Lecture 1: Derivation of the Boltzmann Equation

Introduction

1. The basic model describing MHD and transport theory in a plasma is the Boltzmann-Maxwell equations.

2. This is a coupled set of kinetic equations and electromagnetic equations.

3. Initially the full set of Maxwell's equation is maintained.

4. Also, each species is described by a distribution function $f_j(\mathbf{r}, \mathbf{v}, t)$ satisfying a kinetic equation including collisions.

5. The equations are incredibly general and (incredibly)^2 complicated to solve.

6. Our basic approach is to start from this general set of equations and then simplify them by restricting attention to the relatively slow time scales and large length scales associated with MHD and transport.

7. The end result is a set of "simpler" fluid equations which evolve the electromagnetic fields E and B and the fluid quantities $n, \mathbf{v}, p$ in time.

8. The fluid equations will contain certain transport coefficients (e.g. D, κ, η) which are calculated by means of a small gyroradius expansion of the kinetic equations. (More about this later in the term).

9. Keep in mind that the "simpler" fluid model turns out to be a set of nonlinear, three dimensional, time dependent equations. Thus, the model is still enormously difficult to solve.

10. As the models are developed during the lectures, there will be a large number of applications, almost entirely aimed at magnetic fusion. This is important in helping to understand how fusion plasmas behave, as well as providing a down to earth foundation for the model development which tends to be somewhat formal at times.

11. In the first part of model development, no distinction is necessary between MHD and transport. However, after the fluid equations are derived, MHD and transport are separated by making a finer scale distinction between the long time and large length scales involved.

12. Since the entire MHD-transport model is based on the Boltzmann-Maxwell equations, the first step in the theoretical development is a derivation of the Boltzmann equation. For simplicity a simple heuristic derivation is presented.

Heuristic Derivation of the Boltzmann Equation

1. Plan of attack

   a. Derivation is based on a common sense conservation of particle relation.

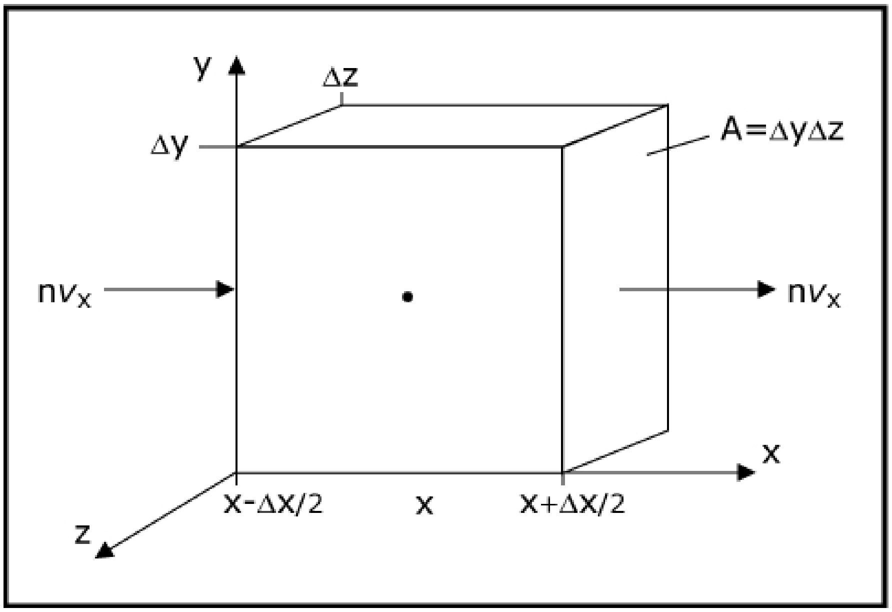
   b. Derive conservation of particles for a simple fluid in physical space.

c. Generalize to a 6-D phase space assuming only long range forces are present- that is, neglect collisions (Vlasov equation).

d. Add in the effect of collisions (Boltzmann equation).

e. Discuss general conservation properties of the collision operator.

2. Conservation of particles in a fluid. Consider the 1-D geometry illustrated below



3. Conservation of particles states that the gain in the number of particles in a fixed Eulerian volume element $\Delta x \Delta y \Delta z$ is given by

gain in particles = flow in - flow out + sources - sinks

4. Evaluate terms separately

gain in particles in a time $\Delta t = \left[n(x, t+\Delta t) - n(x,t)\right]\Delta x \Delta y \Delta z$

flow in of particles in a time $\Delta t = (\text{flux})(\text{area})(\Delta t) = \Delta y \Delta z \Delta t \left(n v_x\right)\big|_{x-\Delta x/2}$

flow out of particles in a time $\Delta t = (\text{flux})(\text{area})(\Delta t) = \Delta y \Delta z \Delta t \left(n v_x\right)\big|_{x+\Delta x/2}$

sources – sinks in a time $\Delta t =$ (source density/time) (Vol)$(\Delta t) = S_n \Delta x \Delta y \Delta z \Delta t$

5. Taylor expand assuming small $\Delta t, \Delta x, \Delta y, \Delta z$

a. $$\Delta x \Delta y \Delta z \Delta t \frac{\partial n}{\partial t} = \Delta y \Delta z \Delta t \left[n v_x \big|_x - \frac{\Delta x}{2} \frac{\partial (n v_x)}{\partial x}\bigg|_x \right] - \Delta y \Delta z \Delta t \left[n v_x \big|_x + \frac{\Delta x}{2} \frac{\partial (n v_x)}{\partial x}\bigg|_x \right] + S_n \Delta x \Delta y \Delta z \Delta t$$

b. $$\frac{\partial n}{\partial t} + \frac{\partial}{\partial_x} n v_x = S_n(x,t)$$

conversion of particles in 1-D.

6. Generalize to 3-D

a. $\frac{\partial n}{\partial t} + \frac{\partial(nv_x)}{\partial x} + \frac{\partial(nv_y)}{\partial y} + \frac{\partial(nv_z)}{\partial z} = S_n$

b. $\frac{\partial n}{\partial t} + \nabla \cdot n\mathrm{v} = S_n(\mathrm{x}, t)$

Kinetic Generalization

1. Now generalize the concept to a 6-D phase space.

   a. $n(x, t) \to f(x, v, t) = 6$ dimensional density.

   b. $\Delta x \Delta y \Delta z \to \Delta x \Delta y \Delta z \Delta v_x \Delta v_y \Delta v_z = 6$ -D volume element.

2. Particles flow into or out of the physical part of the volume element with velocity *v*.

3. Particles accelerate into or out of the velocity part of the volume element with an acceleration *a*.

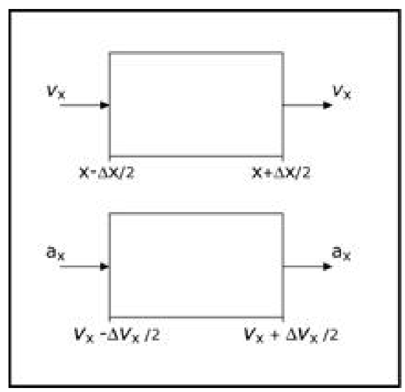
4. Note that *v* and *x* are independent coordinates (variables). At any point *x*, a particle can have any velocity *v*.

5. However, in general $a = a(x, v, t)$. A particle at *x* moving with velocity *v* will have a known acceleration as determined by Newton's law and the specific force field under consideration.

6. Same conservation law applies

   gain in particles = flow in − flow out + sources − sinks

7. Evaluate terms separately for 1-D case : $\Delta x \equiv \Delta x \Delta y \Delta z,\ \Delta v \equiv \Delta v_x \Delta v_y \Delta v_z$



   gain in a time $\Delta t = \left[f(x, v, t + \Delta t) - f(x, v, t)\right] \Delta x \Delta v$

   flow into physical space = $\Delta y \Delta z \Delta t (v_x f \Delta v)\big|_{x - \Delta x/2}$

   flow out of physical space = $\Delta y \Delta z \Delta t (v_x f \Delta v)\big|_{x + \Delta x/2}$

   flow into velocity space = $\Delta v_y \Delta v_z \Delta t (a_x f \Delta x)\big|_{v_x - \Delta v_x/2}$

   flow out of velocity space = $\Delta v_y \Delta v_z \Delta t (a_x f \Delta x)\big|_{v_x + \Delta v_x/2}$

   sources − sinks = $s \Delta x \Delta v \Delta t$

8. Taylor expand as before

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(v_x f) - \frac{\partial}{\partial v_x}(a_x f) + s(x, v_x, t)$$

   Note that $\partial v_x / \partial x = 0$ (independent coordinates)

9. Generalize to 3-D

   a. $\frac{\partial f}{\partial t} + \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial_y} + v_z \frac{\partial}{\partial z}\right) f + \left(a_x \frac{\partial}{\partial v_x} + a_y \frac{\partial}{\partial v_y} + a_z \frac{\partial}{\partial v_z}\right) f = -f\left(\frac{\partial a_x}{\partial v_x} + \frac{\partial a_y}{\partial v_y} + \frac{\partial a_z}{\partial v_z}\right) + s$

b. $\frac{\partial f}{\partial t} + v \cdot \nabla f + a \cdot \nabla_v f = s - f \nabla_v \cdot a$

The Vlasov Equation

1. Now consider a for a plasma. In most cases of interest a can be divided into two parts: $a = a_S$ (short range) $+ a_l$ (long range)

2. Short range forces act over a distance $\lesssim \lambda_d$. These are the collisions.

3. Long range forces act over distances $\gtrsim \lambda_d$. For $x \gtrsim \lambda_d$, the short range collisional forces are shielded out by the Debye effect.

4. Assume for now that collisional effects are negligible. Also assume that the number of particles in a Debye sphere is large so that a statistical description makes sense. Even so, the Debye length is assumed small compared to the gradient lengths of interest. Similarly, the time scales of interest are assumed much slower than ω_{pe}.

5. All but the first collisional assumptions are well satisfied in fusion plasmas.

6. In the collisionless limit

$$a_l\,(x, v, t) = \frac{q}{m}\left(E + v \times B\right)$$

where E(x, t) and B(x, t) are the smooth long range electric and magnetic fields.

7. The quantity $\nabla_v \cdot a$ reduces to

$$\nabla_v \cdot a_l = \frac{q}{m} \nabla_v \cdot \left(E + v \times B\right)$$

$$= \frac{q}{m}\left(B \cdot \nabla_v \times v - v \cdot \nabla_v \times B\right)$$

$$= \frac{q}{m} B \cdot \nabla_v \times v$$

$$= 0$$

8. When there are no sources or sinks present $\left(s = 0\right)$, the 6-D conservation relation reduces to

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{q}{m}\left(E + v \times B\right) \cdot \nabla_v f = 0$$

9. This is the Vlasov equation. It has the simple interpretation that

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \frac{dx}{dt} \cdot \nabla f + \frac{dv}{dt} \cdot \nabla_v f = 0$$

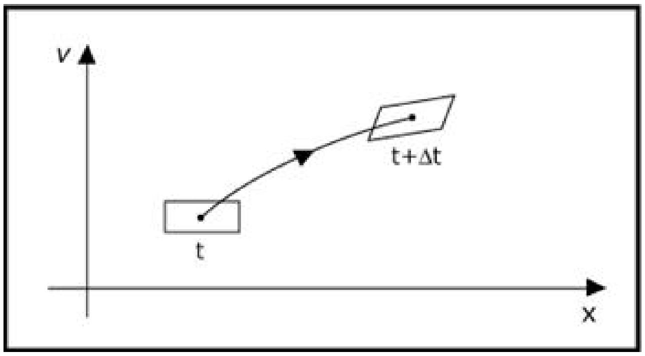
along the trajectory

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \frac{q}{m}\left(E + v \times B\right)$$

10. The density in phase space is conserved moving with the particle orbits.

11. As *f* evolves in phase space the volume element $\Delta x \Delta v$ varies smoothly as the particles move



12. With only long range forces acting, no particles suddenly enter or leave the phase space volume element due to collisions. Thus, moving with the particles, the quantity $f \Delta x \Delta v$ is by definition conserved.

13. Furthermore, the condition $\nabla_v \cdot a = 0$ is equivalent to the fact that $\Delta x \Delta v$ is by itself conserved. Thus, not only is the total number of particles but the $f \Delta x \Delta v$ conserved, but the density *f* as well. This is the significance of the Vlasov equation.

The Boltzmann Equation

1. The Boltzmann equation makes use of the same assumption as the Vlasov equation concerning the number of particles in a Debye sphere and the smallness of λ_d and $1/\omega_{pe}$. However, it does not neglect the effect of collisions.

2. The conventional approach is to place all terms with a, on the right-hand side of the equation and simply call them $(\partial f / \partial t)_c$.

3. Since collisions can occur between both like and unlike particles, the collision term is usually written as a sum over all species. Specifically, we write

$$\left(\frac{\partial f_j}{\partial t}\right)_c \equiv \sum_k C_{jk}\left(x,\ v,\ t\right)$$

where C_{jk} represents the change in f_j due to collisions with species *k*.

4. In this formalism the Boltzmann equation for species *j* can be written as

$$\frac{\partial f_j}{\partial t} + v \cdot \nabla f_j + \frac{q_j}{m_j}\left(E + v \times B\right) \cdot \nabla_V f_j = \sum_k C_{jk} + s_j$$

5. These kinetic equations are coupled to Maxwell's equation as follows

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$\nabla \cdot E = \frac{\sigma}{\epsilon_0}$$

$$\nabla \cdot B = 0$$

6. Here, the smoothed charge and current densities determining the long range electric and magnetic fields are calculated as follows

$$\sigma(x,t) = \sum_j q_j n_j = \sum_j q_j \int f_j dv$$

$$J(x,t) = \sum_j q_j n_j u_j = \sum_j q_j \int v f_j dv$$

7. The Boltzmann-Maxwell equations are a set of nonlinear, 6-D, time dependent integro-differential equations – indeed a complex model.

The Collision Operator

1. At this point the collision operators C_{jk} have yet to be determined. In fact a good part of the course will involve procedures for determining C_{jk} and then solving for the corresponding f_j using appropriate expansions.

2. However, even without giving specific forms for C_{jk}, it is possible to proceed, at least in a formal manner, and determine a set of "simplified" fluid equations by taking moments of the Boltzmann equation.

3. Certain terms in the moment equations are written in terms of the C_{jk}. These terms can be somewhat simplified by invoking general conservation relations involving C_{jk}.

4. These conservation laws arise from the assumption that the collisions characterized by C_{jk} are purely elastic – Coulomb collision to be specific.

5. Inelastic collisions representing ionization, recombination, charge exchange, alpha production, etc. are assumed to be contained in the source term s_j. Collisions of this type do not play as dominant a role in fusion plasmas as Coulomb collisions and hence do not need to be modelled with as much detail and accuracy.

6. For purely elastic two body Coulomb collisions it is accurately assumed that collisions take place locally, at a single point in space. It then follows that the following conservation laws are exactly satisfied.

   a. Conservation of particles in like and unlike particle collisions

   $$\int C_{jk} dv = 0 \quad \text{all } j \text{ and } k$$

   b. Conservation of momentum between like particle collisions

   $$\int m_j v C_{jj} dv = 0$$

   c. Conservation of energy between like particle collisions

$$\int \frac{m_j v^2}{2} C_{jj} d\mathbf{v} = 0$$

d. Conservation of total momentum between unlike particle collisions

$$\sum_{j,k} \int m_j \mathbf{v} C_{jk} d\mathbf{v} = 0 \quad j \neq k$$

e. Conservation of total energy between unlike particle collisions

$$\sum_{j,k} \int \frac{m_j v^2}{2} C_{jk} d\mathbf{v} = 0 \quad j \neq k$$

7. With this introduction we are now ready to calculate the moment equations.