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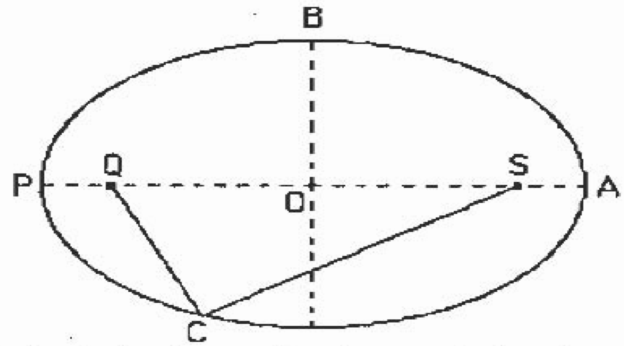
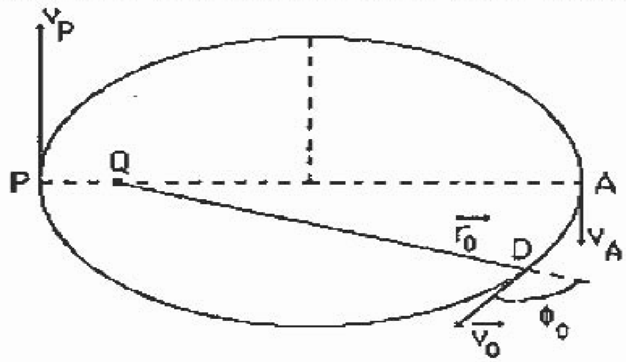
Some Notes on Kepler Orbits.

The figure shows an ellipse; Q and S are the foci.



The semi-major axis a = OA (= OP), the semi-minor axis b = OB. QO (= OS) = $a\varepsilon$ (ε is called the eccentricity). For the special case that the eccentricity is zero, Q, S and O coincide and we have a circle with radius $R = a = b$. If two of the three parameters a, b, and ε are known, the third can be calculated. Convince yourself of this. Keep in mind that QC + CS = QB + BS = QA + AS = $2a$; that is a property of an ellipse.

A satellite (planet) with mass m goes around the earth (sun) in an elliptical orbit. The earth (sun) has a mass M and is at one of the foci, Q, of the ellipse.



The orbital period T is only a function of M and of the semi-major axis a (see Example 6, page 222 in Ohanian):

$$T^2 = 4\,\pi^2\,a^3 / MG \qquad (1)$$

For a circular orbit, the semi-major axis is replaced by the radius (R) of the circle (see eq. 15 on page 217 of Ohanian). The total energy of the mass m ***anywhere*** in orbit is:

$$E_{tot} = -MmG/2a \qquad (2)$$

For a circular orbit, a is replaced by R (see eq. 27 on page 226 of Ohanian).

At D: $$E_{tot} = mv_0^2/2 - MmG/r_0 = -MmG/2a \qquad (3)$$

The angular momentum (about Q where M is located) of m is conserved; there is no external torque on m relative to point Q (the cross product of the gravitational force and r_0 equals zero everywhere in the orbit). Thus,

$$L_Q = m\, v_0\, r_0 \sin \phi_0 \text{ is conserved} \qquad (4)$$

The point P of closest approach has a name. In case we deal with the earth and its satellites, it is called perigee, when we deal with the sun and planets it is called perihelion, and in the case of a binary star system (lecture of Nov. 9) it is called periastron. The point A is called: apogee (earth), aphelion (sun), and apastron (binary). Since at A and P the angle $\phi = \pi/2$:

$$L_Q = mv_A \times (QA) = mv_P \times (QP) = m\, v_0\, r_0 \sin \phi_0 \qquad (5)$$

How to calculate the elliptical orbit from initial conditions?

If the position of m relative to Q is given and if one knows the velocity v_0 of m at that point (magnitude and direction), the elliptical orbit is determined. You first calculate the total energy (eq. 3) from the initial conditions (at D), and you find a. You now apply eq. (3) to point P and you find an equation in v_P^2 [in the potential energy term in eq. 3, substitute for the distance $QP = L_Q / mv_P$]. This leads to two solutions which are v_P and v_A (why?). With eq. (5) you now solve for QA and QP (you know L_Q from the initial conditions; see eq. 4). Once you know QA and QP you know the eccentricity and thus *the* shape of the ellipse.

Example (check!)

A satellite of mass 10^4 kg is in orbit around the earth. Its velocity is 9.00 km/sec and $\phi = 120°$ at the moment that its distance to the center of the earth is 9000 km. What is the speed at perigee and at apogee? What is the distance of closest approach (perigee) to the earth's center and what is the distance to apogee? What is its orbital period, and what the eccentricity? What are the answers if the satellite's mass were 10 times less?

I found: $a \approx 5\ 10^4$ km, $v_P \approx 10.7$ km/sec, $v_A \approx 0.75$ km/sec (notice the enormous difference in speed at apogee and perigee!), QP $\approx 6.6\ 10^3$ km, and QA $\approx 9.3\ 10^4$ km. QA + QP should add up to $2a$. The orbital period $T \approx 31$ hour; $\varepsilon \approx 0.87$ (the orbit is highly eccentric). Notice that the satellite comes within about 200 km of the surface of the earth (radius about 6,400 km). At that altitude, the atmosphere will decelerate the satellite; the orbit will ultimately decay and the satellite will reenter.

Much Ado About a Ham Sandwich.

There are two spacecrafts in exactly the same circular orbit with radius R. They are separated (measured distance along the circumference) by a distance $f2\pi R$ (f is the fraction of the circumference). An astronaut (Peter) in the trailing spacecraft wants to throw a ham sandwich to his friend Mary in front. We will only explore solutions whereby the sandwich is released (at point X) in *exactly* the same or in *exactly* the opposite direction of the velocity of Peter's spacecraft. The period T of the circular orbit follows from eq. (1); the orbital speed v_a is $2\pi R/T$. Let the orbital period of the ***elliptical*** orbit of the sandwich (after it is thrown) be T_s. We can now try to arrange matters such that Mary will catch the sandwich at the very first time that she reaches point X; the sandwich will then have completed one elliptical orbit. Thus:

$$T_s = T(1- f) = (4\,\pi^2\,a^3\,/\,MG)^{0.5}$$

For given f, you can calculate a, and that will lead you to the speed v_s (using eq. 3) in X which is *the only point* (why?) where the sandwich can be caught (as discussed in lectures, in this case, X will be apogee of the elliptical orbit). This speed, v_s, can be compared with v_a, and you can find the velocity $v_s - v_a$ with which Peter should throw the sandwich relative to his moving spacecraft. In this case $v_s < v_a$, and thus Peter should throw the sandwich in the opposite direction of his velocity.

There is an infinite number of solutions. We can let Mary make the catch after n_a passes through point X and we can let the sandwich in the meantime go around in its orbit n_s times. A necessary condition for the catch would then be:

$$n_s T_s = T\,(n_a - f) \quad \text{(verify this).}$$

This leads to a required value for the semi-major axis of the elliptical orbit of the sandwich of: $a = R\,[(n_a - f)/n_s]^{2/3}$ (verify this).

You can now try various combinations of n_s and n_a and see whether a solution exists. For $f = 0.05$, $n_s = 1$ and $n_a = 3$ the orbital period of the sandwich will have to become larger than T and thus the sandwich will have to be thrown forward (this, and several other possibilities; see page 4), were demonstrated in lectures using Prof. George Clark's computer program which he kindly wrote for this occasion.