22.615, MHD Theory of Fusion Systems
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Lecture 9: The High Beta Tokamak

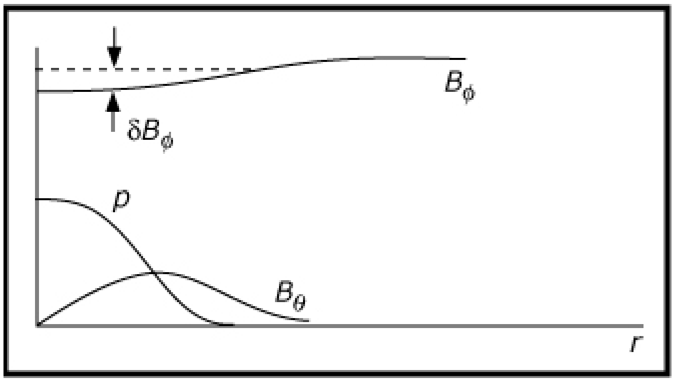
Summary of the Properties of an Ohmic Tokamak

1. Advantages:
    a. good equilibrium (small shift)
    b. good stability $\left(q \sim 1\right)$
    c. good confinement $\left(\tau \sim naR^2\right)$
    d. good ohmic heating $\left(T_e \sim T_i \sim 2\text{keV}\right)$
2. Disadvantages:
    a. low β : $\beta \sim \in^2, \beta_t = \dfrac{\in^2 \ \beta_p}{q_a^2} \sim \dfrac{0.3^2 \times 0.5}{3^2} = 1/2\%$
    b. external heating is required: joule heating is not adequate. External heating is expensive, but also raises β
    c. tight aspect ratio is required to raise β : this is technologically difficult
    d. pulsed operation is required unless current drive works efficiently
3. The high β tokamak resolves the problem of low β
    a. It allows a tokamak to operate at higher $\beta \sim 5 - 10\%$. This leads to more economic devices
    b. How do we achieve higher β? We apply additional auxiliary heating, keeping B_ϕ fixed. This raises p relative to $B_\phi^2 / 2\mu_0$.

High β Tokamak Expansion

1. We again assume large aspect ratio: $a/R_0 \ll 1 \ \in \equiv a/R_0$
2. Stability is produced by a large toroidal field: $q \sim 1$
3. Thus, as in the ohmic tokamak $q \sim rB_\phi / RB_\theta$, implying that $B_\theta / B_\phi \sim \in$

4. Radial pressure balance, however is produced by the *toroidal* field



$$\underset{\substack{|\\ \text{diamagnetic}}}{B_\phi = B_0 + \delta B_\phi}$$

$$\left(p+\frac{B_\phi^2}{2\mu_0}\right)' + \frac{B_\theta}{\mu_0 r}\left(rB_\theta\right)' = 0$$

$$\left(p+\frac{B_0\delta B_\phi^2}{2\mu_0}\right)' + \underbrace{\frac{B_\theta}{\mu_0 r}}_{\text{neglect, confines only } \beta\sim\in^2}\left(rB_\theta\right)' = 0$$

5. To improve β over that achievable in the ohmic tokamak we need

$$p \sim \frac{B_0\delta B_\phi}{\mu_0} \gg \frac{B_\theta^2}{\mu_0}$$

6. Or in terms of β

$$\beta \sim \frac{\delta B_\phi}{B_0} \gg \frac{B_\theta^2}{B_\phi^2} \sim \in^2$$

7. How large can β and $\delta B_\phi / B_0$ get?

8. The limiting condition is determined by toroidal force balance which is still accomplished by a combination of I and B_v

9. Increasing β increases the toroidal shift. The largest possible β occurs when the shift becomes of order unity $\Delta/a \sim 1$. Recall that in an ohmic tokamak $\Delta/a \sim a/R_0$.

10. Let us estimate the shift using the small shift relation

$$\psi_1 = -B_\theta \int_r^b \frac{dr'}{r' B_\theta^2} \int_0^{r'} \left(\underset{\substack{\Big| \\ \text{neglect since } \mu_0 p \gg B_\theta^2}}{y B_\theta^2} - 2\mu_0 y^2 \frac{dp}{dy} \right) dy$$

$$\psi_1 \sim \frac{\mu_0 a^2 p}{B_\theta}$$

11. Therefore

$$\frac{\Delta}{a} \sim -\frac{\psi_1}{a\psi_0'} \sim \frac{1}{aRB_\theta}\left(\frac{\mu_0 a^2 p}{B_\theta}\right) \sim \frac{a}{R}\left(\frac{\mu_0 p}{B_\theta^2}\right) \sim \in \beta_p \sim \frac{\beta_t q^2}{\in}$$

12. For $q \sim 1$, then $\Delta/a \sim 1$ when $\beta_t/\in \sim 1$

13. This suggests the following ordering for the high β tokamak

$$\beta \sim \in, \frac{\delta B_\phi}{B_\phi} \sim \in, \quad \Delta/a \sim 1$$

Comparison of Expansions

Ohmic Tokamak	High Beta Tokamak
$q \sim 1$	$q \sim 1$
$B_\theta/B_\phi \sim \in$	$B_\theta/B_\phi \sim \in$
$\beta \sim 2\mu_0 p/B_0^2 \sim \in^2$	$\beta \sim 2\mu_0 p/B_0^2 \sim \in^2$
$\delta B_\phi/B_0 \sim \in^2$ (*para*)	$\delta B_\phi/B_0 \sim \in$ (*dia*)
$\beta_p \sim 2\,\mu_0 p/B_\theta^2 \sim 1$	$\beta_p \sim 2\,\mu_0 p/B_\theta^2 \sim 1/\in$

Expansion of Grad-Shafranov Equation

1. Since $\Delta/a \sim 1$, toroidal force balance and radial pressure balance enter together in zeroth order.
2. Good news: we need only the zeroth order equations. No first order corrections are required.
3. Bad news: The zero other equations are still nonlinear partial differential equations.

4. Expansion:

   a. $\psi = \psi_0\left(r,\theta\right) + \ldots, \psi \sim rRB_\theta \sim a^2 B_0$

   b. $p\left(\psi\right) = p\left(\psi_0\right) + \ldots, \mu_0 p \sim \in B_0^2$

   c. $F \equiv RB_\phi = F\left(\psi\right)$

$$F^2 = R_0^2\left[\underbrace{B_0^2}_{\sim 1} - \underbrace{2\mu_0 p\left(\psi\right)}_{\sim \in} + \underbrace{2B_0 \overset{\text{new free function}}{B_2\left(\psi\right)}}_{\sim \in^2}\right] \qquad B_2/B_0 \sim \in^2$$

   d. This automatically produces a θ pinch pressure balance

$$p + \frac{B_\phi^2}{2\mu_0} \approx \text{const}$$

5. Substitute the expansion into the Grad-Shafranov equation

$$\nabla^2\psi = -\mu_0\left(R_0 + r\cos\theta\right)^2 \frac{dp}{d\psi} - \frac{d}{d\psi}\frac{F^2}{2} + \frac{1}{R}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right)$$

$$T_1 = \nabla^2\psi \sim \frac{\psi}{a^2} \sim B_0$$

$$T_3 = \frac{1}{R}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right) \sim \frac{\psi}{aR} \sim \in B_0 \ \text{neglect}$$

$$T_2 = -\mu_0\left(R_0 + r\cos\theta\right)^2 \frac{dp}{d\psi} - \frac{d}{d\psi}\frac{F^2}{2}$$

$$\approx -\frac{d}{d\psi}\left(\frac{F^2}{2} + \mu_0 R_0^2 p\right) - 2\mu_0 R_0 r\cos\theta\frac{dp}{d\psi} + \ldots$$

$$= -R_0^2\frac{d}{d\psi}\left[\left(\overset{=0}{\frac{B_0^2}{2}} - \cancel{\mu_0 p} + B_0 B_2\right) + \cancel{\mu_0 p}\right] - 2\mu_0 R_0 r\cos\theta\frac{dp}{d\psi}$$

$$= -R_0^2\frac{d}{d\psi}\left(B_0 B_2\right) \qquad \sim \frac{R_0^2 B_0 B_2}{\psi} \sim B_0$$

$$-2\mu_0 R_0 r\cos\theta\frac{dp}{d\psi} \qquad \sim \frac{\mu_0 p B_0 a}{\psi} \sim B_0$$

6. Therefore, to leading order the Grad-Shafranov equation reduces to

$$\nabla^2\psi_0 = -R_0^2 \frac{d}{d\psi_0} B_0 B_2 \left(\psi_0\right) - 2\mu_0 R_0 r \cos\theta \frac{dp\left(\psi_0\right)}{d\psi_0}$$

7. Note that $\mu_0 R J_\phi \approx -\nabla^2\psi$, so that on a circular plasma flux surface

$$\underbrace{\left\langle \mu_0 R J_\phi \right\rangle}_{\text{average over } \theta} = R_0^2 \frac{d}{d\psi} B_0 B_2$$

   We see that $dB_2/d\psi$ is proportional to the average toroidal current within a given flux surface.

8. Even though the equation is simpler, it is still a nonlinear PDE.

9. In general, it must be solved numerically.

10. The difficulty arises because the shifts are finite and cannot be treated perturbatively.

11. We shall determine general features of high β tokamak by examining a special case.

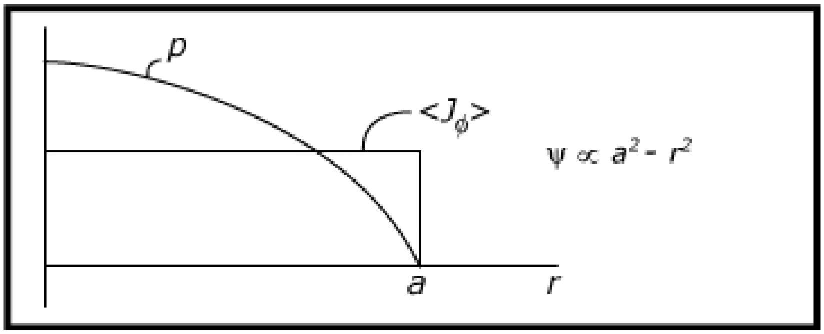
Special Case

1. Choose

$$2\mu_0 R_0 \frac{dp}{d\psi_0} = -C \quad C = \text{const}$$

$$R_0^2 B_0 \frac{dB_2}{d\psi_0} = -A \quad A = \text{const}$$

2. This implies $p \sim -C\psi$ (assume ψ (boundary)=0) and $\left\langle J_\phi \right\rangle \sim -A$



3. Solution: with these choices the Grad-Shafranov equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \psi_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi_0}{\partial \theta^2} = A + Cr\cos\theta$$

4. Boundary conditions: We assume a *circular* plasma of radius $r = a$

$\psi\left(a,\theta\right) = 0$ (normalization of flux function is arbitrary)

$\psi\left(r,\theta\right)$ regular for $r \le a$

The circular assumption is made for simplicity and can be generalized to other cross sections.

5. Solution: (We need only $\cos n\theta$ terms because of up-down symmetry.)

$$\psi_{part} = A\frac{r^2}{4} + C\frac{r^3}{8}\cos\theta$$

$$\psi_{hom} = k_1 + \underbrace{k_2 \ln r}_{\text{not regular}} + k_3 r\cos\theta + \underbrace{\frac{k_4\cos\theta}{r}}_{\text{not regular}} + \sum_{n=2}^{\infty}\left(a_n r^n + \underbrace{b_n r^{-n}}_{\text{not regular}}\right)\cos n\theta$$

$$= \underbrace{k_3 r\cos\theta + k_1}$$

only terms which are regular and required to balance ψ_{part} on the boundary $r = a$

For all $n \ge 2$, it follows that $a_n = 0$

6. Choose k_1 and k_3 to make $\psi\left(a,\theta\right) = 0$

$$\psi\left(r,\theta\right) = \frac{A}{4}\left(r^2 - a^2\right) + \frac{C}{8}\left(r^3 - a^2 r\right)\cos\theta$$

7. Then

$$B_\theta = \frac{1}{R_0}\frac{\partial \psi_0}{\partial r} = \frac{1}{2R_0}\left[Ar + \frac{C}{4}\left(3r^2 - a^2\right)\cos\theta\right]$$

$$\mu_0 p = -\frac{C}{2R_0}\psi_0 = \frac{C}{8R_0}\left[A\left(a^2 - r^2\right) + \frac{C}{2}\left(a^2 r - r^3\right)\cos\theta\right]$$

Physical Interpretation

Let us express A and C in terms of more physical quantities: β_t, I or equivalently $\beta_t, q_* \propto 1/I$

1. q_* is a parameter related to kink stability and the surface MHD safety factor q_a

$$q_* \equiv \frac{2A_pB_0}{\mu_0R_0I}$$

$$= \frac{2\pi a^2B_0}{\mu_0R_0I}$$

2. $\mu_0 I = \int \mathrm{B}_p \cdot dl = \int B_\theta\left(a,\theta\right)a d\theta$

ON A CIRCLE

$$= \frac{a}{2R_0}\int_0^{2\pi}\left[Aa + \frac{Ca^2}{2}\cancel{\cos}\,\theta\right]d\theta = \frac{Aa^2}{2R_0}\cdot 2\pi = \frac{\pi Aa^2}{R_0}$$

3. $\dfrac{1}{q_*} = \dfrac{\mu_0R_0}{2\pi a^2B_0}\dfrac{\pi Aa^2}{R_0\mu_0} = \dfrac{A}{2B_0}$

4. $\beta_t = \dfrac{2\mu_0}{B_0^2}\left\langle p\right\rangle = \dfrac{2\mu_0}{B_0^2}\dfrac{1}{\pi a^2}\int pr\ \ dr\ \ d\theta$

$$= \frac{2\mu_0}{\pi a^2B_0}\frac{C}{8\mu_0R_0}\int_0^a\int_0^{2\pi} r\ \ dr\ \ d\theta\left[\underbrace{A\left(a^2-r^2\right)}_{\frac{2\pi a^4}{4}} + \frac{C}{2}\left(a^2r-r^3\right)\cancel{\cos}\,\theta\right]$$

$$= \frac{a^2}{8R_0B_0^2}AC = \frac{a^2}{4R_0B_0}\frac{C}{q_*}$$

5. Thus

$$\frac{A}{2B_0} = \frac{1}{q_*}$$

$$\frac{a^2C}{4R_0B_0} = q_*\beta_t$$

6. General equilibrium relation for a high β circular tokamak (minor degression)

a. $$\beta_p = \frac{\int pr\ dr\ d\theta}{\mu_0 I^2 / 8\pi}$$

$$= \frac{8\pi}{\mu_0 I^2} \frac{\pi a^2 B_0^2}{2\mu_0} \beta_t$$

$$= \left(\frac{2\pi a B_0}{\mu_0 I} \right)^2 \beta_t$$

$$= \frac{q_*^2 \beta_t}{\in^2},\ \in \equiv a/R_0$$

b. The general equilibrium relation is given by

$$\beta_t = \frac{\in^2\ \beta_p}{q_*^2}$$

7. Substitute A and C back into the solutions. Define $\nu = \frac{\beta_t q_*^2}{\in} = \in \beta_p \sim 1, \rho = r/a$

$$\frac{\Psi_0}{a^2 B_0} = \frac{1}{2q_*}\left[\rho^2 - 1 + \nu\left(\rho^3 - \rho\right)\cos\theta\right]$$

$$\frac{B_\theta}{\in B_0} = \frac{1}{q_*}\left[\rho + \frac{\nu}{2}\left(3\rho^2 - 1\right)\cos\theta\right]$$

$$\frac{B_r}{\in B_0} = -\frac{\nu}{2q_*}\left(\rho^2 - 1\right)\sin\theta$$

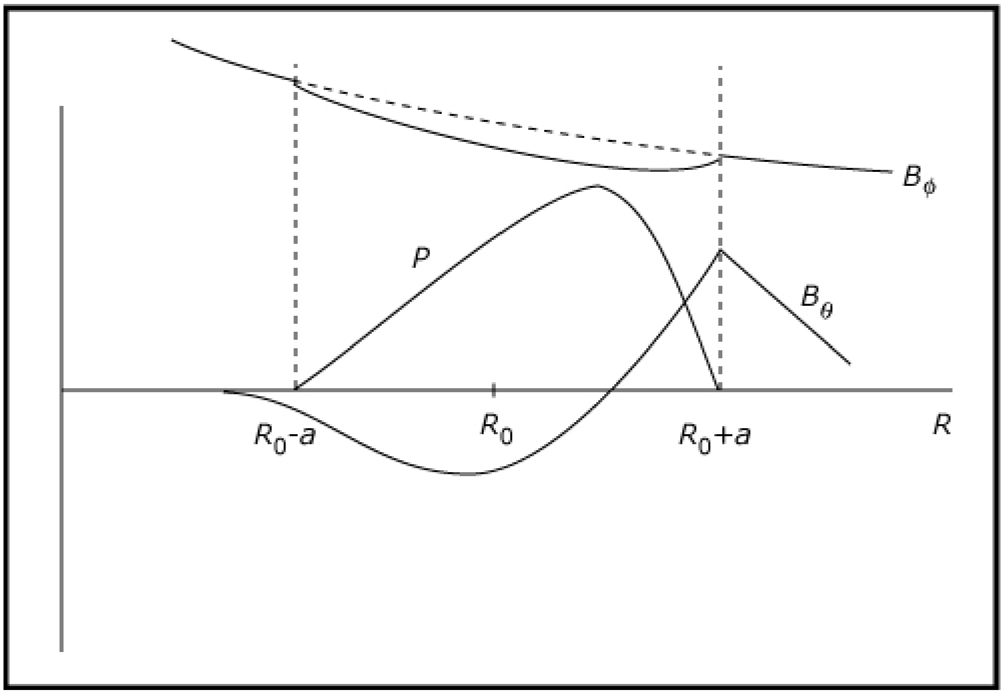
$$\frac{2\mu_0 p}{B_0^2} = 2\beta_t \left(1 - \rho^2\right)\left(1 + \nu\rho\cos\theta\right)$$

$$\frac{\mu_0 R_0 J_\phi}{B_0} = -\frac{2}{q_*}\left(1 + 2\nu\rho\cos\theta\right)$$

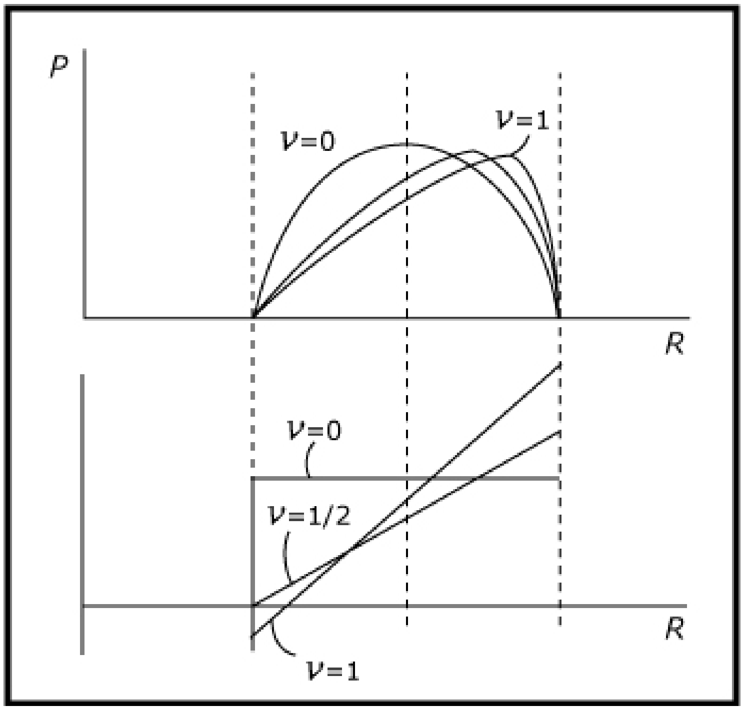
$$\frac{B_\phi}{B_0} = 1 - \in\rho\cos\theta - \beta_t\left(1 - \rho^2\right)\left(1 + \nu\rho\cos\theta\right)$$

Properties of High Beta Tokamak Equilibria

1. Sketched below are typical midplane profiles ($Z=0$) showing radial pressure balance.



2. Shown here are p and J_ϕ profiles along the midplane for different ν. Increasing ν implies higher β.



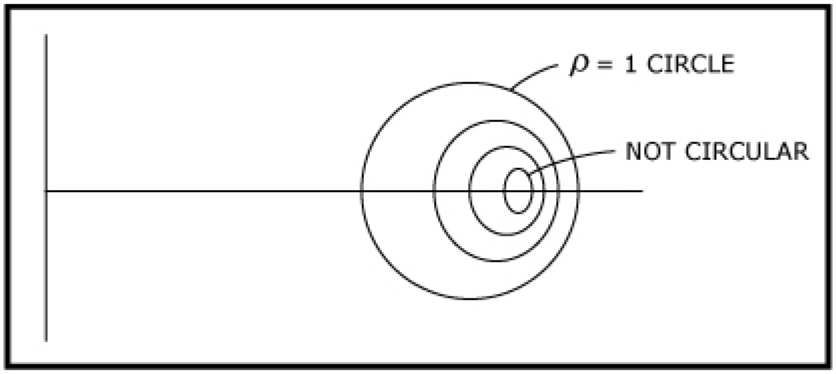
a. Observe the increased shift of the magnetic axis as β increases.

b. Observe the buildup of current on the outside of the torus to produce toroidal force balance at higher β.

c. Observe J_ϕ reversing on the inside of the torus when $\nu > 1/2$.

3. The flux surfaces are "round", but are not circles except for the boundary

$$\rho^2 - 1 + \nu\left(\rho^3 - \rho\right)\cos\theta = \text{const}$$



4. Let us calculate the magnetic axis shift Δ_0 by finding the value of r where $\frac{\partial\psi}{\partial r} \propto \frac{\partial p}{\partial r} = 0$. By symmetry this occurs when $\theta = 0\left(\text{or}\ \pi\right)$.

a. At $\theta = 0$

$$\psi \propto \rho^2 - 1 + \nu\left(\rho^3 - \rho\right)$$

b. Set

$$\frac{\partial\psi}{\partial\rho}\left(\rho = \frac{\Delta_0}{a}, \theta = 0\right) = 0$$

c. This yields

$$\frac{2\Delta_0}{a} + \nu\left(3\frac{\Delta_0^2}{a^2} - 1\right) = 0$$

d. The value of Δ_0 is given by

$$\frac{\Delta_0}{a} = \frac{\nu}{1 + \left(1 + 3\nu^2\right)^{1/2}} \sim 1$$

e. For the HBT $\nu \sim 1 \to \frac{\Delta_0}{a} \sim 1$

Ohmic T$\nu \sim \in \to \frac{\Delta_0}{a} \sim \frac{\nu}{2} \sim \in \ll 1$

5. Find the shape of the flux surfaces near the magnetic axis.

a. Let $x = \rho\cos\theta \quad y = \rho\sin\theta$

b. Then $\psi \propto \left(x^2 + y^2 - 1\right)\left[1 + \nu x\right]$

c. Expand $x = \frac{\Delta_0}{a} + \delta x \quad y = \delta y \quad \delta x, \delta y \ll 1$

and define $x_0 = \Delta_0/a$

d. Substitute

$$\psi \propto \left[x_0^2 + 2x_0\delta x + (\delta x)^2 + (\delta y)^2 - 1\right]\left[1 + \nu x_0 + \nu\delta x\right] = \text{const}$$

$$= \left(x_0^2 - 1\right)\left(1 + \nu x_0\right) + \delta x\left[2x_0 + 2\nu x_0^2 + \nu x_0^2 - \nu\right]$$

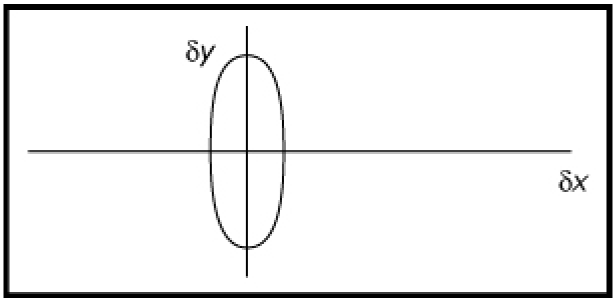
0 definition of x_0

$$+ (\delta y)^2\left(1 + \nu x_0\right) + (\delta x)^2\left[1 + \nu x_0 + 2\nu x_0\right]$$

or

$$\left(1+3\nu x_0\right)\left(\delta x\right)^2+\left(1+\nu x_0\right)\left(\delta y\right)^2=\text{const}$$

e. This is the equation of an ellipse



elongated flux surfaces, squashed near the outside

f. The elongation κ_0 is defined by

$$\kappa_0^2=\frac{b^2}{a^2}=\frac{1+3\nu x_0}{1+\nu x_0}$$

$$=\left(1+3\nu^2\right)^{1/2}\frac{\left[1+\left(1+3\nu^2\right)^{1/2}\right]}{\left[1+\nu^2+\left(1+3\nu^2\right)^{1/2}\right]}$$

$\kappa_0^2\approx 1$ for $\nu\ll 1$ and $\kappa_0^2=3/2$ for $\nu=1$.