shall care about thermal (Maxwellian) distributions (or nearly thermal) of both species. This is harder to calculate and we shall resort to some heuristic calculations.

3.4.1 $e \rightarrow i$

Very rare for thermal ion velocity to be $\sim$ electron. So ignore ion motion.

Average over electron distribution.

Momentum loss to ions from (assumed) drifting Maxwellian electron distribution:

$$f_e(\mathbf{v}) = n_e \left(\frac{m_e}{2\pi T_e}\right)^{\frac{3}{2}} \exp\left[-\frac{m\,(\mathbf{v}-\mathbf{v}_d)^2}{2T}\right] \qquad (3.97)$$

Each electron in this distribution is losing momentum to the ions at a rate given by the collision frequency

$$\nu_p = n_i \frac{q_e^2 q_i^2}{(4\pi\epsilon_0)^2} \frac{4\pi\,(m_e+m_i)}{m_i m_e^2 v^3} \ln\Lambda \qquad (3.98)$$

so total rate of loss of momentum is given by (per unit volume)

$$-\frac{dp}{dt} = \int f_e(\mathbf{v}) \ \nu_p(v) \ m_e \mathbf{v} \ d^3\mathbf{v} \qquad (3.99)$$

To evaluate this integral approximately we adopt the following simplifications.

1. Ignore variations of $\ln \Lambda$ with v and just replace a typical thermal value in $\Lambda = \lambda_D / b_{90}(v_1)$.

2. Suppose that drift velocity $\mathbf{v}_d$ is small relative to the typical thermal velocity, written $v_e \equiv \sqrt{t_e/m_e}$ and express f_e in terms of $\mathbf{u} \equiv \frac{\mathbf{v}}{v_e}$ to first order in $\mathbf{u}_d \equiv \frac{\mathbf{v}_d}{v_e}$:

$$f_e = n_e \frac{1}{(2\pi)^{\frac{3}{2}} v_e^3} \exp\left[\frac{-1}{2} (\mathbf{u} - \mathbf{u}_d)^2\right] \tag{3.100}$$

$$\simeq \frac{n_e}{(2\pi)^{\frac{3}{2}} v_e^3} (1 + \mathbf{u}.\mathbf{u}_d) \exp\left[\frac{-u^2}{2}\right] = (1 + u_x u_d) f_o \tag{3.101}$$

taking x-axis along $\mathbf{u}_d$ and denoting by f_o the unshifted Maxwellian.

Then momentum loss rate per unit volume

$$-\frac{dp_x}{dt} = \int f_e \nu_p m_e v_x d^3 v$$

$$= \nu_p(v_t) m_e \int (1 + u_x u_d) f_o \frac{v_e^3}{v^3} v_x \, d^3 \, \mathbf{v} \tag{3.102}$$

$$= \nu_p(v_t) m_e v_d \int \frac{u_x^2}{u^3} f_o \, d^3 \, \mathbf{v}$$

To evaluate this integral, use the spherical symmetry of f_o to see that:

$$\int \frac{u_x^2}{u^3} f_o d^3\mathbf{v} = \frac{1}{3}\int \frac{u_x^2 + u_y^2 + u_z^3}{u^3} f_o d^3\mathbf{v} = \frac{1}{3}\int \frac{u^2}{u^3} f_o d^3\mathbf{v}$$

$$= \frac{1}{3}\int_0^\alpha \frac{v_e}{v} f_o 4\pi v^2 dv$$

$$= \frac{2\pi}{3} v_e \int_0^\alpha f_o 2v dv$$

$$= \frac{2\pi}{3} v_e \frac{n_e}{(2\pi)^{\frac{3}{2}} v_e^3} \int_0^\alpha \exp\left(\frac{-v^2}{2v_e^2}\right) dv^2$$

$$= \frac{2\pi}{3} \frac{n_e}{(2\pi)^{\frac{3}{2}}} 2 = \frac{2}{3\,(2\pi)^{\frac{1}{2}}} n_e \,. \tag{3.103}$$

Thus the Maxwell-averaged momentum-loss frequency is

$$-\frac{1}{p}\frac{dp}{dt} \equiv \overline{\nu}_{ei} = \frac{2}{3\,(2\pi)^{\frac{1}{2}}} \nu_p(v_t) \tag{3.104}$$

(where $p = m_e v_d n_e$ is the momentum per unit volume attributable to drift).

$$\overline{\nu}_{ei} = \frac{2}{3\,(2\pi)^{\frac{1}{2}}} n_i \frac{q_e^2 q_i^2}{(4\pi\epsilon_0)^2} \frac{4\pi\,(m_e + m_i)}{m_i m_e^2 v_e^3} \ln \Lambda_e \tag{3.105}$$

$$= \frac{2}{3\,(2\pi)^{\frac{1}{2}}} n_i \left(\frac{ze^2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e^{\frac{1}{2}} T_e^{\frac{3}{2}}} \ln \Lambda_e \tag{3.106}$$

(substituting for thermal electron velocity, v_e, and dropping $\frac{m_e}{m_i}$ order term), where $Ze = q_i$. This is the standard form of electron collision frequency.

3.4.2 $i \to e$

Ion momentum loss to electrons can be treated by a simple Galilean transformation of the $e \to i$ case because it is still the electron thermal motions that matter.

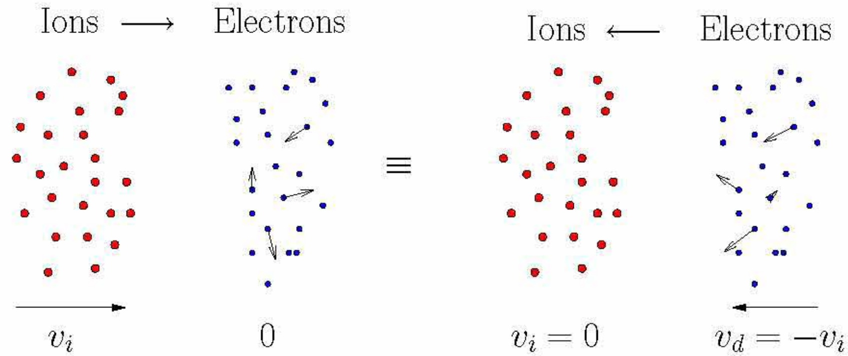

Figure 3.6: Ion-electron collisions are equivalent to electron-ion collisions in a moving reference frame.

Rate of momentum transfer, $\frac{dp}{dt}$, is same in both cases:

$$\frac{dp}{dt} = -p\nu \tag{3.107}$$

Hence $p_e\nu_{ei} = p_i\nu_{ie}$ or

$$\overline{\nu}_{ie} = \frac{p_e}{p_i}\overline{\nu}_{ei} = \frac{n_e m_e}{n_i m_i}\,\overline{\nu}_{ei} \tag{3.108}$$

(since drift velocities are the same).

Ion momentum loss to electrons is much lower collision frequency than $e \to i$ because ions possess so much more momentum for the same velocity.

3.4.3 $i \to i$

Ion-ion collisions can be treated somewhat like $e \to i$ collisions except that we have to account for *moving targets* i.e. their thermal motion.

Consider two different ion species moving relative to each other with drift velocity v_d; the targets' thermal motion affects the momentum transfer cross-section.

Using our previous expression for momentum transfer, we can write the average rate of transfer per unit volume as: [see 3.74 "note for future reference"]

$$-\frac{d\mathbf{p}}{dt} = \int\int \mathbf{v}_r\,\frac{m_1 m_2}{m_1+m_2}\,v_r 4\pi\,b_{90}^2\,\ln\Lambda\;f_1 f_2 d^2v_1 d^3v_2 \tag{3.109}$$

where $\mathbf{v}_r$ is the relative velocity $(\mathbf{v}_1 - \mathbf{v}_2)$ and b_{90} is expressed

$$b_{90} = \frac{q_1 q_2}{4\pi\epsilon_0}\,\frac{1}{m_r v_r^2} \tag{3.110}$$

and m_r is the reduced mass $\frac{m_1 m_2}{m_1+m_2}$.
Since everything in the integral apart from $f_1 f_2$ depends only on the relative velocity, we proceed by transforming the velocity coordinates from $\mathbf{v}_1, \mathbf{v}_2$ to being expressed in terms of relative ($\mathbf{v}_r$) and average ($\mathbf{V}$ say)

$$\mathbf{v}_r \equiv \mathbf{v}_1 - \mathbf{v}_2 \quad ; \quad \mathbf{V} \equiv \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1+m_2} \quad . \tag{3.111}$$

Take f_1 and f_2 to be shifted Maxwellians in the overall C of M frame:

$$f_j = n_j \left(\frac{m_j}{2\pi T}\right)^{\frac{3}{2}} \exp\left[-\frac{m_j\left(\mathbf{v}_j - \mathbf{v}_{dj}\right)^2}{2T}\right] \quad (j = 1, 2) \tag{3.112}$$

where $m_1\mathbf{v}_{d1} + m_2\mathbf{v}_{d2} = 0$. Then

$$\begin{aligned} f_1 f_2 &= n_1 n_2 \left(\frac{m_1}{2\pi T}\right)^{\frac{3}{2}} \left(\frac{m_2}{2\pi T}\right)^{\frac{3}{2}} \exp\left[-\frac{m_1 v_1^2}{2T} - \frac{m_2 v_2^2}{2T}\right] \\ &\quad \times \left\{1 + \frac{\mathbf{v}_1.m_1\mathbf{v}_{d1}}{T} + \frac{\mathbf{v}_2.m_2\mathbf{v}_{d2}}{T}\right\} \end{aligned} \tag{3.113}$$

to first order in $\mathbf{v}_d$. Convert CM coordinates and find (after algebra)

$$\begin{aligned} f_1 f_2 &= n_1 n_2 \left(\frac{M}{2\pi T}\right)^{\frac{3}{2}} \left(\frac{m_r}{2\pi T}\right)^{\frac{3}{2}} \exp\left[-\frac{MV^2}{2T} - \frac{m_r v_r^2}{2T}\right] \\ &\quad \times \left\{1 + \frac{m_r}{T}\mathbf{v}_d.\mathbf{v}_r\right\} \end{aligned} \tag{3.114}$$

where $M = m_1 + m_2$. Note also that (it can be shown) $d^3v_1 d^3v_2 = d^3v_r d^3V$. Hence

$$\begin{aligned} -\frac{d\mathbf{p}}{dt} = \int\int \quad & \mathbf{v}_r m_r v_r 4\pi\ b_{90}^2\ \ln \Lambda n_1 n_2 \left(\frac{M}{2\pi T}\right)^{\frac{3}{2}} \left(\frac{m_r}{2\pi T}\right)^{\frac{3}{2}} \\ & \exp\left(-\frac{MV^2}{2T}\right) \exp\left(-\frac{m_r v_r^2}{2T}\right) \left\{1 + \frac{m_r}{T}\mathbf{v}_d.\mathbf{v}_r\right\} d^3v_r d^3V \end{aligned} \tag{3.115}$$

and since nothing except the exponential depends on V, that integral can be done:

$$-\frac{d\mathbf{p}}{dt} = \int \mathbf{v}_r\ m_r v_r 4\pi\ \ln\Lambda\ n_1 n_2 \left(\frac{m_r}{2\pi T}\right)^{\frac{3}{2}} \exp\left(\frac{-m_r v_r^2}{2\pi}\right) \left\{1 + \frac{m_r}{T}\mathbf{v}_d.\mathbf{v}_r\right\} d^3v_r \tag{3.116}$$

This integral is of just the same type as for $e - i$ collisions, i.e.

$$\begin{aligned} -\frac{dp}{dt} &= v_d v_{rt} m_r\ 4\pi\ b_{90}^2(v_{rt})\ \ln\Lambda_t\ n_1 n_2 \int \frac{u_x^2}{u_3} \hat{f}_o(v_r) d^3\mathbf{v}_r \\ &= v_d v_{rt} m_r 4\ \pi\ b_{90}^2(v_{rt})\ \ln\ \Lambda_t\ n_1 n_2\ \frac{2}{3\left(2\pi\right)^{\frac{2}{2}}} \end{aligned} \tag{3.117}$$

where $v_{rt} \equiv \sqrt{\frac{T}{m_r}}$, $b_{90}^2(v_{rt})$ is the ninety degree impact parameter evaluated at velocity v_{tr}, and $\hat{f}_o$ is the normalized Maxwellian.

$$-\frac{dp}{dt} = \frac{2}{3\,(2\pi)^{\frac{1}{2}}} \left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_r^2 v_{rt}^3} \;\ln\Lambda_t \; n_1 n_2 \; m_r v_d \tag{3.118}$$

This is the general result for momentum exchange rate between two Maxwellians drifting at small relative velocity v_d.

To get a collision frequency is a matter of deciding which species is stationary and so what the momentum density of the moving species is. Suppose we regard 2 as targets then momentum density is $n_1 m_1 v_d$ so

$$\overline{\nu}_{12} = \frac{1}{n_1 m_1 v_d}\frac{dp}{dt} = \frac{2}{3\,(2\pi)^{\frac{1}{2}}} n_2 \left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_r v_{rt}^3}\frac{\ln\Lambda_t}{m_1} \quad . \tag{3.119}$$

This expression works immediately for electron-ion collisions substituting $m_r \simeq m_e$, recovering previous.

For equal-mass ions $m_r = \frac{m_i^2}{m_i+m_i} = \frac{1}{2}m_i$ and $v_{rt} = \sqrt{\frac{T}{m_r}} = \sqrt{\frac{2T}{m_i}}$.
Substituting, we get

$$\overline{\nu}_{ii} = \frac{1}{3\pi^{\frac{1}{2}}} \; n_i \; \left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_i^{\frac{1}{2}} T_i^{\frac{3}{2}}} \;\ln\Lambda \tag{3.120}$$

that is, $\frac{1}{\sqrt{2}}$ times the $e-i$ expression but with ion parameters substituted. [Note, however, that we have considered the ion species to be different.]

3.4.4 $e \to e$

Electron-electron collisions are covered by the same formalism, so

$$\overline{\nu}_{ee} = \frac{1}{3\pi^{\frac{1}{2}}} \; n_e \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e^{\frac{1}{2}} T_e^{\frac{3}{2}}} \;\ln\Lambda \quad . \tag{3.121}$$

However, the physical case under discussion is not so obvious; since electrons are indistiguishable how do we define two different "drifting maxwellian" electron populations? A more specific discussion would be needed to make this rigorous.

Generally $\nu_{ee} \sim \nu_{ei}/\sqrt{2}$: electron-electron collision frequency $\sim$ electron-ion (for momentum loss).

3.4.5 Summary of Thermal Collision Frequencies

For *momentum loss*:

$$\overline{\nu}_{ei} \;=\; \frac{\sqrt{2}}{3\sqrt{\pi}} \; n_i \left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_e^{\frac{1}{2}} T_e^{\frac{3}{2}}} \;\ln\Lambda_e \quad . \tag{3.122}$$

$$\overline{\nu}_{ee} \simeq \frac{1}{\sqrt{2}}\,\overline{\nu}_{ei} \quad . \qquad \text{(electron parameters)} \tag{3.123}$$

$$\overline{\nu}_{ie} = \frac{n_e m_e}{n_i m_i}\,\overline{\nu}_{ei} \quad . \tag{3.124}$$

$$\overline{\nu}_{ii\prime} = \frac{\sqrt{2}}{3\sqrt{\pi}}\, n_{i\prime} \left(\frac{q_i q_{i\prime}}{4\pi\epsilon_0}\right)^2 \frac{4\pi}{m_i^{\frac{1}{2}} T_i^{\frac{3}{2}}} \left(\frac{m_{i\prime}}{m_i + m_{i\prime}}\right)^{\frac{1}{2}} \ln \Lambda_i \tag{3.125}$$

Energy loss ${}^K\nu$ related to the above (${}^p\nu$) by

$$^K\nu = \frac{2m_i}{m_1 + m_2}\;{}^p\nu \quad . \tag{3.126}$$

Transverse 'diffusion' of momentum ${}^\perp\nu$, related to the above by:

$$^\perp\nu = \frac{2m_2}{m_1 + m_2}\;{}^p\nu \quad . \tag{3.127}$$

3.5 Applications of Collision Analysis

3.5.1 Energetic ('Runaway') Electrons

Consider an energetic ($\frac{1}{2}m_e v_1^2 >> T$) electron travelling through a plasma. It is slowed down (loses momentum) by collisions with electrons and ions (Z), with collision frequency:

$$^p\nu_{ee} = \nu_{ee} = n_e \frac{e^4}{(4\pi\epsilon_0)^2} \frac{8\pi}{m_e^2 v_1^3} \ln \Lambda \tag{3.128}$$

$$^p\nu_{ei} = \frac{1}{2}\, Z\, \nu_{ee} \tag{3.129}$$

Hence (in the absence of other forces)

$$\frac{d}{dt}(m_e v) = -\left({}^p\nu_{ee} + {}^p\nu_{ei}\right) m_p v \tag{3.130}$$

$$= -\left(1 + \frac{Z}{2}\right) \nu_{ee}\, m_e v \tag{3.131}$$

This is equivalent to saying that the electron experiences an effective 'Frictional' force

$$F_f = \frac{d}{dt}(m_e v) = -\left(1 + \frac{Z}{2}\right) \nu_{ee}\, m_e v \tag{3.132}$$

$$F_f = -\left(1 + \frac{Z}{2}\right) n_e \frac{e^4}{(4\pi\epsilon_0)^2} \frac{8\pi \ \ln \Lambda}{m_e v^2} \tag{3.133}$$

Notice

1. for $Z = 1$ slowing down is $\frac{2}{3}$ on electrons $\frac{1}{3}$ ions
2. F_f decreases with v increasing.

Suppose now there is an electric field, E. The electron experiences an accelerating Force. Total force

$$F = \frac{d}{dt}(mv) = -eE + F_f = -eE - \left(1 + \frac{Z}{2}\right) n_e \frac{e^4}{(4\pi\epsilon_0)^2} \frac{8\pi \ln \Lambda}{m_e v^2} \tag{3.134}$$

Two Cases (When E is accelerating)

1. $|eE| < |F_f|$: Electron Slows Down
2. $|eE| > |F_f|$: Electron Speeds Up!

Once the electron energy exceeds a certain value its velocity increases continuously and the friction force becomes less and less effective. The electron is then said to ahve become a *'runaway'*.

Condition:

$$\frac{1}{2} m_e v^2 > \left(1 + \frac{Z}{2}\right) n_e \frac{e^4}{(4\pi\epsilon_0)^2} \frac{8\pi \ln \Lambda}{2eE} \tag{3.135}$$

3.5.2 Plasma Resistivity (DC)

Consider a bulk distribution of electrons in an electric field. They tend to be accelerated by E and decelerated by collisions.

In this case, considering the electrons as a whole, no loss of total electron momentum by $e - e$ collisions. Hence the friction force we need is just that due to $\overline{\nu}_{ei}$.

If the electrons have a mean drift velocity $v_d (<< v_{the})$ then

$$\frac{d}{dt}(m_e v_d) = -eE - \overline{\nu}_{ei} m_e v_d \tag{3.136}$$

Hence in steady state

$$v_d = \frac{-eE}{m_e \overline{\nu}_{ei}} \quad . \tag{3.137}$$

The current is then

$$j = -n_e e v_d = \frac{n_e e^2 E}{m_e \overline{\nu}_{ei}} \tag{3.138}$$

Now generally, for a conducting medium we define the conductivity, σ, or resistivity, η, by

$$j = \sigma E \quad ; \quad \eta j = E \qquad \left(\sigma = \frac{1}{\eta}\right) \tag{3.139}$$

Therefore, for a plasma,

$$\sigma = \frac{1}{\eta} = \frac{n_e e^2}{m_e \overline{\nu}_{ei}} \tag{3.140}$$

Substitute the value of $\overline{\nu}_{ei}$ and we get

$$\eta \simeq \frac{n_i Z^2}{n_e} \cdot \frac{e^2 m_e^{\frac{1}{2}}\, 8\pi\ \ln\Lambda}{(4\pi\epsilon_0)^2\ 3\sqrt{2\pi}\ T_e^{\frac{3}{2}}} \tag{3.141}$$

$$= \frac{Z e^2 m_e^{\frac{1}{2}}\, 8\pi\ \ln\Lambda}{(4\pi\epsilon_0)^2\ 3\sqrt{2\pi}\ T_e^{\frac{3}{2}}} \quad \text{(for a single ion species).} \tag{3.142}$$

Notice

1. Density cancels out because more electrons means (a) more carriers but (b) more collisions.
2. Main dependence is $\eta \propto T_e^{-3/2}$. High electron temperature implies low resistivity (high conductivity).
3. This expression is only approximate because the current tends to be carried by the more energetic electrons, which have smaller ν_{ei}; thus if we had done a proper average over $f(v_e)$ we expect a lower numerical value. Detailed calculations give

$$\eta = 5.2 \times 10^{-5} \frac{\ln\Lambda}{(T_e/eV)^{\frac{3}{2}}}\ \Omega m \tag{3.143}$$

for $Z = 1$ (vs. $\simeq 10^{-4}$ in our expression). This is 'Spitzer' resistivity. The detailed calculation value is roughly a factor of two smaller than our calculation, which is not a negligible correction!

3.5.3 Diffusion

For motion *parallel* to a magnetic field if we take a typical electron, with velocity $v_\parallel \simeq v_{te}$ it will travel a distance approximately

$$\ell_e = v_{te}/\overline{\nu}_{ei} \tag{3.144}$$

before being pitch-angle scattered enough to have its velocity randomised. [This is an order-of-magnitude calculation so we ignore $\overline{\nu}_{ee}$.] ℓ is the mean free path.

Roughly speaking, any electron does a random walk along the field with step size ℓ and step frequency $\overline{\nu}_{ei}$. Thus the diffusion coefficient of this process is

$$D_{e\parallel} \simeq \ell_e^2\ \overline{\nu}_{ei} \simeq \frac{v_{te}^2}{\overline{\nu}_{ei}}\ . \tag{3.145}$$

Similarly for ions

$$D_{i\parallel} \simeq \ell_i^2 \overline{\nu}_{ii} \simeq \frac{v_{ti}^2}{\overline{\nu}_{ii}} \tag{3.146}$$

Notice

$$\overline{\nu}_{ii}/\overline{\nu}_{ei} \simeq \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \simeq \frac{v_{ti}}{v_{te}} \qquad (\text{if } T_e \simeq T_i) \tag{3.147}$$

Hence $\quad \ell_e \simeq \ell_i$

Mean free paths for electrons and ions are $\sim$ same.

The diffusion coefficients are in the ratio

$$\frac{D_i}{D_e} \simeq \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \quad : \quad \text{Ions diffuse slower in parallel direction.} \tag{3.148}$$

Diffusion Perpendicular to Mag. Field is different

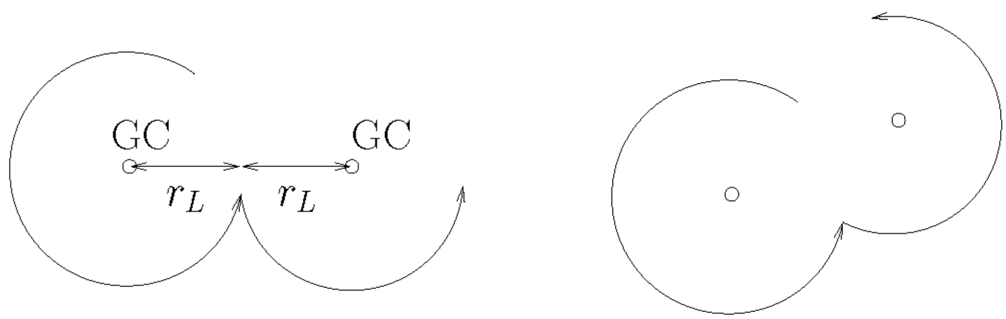

Figure 3.7: Cross-field diffusion by collisions causing a jump in the gyrocenter (GC) position.

Roughly speaking, if electron direction is changed by $\sim 90°$ the Guiding Centre moves by a distance $\sim r_L$. Hence we may think of this as a random walk with step size $\sim r_L$ and frequency $\overline{\nu}_{ei}$. Hence

$$D_{e\perp} \simeq r_{Le}^2 \overline{\nu}_{ei} \simeq \frac{v_{te}^2}{\Omega_e^2} \overline{\nu}_{ei} \tag{3.149}$$

Ion transport is similar but requires a discussion of the effects of *like* and *unlike* collisions.

Particle transport occurs only via *unlike* collisions. To show this we consider in more detail the change in guiding center position at a collision. Recall $m\dot{\mathbf{v}} = q\mathbf{v} \wedge \mathbf{B}$ which leads to

$$\mathbf{v}_\perp = \frac{q}{m}\mathbf{r}_L \wedge \mathbf{B} \qquad (\text{perp. velocity only}). \tag{3.150}$$

This gives

$$\mathbf{r}_L = \frac{\mathbf{B} \wedge m\mathbf{v}_\perp}{qB^2} \tag{3.151}$$

At a collision the particle position does not change (instantaneously) but the guiding center position ($\mathbf{r}_0$) does.

$$\mathbf{r}_0' + \mathbf{r}_L' = \mathbf{r}_0 + \mathbf{r}_L \Rightarrow \Delta\mathbf{r}_0 \equiv \mathbf{r}_0' - \mathbf{r}_0 = -(\mathbf{r}_L' - \mathbf{r}_L) \tag{3.152}$$

Change in $\mathbf{r}_L$ is due to the momentum change caused by the collision:

$$\mathbf{r}'_L - \mathbf{r}_L = \frac{\mathbf{B}}{qB^2} \wedge m(\mathbf{v}' - \mathbf{v}) \equiv \frac{\mathbf{B}}{qB^2} \wedge \boldsymbol{\Delta}(m\mathbf{v}) \tag{3.153}$$

So

$$\boldsymbol{\Delta}\mathbf{r}_0 = -\frac{\mathbf{B}}{qB^2} \wedge \boldsymbol{\Delta}(m\mathbf{v}). \tag{3.154}$$

The total momentum conservation means that $\boldsymbol{\Delta}(m\mathbf{v})$ for the two particles colliding is equal and opposite. Hence, from our equation, for *like* particles, $\boldsymbol{\Delta}\mathbf{r}_0$ is equal and opposite. The mean position of guiding centers of two colliding like particles $(\mathbf{r}_{01} + \mathbf{r}_{02})/2$ does not change.

No net cross field particle (guiding center) shift.

Unlike collisions (between particles of different charge q) *do* produce net transport of particles of either type. And indeed may move $\mathbf{r}_{01}$ and $\mathbf{r}_{02}$ in same direction if they have opposite charge.

$$D_{i\perp} \simeq r_{Li}^2\ \overline{{}^p\nu}_{ie} \simeq \frac{v_{ti}^2}{\Omega_i^2}\ \overline{{}^p\nu}_{ie} \tag{3.155}$$

Notice that $r_{Li}^2/r_{Le}^2 \simeq m_i/m_e \quad ; \quad \overline{{}^p\nu}_{ie}/\overline{\nu}_{ei} \simeq \frac{m_e}{m_i}$

So $D_{i\perp}/D_{e\perp} \simeq 1$ (for equal temperatures). Collisional diffusion rates of *particles* are same for ions and electrons.

However *energy* transport is different because it *can* occur by like-like collisins.

Thermal Diffusivity:

$$\chi_e \sim r_{Le}^2\,(\overline{\nu}_{ei} + \overline{\nu}_{ee}) \sim r_{Le}^2\ \overline{\nu}_{ei} \quad (\overline{\nu}_{ei} \sim \overline{\nu}_{ee}) \tag{3.156}$$

$$\chi_i \sim r_{Li}^2\,(\overline{{}^p\nu}_{ie} + \overline{\nu}_{ii}) \simeq r_{Li}^2\ \overline{\nu}_{ii} \quad (\overline{\nu}_{ii} >> \overline{\nu}_{ie}) \tag{3.157}$$

$$\chi_i/\chi_e \sim \frac{r_{Li}^2}{r_{Le}^2}\ \frac{\overline{\nu}_{ii}}{\overline{\nu}_{ei}} \simeq \frac{m_i}{m_e}\ \frac{m_e^{\frac{1}{2}}}{m_i^{\frac{1}{2}}} = \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \quad \text{(equal T)} \tag{3.158}$$

Collisional *Thermal* transport by *Ions* is *greater* than by *electrons* [factor $\sim (m_i/m_e)^{\frac{1}{2}}$].

3.5.4 Energy Equilibration

If $T_e \neq T_i$ then there is an exchange of enegy between electrons and ions tending to make $T_e = T_i$. As we saw earlier

$${}^K\nu_{ei} = \frac{2m_e}{m_i}\ {}^p\nu_{ei} = \frac{m_e}{m_i}\ {}^{\perp}\nu_{ei} \tag{3.159}$$

So applying this to averages.

$$\overline{{}^K\nu}_{ei} \simeq \frac{2m_e}{m_i}\ \overline{\nu}_{ei} \qquad (\simeq \overline{\nu}_{ie}) \tag{3.160}$$

Thermal energy exchange occurs $\sim m_e/m_i$ *slower* than momentum exchange. (Allows $T_e \neq T_i$). So

$$\frac{dT_e}{dt} = -\frac{dT_i}{dt} = -\overline{^K\nu}_{ei}\left(T_e - T_i\right) \tag{3.161}$$

From this one can obtain the heat exchange rate (per unit volume), H_{ei}, say:

$$\begin{aligned} H_{ei} &= -\frac{d}{dt}\left(\frac{3}{2}n_e T_e\right) = \frac{d}{dt}\left(\frac{3}{2}n_i T_i\right) && (3.162) \\ &= -\frac{3}{4}n\frac{d}{dt}\left(T_e - T_i\right) = \frac{3}{2}n\overline{^K\nu}_{ei}\left(T_e - T_i\right) && (3.163) \end{aligned}$$

Important point:

$$\overline{^K\nu}_{ei} \simeq \frac{m_e}{m_i} Z\nu_{ee} \simeq \frac{1}{Z^2}\left(\frac{M_e}{m_i}\right)^{\frac{1}{2}}\overline{\nu}_{ii}. \tag{3.164}$$

'Electrons and Ions equilibrate among themselves much faster than with each other'.

3.6 Some Orders of Magnitude

1. $\ln\Lambda$ is very slowly varying. Typically has value $\sim$ 12 to 16 for laboratory plasmas.

2. $\overline{\nu}_{ei} \approx 6 \times 10^{-11}(n_i/m^3)\,/\,(T_e/eV)^{\frac{3}{2}}$ $\quad(\ln\Lambda = 15, Z = 1)$.
   e.g. $= 2 \times 10^5 s^{-1}$ (when $n = 10^{20} m^{-3}$ and $T_e = 1keV$.) For phenomena which happen much faster than this, i.e. $\tau \ll 1/\nu_{ei} \sim 5\mu s$, collisions can be ignored.
   Examples: Electromagnetic Waves with high frequency.

3. *Resistivity.* Because most of the energy of a current carrying plasma is in the B field not the K.E. of electrons. Resistive decay of current can be much slower than $\overline{\nu}_{ei}$. E.g.
   Coaxial Plasma: (Unit length)
   Inductance $\quad L = \mu_o \ \ln\frac{b}{a}$
   Resistance $\quad R = \eta \ 1/\pi a^2$
   L/R decay time

$$\begin{aligned} \tau_R &\sim \frac{\mu_o \pi a^2}{\eta}\ \ln\frac{b}{a} \simeq \frac{n_e e^2}{m_e \overline{\nu}_{ei}}\mu_o\ \pi a^2 \ln\frac{b}{a} \\ &\sim \frac{n_e e^2}{m_e \epsilon_0}\,\frac{a^2}{c^2}\,\frac{1}{\overline{\nu}_{ei}} = \frac{\omega_p^2 a^2}{c^2}\cdot\frac{1}{\overline{\nu}_{ei}} >> \frac{1}{\overline{\nu}_{ei}}\ . && (3.165) \end{aligned}$$

Comparison 1 keV temperature plasma has $\sim$ same (conductivity/) resistivity as a slab of *copper* ($\sim 2 \times 10^{-8}\Omega m$).

Ohmic Heating Because $\eta \propto T_e^{-3/2}$, if we try to heat a plasma Ohmically, i.e. by simply passing a current through it, this works well at low temperatures but its effectiveness falls off rapidly at high temperature.

Result for most Fusion schemes it looks as if Ohmic heating does not quite yet get us to the required ignition temperature. We need auxilliary heating, e.g. Neutral Beams. (These slow down by collisions.)