22.615, MHD Theory of Fusion Systems
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Lecture 15: Variational Techniques

1. Variational Procedure
2. Variational formulation of MHD
3. Energy Principle
4. Extended Energy Principle

Variational Procedure

1. Alternate representation of a differential equation
2. Consider classic eigenvalue problem

$$\frac{d}{dx}\left(F\frac{dy}{dx}\right)+(\lambda-g)\,y=0 \qquad y(0)=y(1)=0 \qquad F=F(x),\,g=g(x)$$

3. Methods of solution

   a. inspection if F, g simple

   b. computer

   c. power series (not useful usually)

   d. fourier series (not useful usually)

   e. variational procedure (allows guess and a solution)
      λ is more accurate than guess for y(x)

4. Variational principle: multiply by $\int_0^1 y\,dx$ $\left[\int\rightarrow\int_0^1\right]$ below

$$\int\left[y\frac{d}{dx}\left(F\frac{dy}{dx}\right)+(\lambda-g)\,y^2\right]dx=0$$

$$+\int\left[-Fy'^2+\lambda y^2-gy^2\right]dx+Fy\,y'\Big|_0^1=0$$

$$\lambda=\frac{\int\left(Fy'^2+gy^2\right)dx}{\int y^2dx} \qquad (1)$$

5. Why is this variational? Substitute all allowable y(x) into (1). When resulting λ exhibits an extremum (maximum, minimum, saddle pt) then λ and y are actual eigenvalue and eigenfunction.

6. Proof: assume $y_0 = (x)$ is substituted into (1) yielding λ_0, Modify y a little bit by a small but <u>arbitrary</u> perturbation.

$$y(x) = y_0(x) + \delta y(x) \qquad \delta y(0) = \delta y(1) = 0$$

This produces a change in $\lambda = \lambda_0 + \delta\lambda$ given by

$$\delta\lambda = \frac{\int \left[F\left(y_0 + \delta y\right)'^2 + g\left(y_0 + \delta y\right)^2\right] dx}{\int \left(y_0 + \delta y\right)^2 dx} - \frac{\int \left(F y_0'^2 + g y_0^2\right) dx}{\int y_0^2 dx}$$

7. For small δy

$$\delta\lambda = \frac{\cancel{\int \left(F y_0'^2 + g y_0^2\right) dx}}{\int y_0^2 dx} - \frac{\cancel{\int \left(F y_0'^2 + g y_0^2\right) dx}}{\int y_0^2 dx} + 2\frac{\int \left(F y_0' \delta y' + g y_0 \delta y\right) dx}{\int y_0^2 dx}$$

$$- \underbrace{\frac{2\int \left(F y_0'^2 + g y_0^2\right) dx}{\int y_0^2 dx}}_{\lambda_0} \frac{\int y_0 \delta y \, dx}{\int y_0^2 dx}$$

integrate by parts

$$= \frac{2\int \left[F y_0' \delta y' + g y_0 \delta y - \lambda_0 y_0 \delta y\right] dx}{\int y_0^2 dx}$$

$$\partial\lambda = \frac{-2\int dx \, \delta y \left[\left(F y_0'\right)' + \left(\lambda_0 - g\right) y_0\right]}{\int y_0^2 dx}$$

8. At an extremum $\delta\lambda = 0$ for <u>arbitrary</u> δy, implying

$$\left(F y_0'\right)' + \left(\lambda_0 - g\right) y_0 = 0$$

This is equivalent to original problem

9. One can multiply original equation by h(x) g(x) when h is arbitrary resulting expression

$$\lambda = \frac{\int \left\{ h F y'^2 + \left[\left(hg - \left(F h'\right)^{1/2}\right)\right] y^2 \right\} dx}{\int h y^2 dx}$$

is correct mathematical expression but <u>is not</u> a variational principle unless h=1. Varying y gives

$$\frac{d}{dx}\left(hF\frac{dy}{dx}\right)+\left[\lambda h-hg+\frac{\left(Fh^{'}\right)^{'}}{2}\right]y=0$$

or $$\frac{d}{dx}\left(F\frac{dy}{dx}\right)+F\frac{h^{'}}{h}\frac{dy}{dx}+\left[\lambda-g+\frac{\left(Fh^{'}\right)^{'}}{2h}\right]y=0$$

10. Since $\delta\lambda=0$ when y coincides with a true eigenfunction, this implies that an estimate for λ using a guess (trial function) for g is more accurate than the trial function itself.

11. Proof: Write $y=y_0+\delta y$ where y_0 is true eigenfunction and δy is the error in the guess, assumed of order $\in$. Substitute into (1) yields

$$\lambda=\frac{N_0+N_1+N_2}{D_0+D_1+D_2}=\frac{N_0+N_1+N_2}{D_0}\left[1-\frac{D_1}{D_0}-\frac{D_2}{D_0}+\frac{D_1^2}{D_0^2}+\dots\right]$$

$$=\frac{N_0}{D_0}+\frac{1}{D_0}\underbrace{\left[N_1-D_1\frac{N_0}{D_0}\right]}_{=0}+\frac{1}{D_0}\left[N_2-\cancel{\frac{N_1D_1}{D_0}}-D_2\frac{N_0}{D_0}+\cancel{N_0\frac{D_1^2}{D_0^2}}\right]+\dots$$

$$=\lambda_0 \qquad (\delta\lambda=0) \qquad +\frac{1}{D_0}\left[N_2-D_2\lambda_0\right]$$

$$=\lambda_0+\frac{\int dx\left[F\left(\delta y^{'}\right)^2+\left(g-\lambda_0\right)\left(\delta y\right)^2\right]}{\int dx\, y_0^2}$$

$$=\lambda_0+0\left(\in^2\right)$$

Thus, error is λ is $0\left(\in^2\right)$ while error in y is $0\left(\in\right)$

Generalized Boundary Conditions

1. Problem of practical importance

   new B.C.

   $$\left(Fy'\right)' - gy = 0 \quad y(0) = 1 \quad y'(1) = Ay(1)$$

2. Simple Variational Principle $> \int y\, dx$

   $$\lambda = \frac{\int \left(Fy'^2 - gy^2\right) dx - f\, yy'\big|_{x=1}}{\int y^2\, dx} \qquad (2)$$

3. Let $y = y_0 + \delta y,\ \lambda = \lambda_0 + \delta\lambda$

   $$\delta\lambda = -\frac{2\int \delta y\left[\left(Fy_0'\right)' + (\lambda - g)\, y_0\right] dx + F\left(y_0'\delta y - y_0\delta y'\right)\Big|_1}{\int y_0^2 dx}$$

4. $\delta\lambda$ vanishes if

   a. y_0 satisfies original ODE

   b. y_0 and δy satisfY $y_0'(1) = Ay_0(1)$, $\delta y_0'(1) = A\delta y(1)$

5. Using trial functions which satisfy $g'(1) = Ag(1)$ is not unexpected but can be cumbersome in practice.

6. Alternate, more practical, more elegant varational principle. Replace $y'(1)$ in (2) with $Ay(1)$. Then

   $$\lambda = \frac{\int \left(Fy_1'^2 + gy^2\right) dx + Afy^2\big|_1}{\int y^2\, dx}$$

7. Let $y = y_0 + \delta y$, and $\lambda = \lambda_0 + \delta\lambda$

   $$\delta\lambda = \frac{-2\int \delta y\left[\left(fy_0'\right)' + (\lambda_0 - g)\, y_0\right] dx + 2F\delta y\left(y_0' - Ay_0\right)}{\int y_0^2\, dx}$$

8. $\delta\lambda$ vanishes if

   a. original ODE satisfied

b. $y_0' = Ay_0$

If we choose trial functions which allow $y(1)$ to float freely, then variational principle forces $Ay(1) = y'(1)$. This is a natural boundary condition.

Practical Applications

1. "Exact" solution and eigenvalue. Let

   $$y = \sum a_n Y_n(x)$$ — complete set of orthornormal functions

2. Then

   $$\lambda = \frac{\sum a_n a_m W_{nm}}{\sum a_n^2}, \quad W_{nm} = \int dx \left[F y_n' y_m' + g y_n y_m \right] dx$$

3. Minimize with respect to the a_n. This is equivalent to finding the eigenvalues of $\underline{\underline{W}}$. Simple numerical procedure. In the limit $N \to \infty \left(\sum_1^N \right)$, we obtain exact eigenvalue and eigenfunction. Since Y_n is a complete set.

4. Send "estimate" of eigenvalue. Gives a trial function

   $$y = y(x, c_1, c_2, c_3) \quad \left[\text{e.g. } y = x(1-x)\left[1 + c_1 x + c_2 x^2 + c_3 x^3\right]\right]$$

   $$\text{or } y = x\left(1 - x^2\right) e^{-c_1 x^2} \left[1 + c_2 x^{c_3}\right]$$

   Evaluate $\lambda = \lambda(c_1, c_2, c_3)$

   Find c_1, c_2, c_3 to extremize λ

   $$\frac{\partial \lambda}{\partial c_1} = 0$$

   $$\frac{\partial \lambda}{\partial c_2} = 0$$

   $$\frac{\partial \lambda}{\partial c_3} = 0$$

   Substitute c_1, c_2, c_3 back into λ to get good estimate.

Application of the Variational Principle to MHD

1. Generalize previous analysis to include 3-D and sectors

2. Conceptually the same

$$-\omega^2 \rho \underline{\xi} = \underline{F}\left(\underline{\xi}\right) > \int \underline{\xi}^* \cdot d\underline{r}$$

$$\boxed{\begin{array}{l} \omega^2 = \dfrac{\delta W}{K} \\[2ex] K = \dfrac{1}{2}\int \rho \left|\underline{\xi}\right|^2 dr \\[2ex] \delta W = -\dfrac{1}{2}\int \underline{\xi}^* \cdot \underline{F}\left(\underline{\xi}\right) d\underline{r} \end{array}}$$

3. Proof: Let $\underline{\xi} \rightarrow \underline{\xi} + \delta\underline{\xi}, \quad \omega^2 \rightarrow \omega^2 + \delta\omega^2$

$$\omega^2 + \delta\omega^2 = \frac{\delta W\left(\underline{\xi}^*, \underline{\xi}\right) + \delta W\left(\underline{\delta\xi}^*, \underline{\xi}\right) + \delta W\left(\underline{\xi}^*, \delta\underline{\xi}\right) + \delta W\left(\delta\underline{\xi}^*, \delta\underline{\xi}\right)}{K\left(\underline{\xi}^*, \underline{\xi}\right) + K\left(\underline{\delta\xi}^*, \xi\right) + K\left(\underline{\xi}^*, \delta\underline{\xi}\right) + K\left(\underline{\delta}\xi^*, \delta\xi\right)}$$

4. Assume $\delta\underline{\xi}, \delta\omega^2$ are small

$$\delta\omega^2 = \frac{\delta W\left(\underline{\xi}^*, \delta\,\underline{\xi}\right) + \delta W\left(\delta\underline{\xi}^*, \underline{\xi}\right)}{K\left(\underline{\xi}^*, \xi\right)} - \underbrace{\frac{\delta W\left(\underline{\xi}^*, \xi\right)}{K\left(\underline{\xi}^*, \xi\right)}}_{\omega^2} \frac{\left[K\left(\delta\underline{\xi}^*, \underline{\xi}\right) + K\left(\underline{\xi}^*, \delta\,\underline{\xi}\right)\right]}{K\left(\underline{\xi}^*, \xi\right)}$$

$$= \frac{\delta W\left(\delta\underline{\xi}^*, \underline{\xi}\right) - \omega^2 K\left(\delta\underline{\xi}^*, \xi\right) + \delta W\left(\underline{\xi}^*, \delta\underline{\xi}\right) - \omega^2 K\left(\xi^*, \delta\underline{\xi}\right)}{K\left(\underline{\xi}^*, \underline{\xi}\right)}$$

Use self adjoint property: $K\left(\underline{\xi}^*, \partial\underline{\xi}\right) = K\left(\underline{\delta}\xi, \underline{\xi}^*\right)$

$$\delta W\left(\underline{\xi}^*, \underline{\delta}\xi\right) = \delta W\left(\underline{\delta}\xi, \underline{\xi}^*\right)$$

Then

$$\delta\omega^2 = \int d\underline{r}\ \underline{\delta}\xi^* \cdot \left[F\left(\xi\right) + \omega^2 \rho \underline{\xi}\right] + \delta\underline{\xi}\left[\underline{F}\left(\underline{\xi}^*\right) + \omega^2 \rho \underline{\xi}^*\right] \Big/ K$$

If $\delta\underline{\xi}$ is arbitrary and $\delta\omega^2 = 0$ (extremum) then

$$\boxed{\omega^2 \rho \underline{\xi} = -\underline{F}\left(\underline{\xi}\right)}$$

Simple Interpretation

$$H_2 = -\omega^2 K + \delta W = 0$$ conservation of energy

δW — potential energy

$\omega^2 K$ — kinetic energy