PKT ABSOLUTE

Rigid Body Kinematics

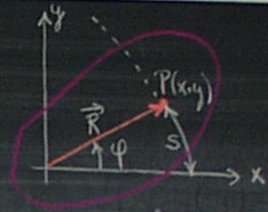
- Real Objects → mass dot. size, shape
- Motion: Translation of CM
  Rotation about axis.
- Inertial Frame

Point P moves in a circle of radius R about a fixed axis.

$S = R\varphi \qquad \varphi(rad) = \frac{\pi}{180}\varphi(deg)$

$\varphi = +$ ccw, $-$ cw $\qquad \varphi = 0$ x-axis, $\varphi = \pi/2$ y-axis, $\varphi = 2\pi$ x-axis again

$\varphi \neq$ vector!!



In general $\varphi = \varphi(t)$

Angular Velocity: $\omega(t)$

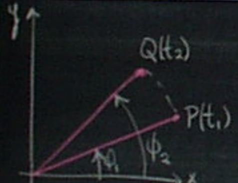
①

$\vec{\omega} = \frac{\varphi_2 - \varphi_1}{t_2 - t_1} = \frac{\Delta\varphi}{\Delta t}\hat{k}\ (s^{-1})$

$\vec{\omega} \rightarrow$ along axis of rotation (RHR)

$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = \frac{d\varphi}{dt}\hat{k}$ rad/s

$\Rightarrow$ Inst. Angular Velocity



Suppose $\omega(t) = \omega_0 =$ constant.

1 revolution $= 2\pi$ radians

Period: $T = \frac{2\pi}{\omega_0}$ (s)

Frequency: $\nu = \frac{1}{T} = \frac{\omega_0}{2\pi}$ (Hz)

$\nu = 1 \Rightarrow 1$ rev/s

$\nu = 10 \Rightarrow 10$ rev/s

etc.

Angular Acceleration: $\alpha(t)$

②

$\vec{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1}\hat{k} = \frac{\Delta\omega}{\Delta t}\hat{k}\ (s^{-2})$

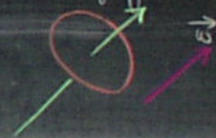
$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}\hat{k} = \frac{d\vec{\omega}}{dt} = \frac{d}{dt}\left(\frac{d\varphi}{dt}\right)\hat{k}$

$\therefore \vec{\alpha}(t) = \frac{d^2\varphi}{dt^2}\hat{k}$ Inst. ang. accel.

Note: Every point: same $\omega(t)$!!! same $\alpha(t)$

Fixed Rotation Axis:
direction $\vec{\alpha} \equiv$ dir $\vec{\omega}$

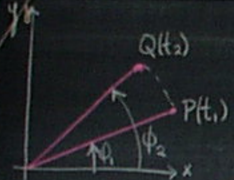
Changing Axis: No!!!



$\vec{\omega}$

$\frac{d\vec{\omega}}{dt} > 0 \quad \vec{\alpha}$ same as $\vec{\omega}$

$\frac{d\vec{\omega}}{dt} < 0 \quad \vec{\alpha}$ opposite $\vec{\omega}$



Suppose $\omega(t) = \omega_0 = \text{constant}$.

1 revolution $= 2\pi$ radians

Period: $T = \dfrac{2\pi}{\omega_0}$ (s)

Frequency: $\nu = \dfrac{1}{T} = \dfrac{\omega_0}{2\pi}$ (Hz)

$\nu = 1 \Rightarrow 1$ rev/s

$\nu = 10 \Rightarrow 10$ rev/s

etc.

Angular Acceleration: $\alpha(t)$

$$\vec{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1}\hat{k} = \frac{\Delta\omega}{\Delta t}\hat{k} \quad (s^{-2})$$

$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}\hat{k} = \frac{d\vec{\omega}}{dt} = \frac{d}{dt}\left(\frac{d\varphi}{dt}\right)\hat{k}$$

$$\therefore \vec{\alpha}(t) = \frac{d^2\varphi}{dt^2}\hat{k} \quad \text{Inst. ang. accel.}$$

Note: Every point: same $\omega(t)$!!!
Same $\alpha(t)$!!!

Fixed Rotation Axis: ②
direction $\vec{\alpha} \equiv$ dir $\vec{\omega}$

Changing Axis: No!!!

$\dfrac{d\vec{\omega}}{dt} > 0$ $\vec{\alpha}$ same as $\vec{\omega}$

$\dfrac{d\vec{\omega}}{dt} < 0$ $\vec{\alpha}$ opposite to $\vec{\omega}$

Rotational Motion: $\vec{\alpha} \equiv$ constant

- Fixed Axes.
  Ignore vector notation ($\pm$ direction)
- Also true for axes in linear translation (rolling)

$$\frac{d\omega}{dt} = \alpha \Rightarrow \int_{\omega_0}^{\omega} d\omega' = \int_0^t \alpha\, dt'$$

$$\boxed{\omega = \omega_0 + \alpha t} \quad ①$$

$$\frac{d\varphi}{dt} = \omega = \omega_0 + \alpha t$$

$$\int_{\varphi_0}^{\varphi} d\varphi' = \int_0^t \omega_0\, dt' + \alpha \int_0^t t\, dt'$$

$$\boxed{\varphi = \varphi_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2} \quad ②$$

From ① $(\omega - \omega_0)/\alpha = t$

$$\rightarrow ② \quad \boxed{\omega^2 = \omega_0^2 + 2\alpha(\varphi - \varphi_0)} \quad ③$$

Angular $\longleftrightarrow$ linear Velocity and Acceleration ③

Every particle moves in a circle about axis of rot.

$s = R\varphi$

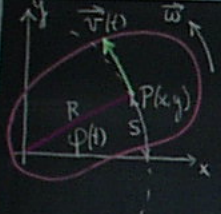
$v = \dfrac{ds}{dt} = R\dfrac{d\varphi}{dt} = R\omega$ [Prop. to distance from axis]

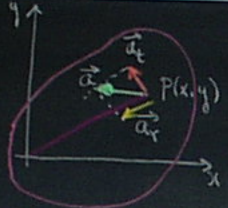
$a_t = a_\parallel = \dfrac{dv}{dt} = R\dfrac{d\omega}{dt} = R\alpha$ [Due to change in mag. of $\vec{v}$]

$a_r = a_\perp = \dfrac{v^2}{R} = \dfrac{R^2\omega^2}{R} = R\omega^2$ [Change in dir. of $\vec{v}$]

a_t, a_r: Prop to R!! If $\alpha = 0$, $a_t = 0$; $a_r \neq 0$

Lecture 21 Blackboard #2





$\vec{a} = \vec{a}_r + \vec{a}_t \qquad |a| = R\sqrt{\alpha^2 + \omega^4}$

Rotational Kinetic Energy

$K = \frac{1}{2}\sum_i m_i v_i^2$

$v_i = r_i \omega$

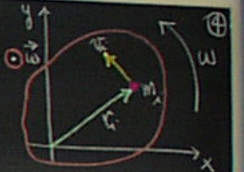
$K = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$

$K = \frac{1}{2} I \omega^2$

$\boxed{I = \sum_i m_i r_i^2}$

- Moment-of-Inertia
- Depends on axis of rotation.
- Particles at large r_i contribute most!

$a, M \longleftrightarrow$ Resis. to Linear Motion
$\alpha, I \longleftrightarrow$ Resis. to Rotational Motion
} Inertial Quantities



4

Example: 4 Rotating Masses

$\omega = \omega_0$: ang velocity constant.

$\alpha = 0$

a) Rotate about y-axis:

$I_y = \sum m_i r_i^2 = Ma^2 + Ma^2 + 0 + 0 = 2Ma^2$

$K_y = \frac{1}{2} I_y \omega^2 = \frac{1}{2}(2Ma^2)\omega_0^2$

b) Rotate about z-axis:

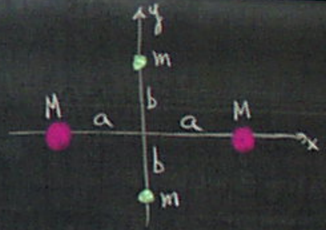
$I_z = \sum m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2$

$= 2Ma^2 + 2mb^2$

$K_z = \frac{1}{2} I_z \omega_0^2 = \frac{1}{2}2(Ma^2 + mb^2)\omega_0^2$

$I_z > I_y$

$K_z > K_y$



5

Lecture 21 Blackboard #3

Example: 4 Rotating Masses

$\omega = \omega_0$: ang. velocity constant.
$\alpha = 0$

a) Rotate about y-axis:

$I_y = \sum m_i r_i^2 = Ma^2 + Ma^2 + 0 + 0 = 2Ma^2$

$K_y = \frac{1}{2} I_y \omega^2 = \frac{1}{2}(2Ma^2)\omega_0^2$

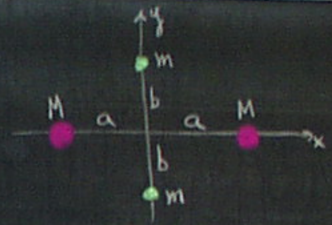
b) Rotate about z-axis:

$I_z = \sum m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2$
$= 2Ma^2 + 2mb^2$

$K_z = \frac{1}{2} I_z \omega_0^2 = \frac{1}{2} 2(Ma^2 + mb^2)\omega_0^2$

$I_z > I_y$

$K_z > K_y$



5

Rigid Bodies: Moments-of-Inertia:

$K = \frac{1}{2} I \omega^2$ KE for rotation about fixed axis.

Δm = element of mass.

$I = \sum r_i^2 \Delta m$ about rot. axis.

$I = \lim_{\Delta m \to 0} \sum r_i^2 \Delta m = \int_V r^2\, dm.$

Let $\rho(x,y,z)$ = local density (mass/volume)

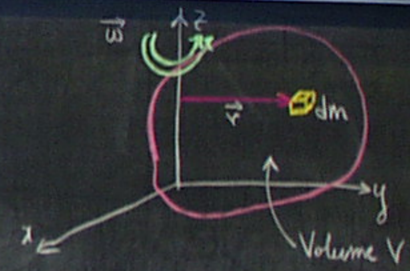
$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$

$\therefore dm = \rho\, dV$

$$\boxed{I = \int_V \rho r^2\, dV}$$

$I \Rightarrow$ 2nd moment of mass dist.

If $\rho(x,y,z)$ = constant $\Longrightarrow$ $\boxed{I = \frac{M}{V}\int_V r^2\, dV}$ where $\rho \equiv \frac{M}{V}$

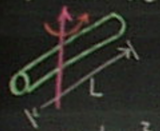


6

Lecture 21 Blackboard #4

Examples: Moments-of-Inertia

a) Thin Rod

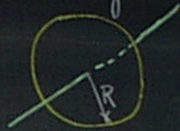


$I = \frac{1}{12} ML^2$

b) Solid Disk Cylinder

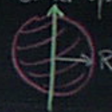
$I = \frac{1}{2} MR^2$

c) Ring



$I = MR^2$

d) Solid Sphere



$I = \frac{2}{5} MR^2$

e) Hollow Sphere/Shell

$I = \frac{2}{3} MR^2$

f) Plate

$I = \frac{M}{12}(a^2 + b^2)$

⑦

Example Uniform-Hollow Cylinder

⑧

Outer Radius R_2
Inner Radius R_1
Axis along center.
Choose cylindrical shell:
Radius r, Thickness dr
Length L

ρ = const: homogeneous/uniform

Volume of Shell $= dV = 2\pi r L (dr)$

Mass of Shell $= dm = \rho\, dV$
$= 2\pi r \rho L r (dr)$

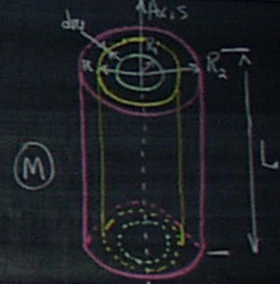
$I = \rho \int r^2 dV = 2\pi\rho L \int_{R_1}^{R_2} r^3 dr$

$I = \frac{\pi \rho L}{2}(R_2^4 - R_1^4)$

$= \frac{\pi \rho L}{2}(R_2^2 - R_1^2)(R_2^2 + R_1^2)$

Mass of Cyl $M = \rho V = \pi L \rho (R_2^2 - R_1^2)$

$\therefore I = \frac{M}{2}(R_2^2 + R_1^2)$



8

Example: Uniform-Hollow Cylinder

Outer Radius R_2
Inner Radius R_1
Axis along center
choose cylindrical shell:
Radius r, Thickness dr
length L

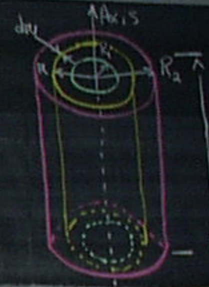
ρ = const: homogeneous/uniform
Volume of Shell: $dV = 2\pi r L (dr)$
Mass of Shell $= dm = \rho\, dV$
$= 2\pi r \rho L (dr)$

$$I = \rho \int r^2\, dV = 2\pi\rho L \int_{R_1}^{R_2} r^3\, dr$$

$$I = \frac{\pi\rho L}{2}\left(R_2^4 - R_1^4\right)$$
$$= \frac{\pi\rho L}{2}\left(R_2^2 - R_1^2\right)\left(R_2^2 + R_1^2\right)$$

Mass of Cyl $M = \rho V = \pi L \rho \left(R_2^2 - R_1^2\right)$

$$\therefore I = \frac{M}{2}\left(R_2^2 + R_1^2\right)$$



If cylinder is solid: $R_1 = 0$
$I = \frac{1}{2} MR^2$
If cylinder is a thin shell:
$R_1 \sim R_2 \sim R$
$I = MR^2$

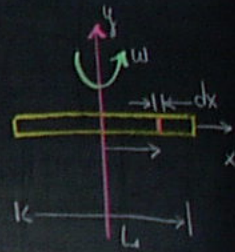
Radius of Gyration: k

$I = Mk^2$
Same I as for a mass M located at a 'radius' = "k"

Uniform Thin Rod

Length = L
Mass = M
Slice dx has mass dm
$dm = M\left(\frac{dx}{L}\right)$

$$I_y = \int r^2\, dm = \int_{x=-L/2}^{L/2} x^2 \frac{M}{L}\, dx = \frac{M}{L}\frac{x^3}{3}\Big|_{-L/2}^{L/2} = \frac{ML^2}{12}$$



Lecture 21 Blackboard #6