Relativistic Momentum

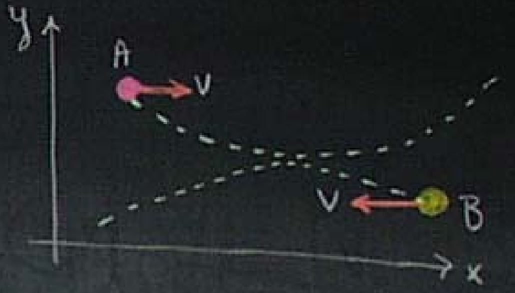
- What does Special Relativity say about Dynamics?
- How do we preserve Cons. of Momentum?

Elastic Collision of two identical particles.
Frame-A moves along x-axis fixed to A.
Frame-B moves along x-axes fixed to B

- Collision completely symmetric

Each particle has speed u_0 along its y-axes.

- Collision changes y-velocities leaves x-motion constant.
- Relative velocity of frames is V



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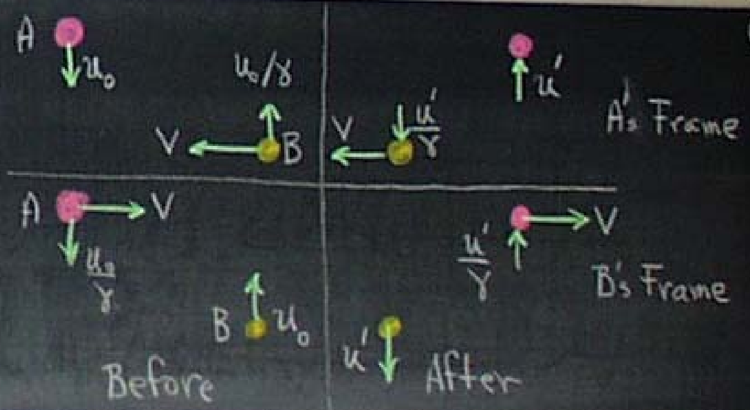
- Law of Trans of Velocities for opposite particle: $\frac{u_0}{\gamma} = u_0\sqrt{1-v^2/c^2}$
- After collision y-velocities reversed.

Assume $\vec{p} = m(w)\,\vec{w}$
m(w): something that may depend on particle speed $\vec{w}$.

Frame-A:
x-momentum only due to particle-B
Before collision B's speed
$$w^2 = V^2 + u_0^2/\gamma^2$$
After collision
$$w^2 = V^2 + \frac{u'^2}{\gamma^2}$$



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Lecture 36 Blackboard #1

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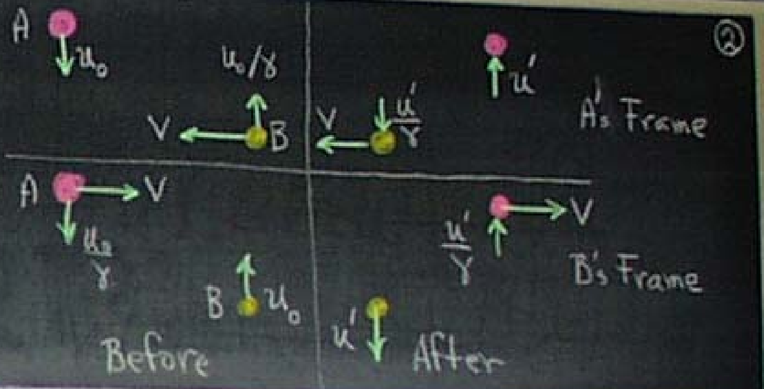

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Conservation of momentum along-x

$$m(w)V = m(w')V$$

$\therefore w = w'$

$\therefore u' = u_0$

Conservation of momentum along-y in Frame-A

$$-m(u_0)u_0 + m(w)\frac{u_0}{\gamma} = m(u_0)u_0 - m(w)\frac{u_0}{\gamma}$$

$$\therefore m(w) = \gamma\, m(u_0)$$

Let $m(u_0) = m_0$ as $u_0 \to 0$

m_0 = rest mass. Mass measured in frame where it is stationary.

In this limit $w = V$.

$$\therefore m(V) = \gamma m_0 = \frac{m_0}{\sqrt{1-V^2/c^2}}$$

Interpret as dependance of mass on speed.

In general:

$$\boxed{\vec{p} = \frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}}} = m\vec{u} = \gamma m_0\vec{u}$$

$$\boxed{m = \frac{m_0}{\sqrt{1-u^2/c^2}}} = \gamma m_0$$

3

Lecture 36 Blackboard #2

Relativistic Energy:

Classical:

$$K_b - K_a = \int_a^b \vec{F}\cdot d\vec{r} = \int_a^b \frac{d\vec{p}}{dt}\cdot d\vec{r}$$

$$= \tfrac{1}{2}mu_b^2 - \tfrac{1}{2}mu_a^2$$

Relativity:

$$\vec{p} = \gamma m_0 \vec{u}$$

$$K_b - K_a = \int_a^b \frac{d\vec{p}}{dt}\cdot d\vec{r}$$

$$= \int_a^b \frac{d}{dt}\frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}}\cdot \vec{u}\,dt$$

$$= \int_a^b \vec{u}\cdot d\left[\frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}}\right]$$

Integrand is $\vec{u}\cdot d\vec{p} = d(\vec{u}\cdot\vec{p}) - \vec{p}\cdot d\vec{u}$

$$K_b - K_a = \vec{u}\cdot\vec{p}\,\Big|_a^b - \int_a^b \vec{p}\cdot d\vec{u}$$

$$= \frac{m_0u^2}{\sqrt{1-u^2/c^2}}\Big|_a^b - \int_a^b \frac{m_0 u\,du}{\sqrt{1-u^2/c^2}}$$

$$= \frac{m_0u^2}{\sqrt{1-u^2/c^2}}\Big|_a^b + m_0c^2\sqrt{1-u^2/c^2}\,\Big|_a^b$$

b is arbitrary point

Assume $u_a = 0$ at a.

$$K = K_b - K_a = \gamma m_0 u^2 + \frac{m_0c^2}{\gamma} - m_0c^2$$

$$K = mc^2 - m_0c^2 \quad \text{where } m = \gamma m_0$$

Suppose $u \ll c$

$$\gamma \sim 1 + \tfrac{1}{2}u^2/c^2 + \dots$$

④

$$K = \frac{m_0c^2}{\sqrt{1-u^2/c^2}} - m_0c^2$$

$$\simeq m_0c^2\left[1 + \frac{1}{2}\frac{u^2}{c^2} - 1\right]$$

$$K = \tfrac{1}{2}m_0u^2$$

Classical Result!!

Rewrite:

$$mc^2 = K + m_0c^2$$

$$= \text{Work Done} + m_0c^2$$

Einstein:

mc^2 = Total energy E of particle.

= Ext. Work + Rest Energy.

$$\therefore \boxed{E = mc^2} = \gamma m_0 c^2$$

If energy ΔE is added to any object, it's mass changes by

$$\Delta m = \frac{\Delta E}{c^2}$$

Energy/Momentum

Classically: $E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$ $\quad (U=0)$

$$\vec{p} = m\vec{u} = \frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}} = \gamma m_0\vec{u}$$

$$E = mc^2 = \gamma m_0 c^2$$

$$p^2 = \gamma^2 m_0^2 u^2 = \frac{1}{1-u^2/c^2}\, m_0^2 u^2$$

⑤

Lecture 36 Blackboard #3

$$K = \frac{m_0c^2}{\sqrt{1-u^2/c^2}} - m_0c^2$$

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Energy/Momentum (5)

Classically: $E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$ $(u=0)$

$$\vec{p} = m\vec{u} = \frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}} = \gamma m_0\vec{u}$$

$$E = mc^2 = \gamma m_0c^2$$

$$p^2 = \gamma^2m_0^2u^2 = \frac{1}{1-u^2/c^2}m_0^2u^2$$

$$\frac{u^2}{c^2} = \frac{p^2}{p^2+m_0^2c^2}$$

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}} = \sqrt{1+\frac{p^2}{m_0^2c^2}}$$

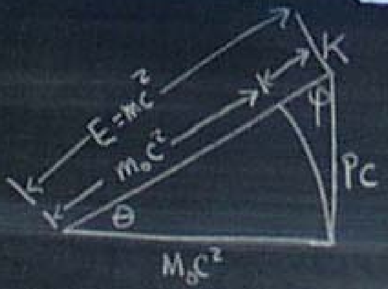
$$\therefore E = m_0c^2\sqrt{1+\frac{p^2}{m_0^2c^2}}$$

$$\therefore E^2 = (pc)^2 + (m_0c^2)^2$$

$$E^2 - (pc)^2 = (m_0c^2)^2$$

4-Vector!

Invariant Quantity



$$\sin\theta = \beta = \frac{u}{c}$$

$$\sin\varphi = \frac{1}{\gamma}$$

Massless Particles (6)

$$E^2 = (pc)^2 + (m_0c^2)^2$$

If $m_0 = 0$

$$E = pc$$

$$p = \gamma m_0u = \frac{m_0}{\sqrt{1-u^2/c^2}}\vec{u}$$

As $m_0 \to 0$ $\vec{p}$ must remain finite!

Only possible if

$u \to c$ as $m_0 \to 0$

$\therefore$ massless particles must travel with c.

Photons have $m_0 = 0$

Neutrinos have $m_0 \simeq 0$

Lecture 36 Blackboard #4

Force and Relativity.

$$\vec F = \frac{d\vec p}{dt} = \frac{d}{dt}(m\vec v) = m\frac{d\vec v}{dt} + \vec v\frac{dm}{dt} \quad ①$$

$$m = E/c^2$$

$$\therefore \frac{dm}{dt} = \frac{1}{c^2}\frac{dE}{dt} = \frac{1}{c^2}\frac{d}{dt}(K + m_0c^2) = \frac{1}{c^2}\frac{dK}{dt}$$

But $\frac{dK}{dt} = \vec F\cdot\frac{d\vec r}{dt} = \vec F\cdot\vec v$

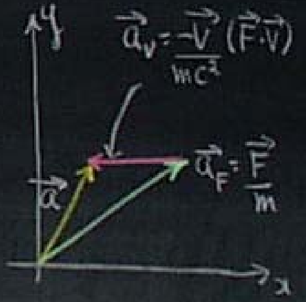
$$\therefore \frac{dm}{dt} = \frac{1}{c^2}\vec F\cdot\vec v$$

Sub in ①

$$\vec F = m\frac{d\vec v}{dt} + \frac{\vec v(\vec F\cdot\vec v)}{c^2}$$

$$\therefore \vec a = \frac{d\vec v}{dt} = \frac{1}{m}\left(\vec F - \frac{\vec v}{c^2}(\vec F\cdot\vec v)\right)$$

$\vec a$ is not // to $\vec F$ has comp. along $\vec v$



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