Module 24: Angular Momentum of a Point Particle

24.1 Introduction

When we consider a system of objects, we have shown that the total external force, acting at the center of mass of the system, is equal to the time derivative of the total momentum of the system,

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \frac{d\vec{\mathbf{p}}^{\text{total}}}{dt}. \tag{24.1.1}$$

We now introduce the rotational analog of this Equation (24.1.1). We will first introduce the concept of angular momentum for a point particle of mass m with linear momentum $\vec{\mathbf{p}}$ about a point S, defined by the equation

$$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}}. \tag{24.1.2}$$

where $\vec{\mathbf{r}}_S$ is the vector from the point S to the point particle. We will show in this chapter that the total torque about the point S acting on the particle is equal to the rate of change of the angular momentum about the point S of the particle,

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \frac{d\vec{\mathbf{L}}_S}{dt}. \tag{24.1.3}$$

Equation (24.1.3) generalizes to any body undergoing rotation.

We shall concern ourselves first with the special case of rigid body undergoing fixed axis rotation about the z-axis with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$. We divide up the rigid body into N elements labeled by the index i, $i = 1,2,\ldots N$, the i_{th} element having mass m_i and position vector $\vec{\mathbf{r}}_{S,i}$. The rigid body has a moment of inertia I_S about some point S on the fixed axis, (often taken to be the *z*-axis, but not always) which rotates with angular velocity ω about this axis. The total angular momentum is then the vector sum of the individual angular momenta,

$$\vec{\mathbf{L}}_S^{\text{total}} = \sum_{i=1}^{i=N} \vec{\mathbf{L}}_{S,i} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{p}}_i \tag{24.1.4}$$

When the rotation axis is the *z*-axis the *z*-component of the total angular momentum about the point S is then given by

$$(\vec{\mathbf{L}}_S^{\text{total}})_z = I_S\,\omega. \tag{24.1.5}$$

We shall show that torque is then the time derivative of the angular momentum, which we can show by differentiation.

$$\tau_{S,z}^{\text{total}} = \frac{d(\vec{\mathbf{L}}_S^{\text{total}})_z}{dt} = I_S \frac{d\omega}{dt} = I_S \alpha . \qquad (24.1.6)$$

24.2 Angular Momentum and Torque

Angular Momentum for a Point Particle

Consider a point particle of mass m moving with a velocity $\vec{\mathbf{v}}$ (Figure 24.1).

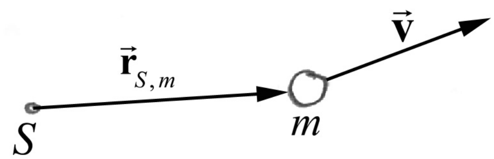


Figure 24.1 A point particle and its angular momentum.

The linear momentum of the particle is $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$. Consider a point S located anywhere in space. Let $\vec{\mathbf{r}}_S$ denote the vector from the point S to the location of the object.

> ***Definition: Angular Momentum about a point*** S
>
> *We define the angular momentum* $\vec{\mathbf{L}}_S$ *about the point* S *of a point particle as the vector cross product of the vector from the point* S *to the location of the object with the momentum of the particle,*
>
> $$\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}} . \qquad (24.2.1)$$

Since angular momentum is defined as a vector, we begin by studying its magnitude and direction.

Magnitude

The magnitude of the angular momentum about S is given by

$$\left|\vec{\mathbf{L}}_S\right| = \left|\vec{\mathbf{r}}_S\right|\left|\vec{\mathbf{p}}\right|\sin\theta , \qquad (24.2.2)$$

where θ is the angle between the vectors and $\vec{\mathbf{p}}$, and lies within the range $[0 \le \theta \le \pi]$ refer to Figures 15.2 and 15.3). The SI units for angular momentum are $\left[\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}\right]$. Like the calculation of torque, there are two ways to visualize the magnitude of the angular momentum.

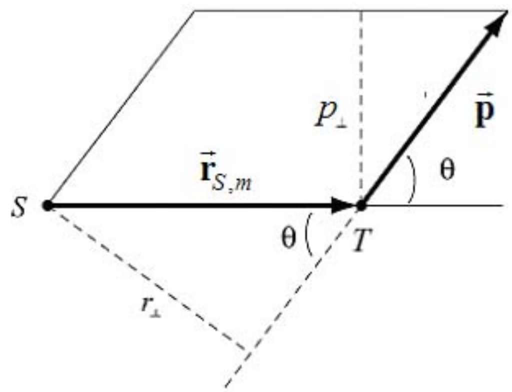


Figure 24.2 Vector diagram for angular momentum.

Define the *moment arm*, $r_\perp$, as the perpendicular distance from the point S to the line defined by the direction of the momentum. Then

$$r_\perp = \left|\vec{\mathbf{r}}_S\right| \sin\theta \,. \tag{24.2.3}$$

Hence the magnitude of the angular momentum is the product of the moment arm with the magnitude of the momentum.

$$\left|\vec{\mathbf{L}}_S\right| = r_\perp \left|\vec{\mathbf{p}}\right| . \tag{24.2.4}$$

Alternatively, define the *perpendicular momentum*, $p_\perp$, to be the magnitude of the component of the momentum perpendicular to the line defined by the direction of the vector $\vec{\mathbf{r}}_{S,m}$. Thus

$$p_\perp = \left|\vec{\mathbf{p}}\right| \sin\theta \,. \tag{24.2.5}$$

We can think of the magnitude of the angular momentum as the product of the distance from S to the point object with the perpendicular momentum,

$$\left|\vec{\mathbf{L}}_S\right| = \left|\vec{\mathbf{r}}_S\right| p_\perp \,. \tag{24.2.6}$$

Right-Hand-Rule for the Direction of the Angular Momentum

Draw the vectors $\vec{\mathbf{r}}_S$ and $\vec{\mathbf{p}}$ so that their tails are touching. Then draw an arc starting from the vector $\vec{\mathbf{r}}_S$ and finishing on the vector $\vec{\mathbf{p}}$. (There are two such arcs; choose the shorter one.) This arc is either in the clockwise or counterclockwise direction. Curl the fingers of your right hand in the same direction as the arc. Your right thumb points in the direction of the angular momentum.

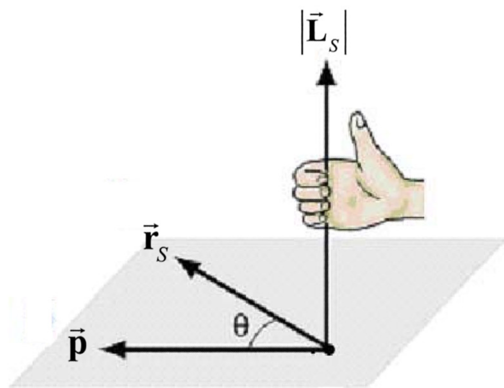
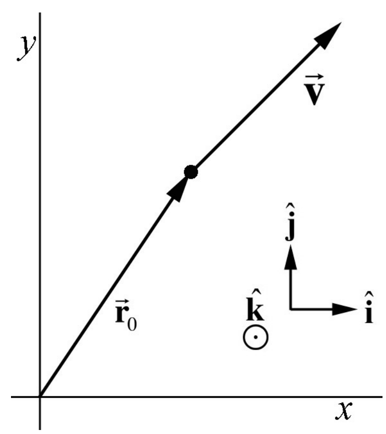


Figure 24.3 The Right Hand Rule.

Remember that, as in all cross products, the direction of the angular momentum is perpendicular to the plane formed by $\vec{\mathbf{r}}_S$ and $\vec{\mathbf{p}}$.

24.2.1 Example Angular Momentum: *Constant Velocity*

A particle of mass $m = 2.0\text{ kg}$ moves as shown in the sketch with a uniform velocity $\vec{\mathbf{v}} = 3.0\text{ m}\cdot\text{s}^{-1}\,\hat{\mathbf{i}} + 3.0\text{ m}\cdot\text{s}^{-1}\,\hat{\mathbf{j}}$. At time t, the particle passes through the point $\vec{\mathbf{r}}_0 = 2.0\text{ m}\,\hat{\mathbf{i}} + 3.0\text{ m}\,\hat{\mathbf{j}}$. Find the direction and the magnitude of the angular momentum about the origin at time t.



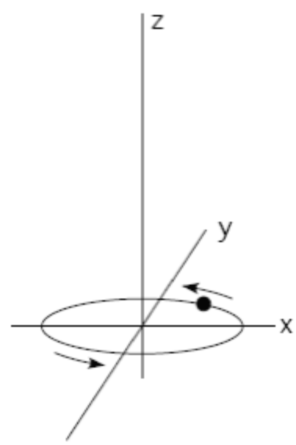
Solution: Choose Cartesian coordinates with unit vectors shown in the figure above. The angular momentum vector $\vec{\mathbf{L}}$ of the particle about the origin is given by:

$$
\begin{aligned}
\vec{\mathbf{L}}_0 &= \vec{\mathbf{r}}_0 \times \vec{\mathbf{p}} = \vec{\mathbf{r}}_0 \times m\vec{\mathbf{v}} \\
&= \left(2.0\,\mathrm{m}\;\hat{\mathbf{i}} + 3.0\,\mathrm{m}\;\hat{\mathbf{j}}\right) \times (2\mathrm{kg})\left(3.0\,\mathrm{m}\cdot\mathrm{s}^{-1}\hat{\mathbf{i}} + 3.0\,\mathrm{m}\cdot\mathrm{s}^{-1}\hat{\mathbf{j}}\right) \\
&= 0 + 12\,\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}\,\hat{\mathbf{k}} - 18\,\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}(-\hat{\mathbf{k}}) + \vec{\mathbf{0}} \\
&= -6\,\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}\,\hat{\mathbf{k}}.
\end{aligned}
$$

In the above, the relations $\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}},\; \vec{\mathbf{j}} \times \vec{\mathbf{i}} = -\vec{\mathbf{k}},\; \vec{\mathbf{i}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}} \times \vec{\mathbf{j}} = \vec{\mathbf{0}}$ were used.

24.2.2 Example Angular momentum and circular motion

A particle of mass m move in a circle of radius r at an angular speed ω about the z axis in a plane parallel to the x-y plane passing thorough the origin. Find the magnitude and the direction of the angular momentum $\vec{\mathbf{L}}_0$ relative to the origin.



Solution:

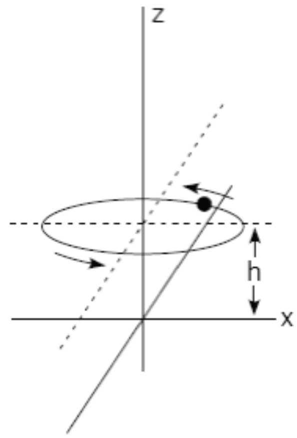
The velocity of the particle is given by $\vec{\mathbf{v}} = r\omega\,\hat{\boldsymbol{\theta}}$. The vector from the center of the circle (the point *S*) to the object is given by $\vec{\mathbf{r}}_S = r\,\hat{\mathbf{r}}$. The angular momentum about the center of the circle is the cross product

$$
\vec{\mathbf{L}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{p}} = \vec{\mathbf{r}}_S \times m\vec{\mathbf{v}} = rmv\,\hat{\mathbf{k}} = rmr\omega\,\hat{\mathbf{k}} = mr^2\omega\,\hat{\mathbf{k}}\,.
$$

The magnitude is $\left|\vec{\mathbf{L}}_S\right| = mr^2\omega$, and the direction is in the $+\hat{\mathbf{k}}$-direction.

24.2.3 Example

Problem: A particle of mass m moves in a circle of radius r at an angular speed ω about the z axis in a plane parallel to but a distance h above the x-y plane.



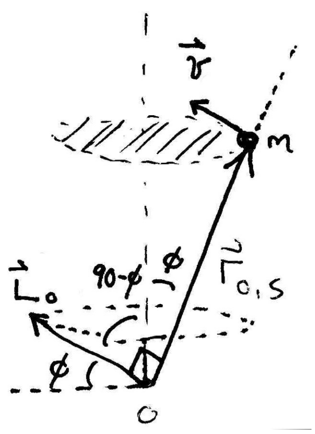
a) Find the magnitude and the direction of the angular momentum $\vec{\mathbf{L}}_0$ relative to the origin.

b) Also find the z component of $\vec{\mathbf{L}}_0$.

Solution: We begin by making a geometric argument. Suppose the particle has coordinates (x, y, h). The angular momentum about the origin is defined as

$$\vec{\mathbf{L}}_0 = \vec{\mathbf{r}}_{0,m} \times m\vec{\mathbf{v}} . \tag{24.2.7}$$

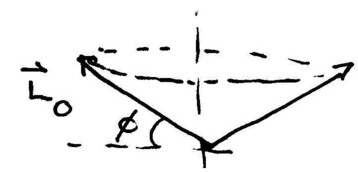
The vectors $\vec{\mathbf{r}}_{0,m}$ and $\vec{\mathbf{v}}$ are perpendicular to each other so the angular momentum is perpendicular to the plane formed by those two vectors. Recall that the speed $v = r\omega$. Suppose the vector $\vec{\mathbf{r}}_{0,m}$ forms an angle ϕ with the z-axis. Then $\vec{\mathbf{L}}_0$ forms an angle $90^\circ - \phi$ with respect to the z-axis or an angle ϕ with respect to the x-y plane as shown in the figure on the right.



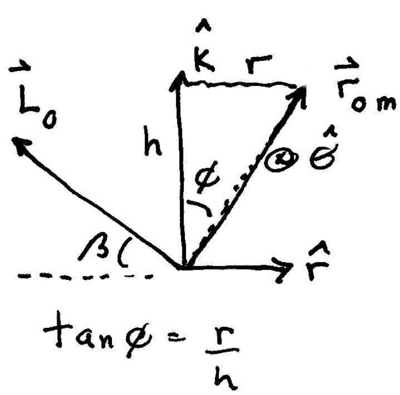
The magnitude of $\vec{\mathbf{L}}_0$ is

$$\left|\vec{\mathbf{L}}_0\right| = \left|\vec{\mathbf{r}}_{0,m}\right| m \left|\vec{\mathbf{v}}\right| = m(h^2 + (x^2 + y^2))^{1/2} r\omega . \qquad (24.2.8)$$

The magnitude of $\vec{\mathbf{L}}_0$ is constant, but its direction is changing as the particle moves in a circular orbit about the z-axis, sweeping out a cone as shown in the figure below. We draw the vector $\vec{\mathbf{L}}_0$ at the origin because it is defined at that point.



We shall now explicitly calculate the cross product. We shall discover that taking the cross product using polar coordinates is the easiest way to calculate $\vec{\mathbf{L}}_0 = \vec{\mathbf{r}}_{0,m} \times m\vec{\mathbf{v}}$. We begin by writing the two vectors that appear in Eq. in polar coordinates . We start with the vector from the origin to the location of the moving object, $\vec{\mathbf{r}}_{0,m} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + h\hat{\mathbf{k}} = r\hat{\mathbf{r}} + h\hat{\mathbf{k}}$ where $r = (x^2 + y^2)^{1/2}$. The velocity vector is tangent to the circular orbit so $\vec{\mathbf{v}} = v\,\hat{\boldsymbol{\theta}} = r\omega\,\hat{\boldsymbol{\theta}}$.



Using the fact that $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}$ and $\hat{\mathbf{k}} \times \hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}$, the angular momentum about the origin $\vec{\mathbf{L}}_0$ is

$$\vec{\mathbf{L}}_0 = \vec{\mathbf{r}}_{0,m} \times m\vec{\mathbf{v}} = (r\hat{\mathbf{r}} + h\hat{\mathbf{k}}) \times mr\omega\hat{\boldsymbol{\theta}} = rmr\omega\hat{\mathbf{k}} - hmr\omega\hat{\mathbf{r}} . \qquad (24.2.9)$$

The magnitude of $\vec{\mathbf{L}}_0$ is given by

$$\left|\vec{\mathbf{L}}_0\right| = ((rmr\omega)^2 + (hmr\omega)^2)^{1/2} = m(h^2 + r^2)^{1/2} r\omega = m(h^2 + (x^2 + y^2))^{1/2} r\omega . \qquad (24.2.10)$$

Agreeing with our geometric argument. The direction of $\vec{\mathbf{L}}_0$ is given by

$$\tan\beta = -\frac{L_{0z}}{L_{0r}} = \frac{r}{h} = \tan\phi \qquad (24.2.11)$$

so $\beta = \phi$ also agreeing with our geometric argument.

The important point to keep in mind regarding this calculation is that for any point along the z-axis not at the center of the circular orbit of a single particle, the angular momentum about that point does not point along the z-axis but it is has a non-zero component in the x-y plane (or in the $-\hat{\mathbf{r}}$ direction if you use polar coordinates). The z-component of the angular momentum about any point along the z-axis is independent of the location of the point along the axis.

Torque and the Time Derivative of Angular Momentum for a Point Particle

We will now show that the torque about a point S is equal to the time derivative of the angular momentum about S.

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \frac{d\vec{\mathbf{L}}_S}{dt}. \qquad (24.2.12)$$

Take the time derivative of the angular momentum about S,

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \frac{d}{dt}\left(\vec{\mathbf{r}}_S \times \vec{\mathbf{p}}\right). \qquad (24.2.13)$$

In this equation we are taking the time derivative of a cross product of two vectors. There are two important facts that will help us simplify this expression. First, the time derivative of the cross product of two vectors satisfies the product rule,

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \frac{d}{dt}(\vec{\mathbf{r}}_S \times \vec{\mathbf{p}}) = \left(\left(\frac{d\vec{\mathbf{r}}_S}{dt}_S\right) \times \vec{\mathbf{p}}\right) + \left(\vec{\mathbf{r}}_S \times \left(\frac{d\vec{\mathbf{p}}}{dt}\right)\right). \qquad (24.2.14)$$

Second, the first term on the right hand side vanishes,

$$\frac{d\vec{\mathbf{r}}_S}{dt} \times \vec{\mathbf{p}} = \vec{\mathbf{v}} \times m\,\vec{\mathbf{v}} = \vec{\mathbf{0}}. \qquad (24.2.15)$$

The rate of angular momentum change about the point S is then

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \vec{\mathbf{r}}_S \times \frac{d\vec{\mathbf{p}}}{dt}. \qquad (24.2.16)$$

From Newton's Second Law, the total force on the particle is equal to the change of linear momentum,

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}. \qquad (24.2.17)$$

Therefore the rate of change in time of angular momentum about the point S is

$$\frac{d\vec{\mathbf{L}}_S}{dt} = \vec{\mathbf{r}}_S \times \vec{\mathbf{F}}. \qquad (24.2.18)$$

Recall that the torque about the point S due to the force $\vec{\mathbf{F}}$ acting on the particle is

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_S \times \vec{\mathbf{F}}. \qquad (24.2.19)$$

Combining the expressions in (24.2.18) and (24.2.19), it is readily seen that the torque about the point S is equal to the rate of change of angular momentum about the point S,

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}. \qquad (24.2.20)$$

24.3 Conservation of Angular Momentum about a Point

So far we have introduced two conservation principles, showing that energy and linear momentum are constants for closed systems (no external forces implies the momentum is constant and no change in energy in the surroundings implies the energy in the system is constant). The conservation of mechanical energy was a consequence of the work-energy theorem; the total non-conservative work done on a system is equal to the change in mechanical energy,

$$W_{\text{nc}} = \Delta E_{\text{mechanical}} = \Delta K + \Delta U^{\text{total}}. \qquad (24.3.1)$$

If the non-conservative work done on a system is zero, then the mechanical energy is conserved,

$$0 = W_{\text{nc}} = \Delta E_{\text{mechanical}} = \Delta K + \Delta U^{\text{total}} \qquad (24.3.2)$$

The conservation of linear momentum arises from Newton's Second Law applied to systems,

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \sum_{i=1}^{N} \frac{d}{dt}\vec{\mathbf{p}}_i = \frac{d}{dt}\vec{\mathbf{p}}^{\text{total}} \tag{24.3.3}$$

Thus if the total external force in any direction is zero, then the component of the total momentum of the system in that direction is a constant, hence conserved. For example, if there are no external forces in the x- and y-directions then

$$\begin{aligned} 0 &= (\vec{\mathbf{F}}_{\text{ext}}^{\text{total}})_x = \frac{d}{dt}(\vec{\mathbf{p}}^{\text{total}})_x \\ 0 &= (\vec{\mathbf{F}}_{\text{ext}}^{\text{total}})_y = \frac{d}{dt}(\vec{\mathbf{p}}^{\text{total}})_y . \end{aligned} \tag{24.3.4}$$

We can now use our relation between torque about a point S and the change of the angular momentum about S,

$$\vec{\boldsymbol{\tau}}_S^{\text{total}} = \frac{d\vec{\mathbf{L}}_S^{\text{total}}}{dt},$$

to introduce a new conservation law. Suppose we can find a point S such that torque about the point S is zero,

$$\vec{\mathbf{0}} = \vec{\boldsymbol{\tau}}_S^{\text{total}} = \frac{d}{dt}\vec{\mathbf{L}}_S^{\text{total}} . \tag{24.3.5}$$

Then the angular momentum about the point S is a constant vector, and so the change in angular momentum is zero,

$$\Delta\vec{\mathbf{L}}_S^{\text{total}} \equiv \vec{\mathbf{L}}_{S,f}^{\text{total}} - \vec{\mathbf{L}}_{S,0}^{\text{total}} = \vec{\mathbf{0}} . \tag{24.3.6}$$

Thus when the total torque about a point S is zero, the final angular momentum about S is equal to the initial angular momentum,

$$\vec{\mathbf{L}}_{S,f}^{\text{total}} = \vec{\mathbf{L}}_{S,0}^{\text{total}} . \tag{24.3.7}$$

24.3.1 Example Meteor Flyby of Earth

A meteor of mass $m = 2.1 \times 10^{13}\ \mathrm{kg}$ is approaching earth as shown on the sketch. The distance h on the sketch below is called the impact parameter. The radius of the earth is $r_e = 6.37 \times 10^{6}\ \mathrm{m}$. The mass of the earth is $m_e = 5.98 \times 10^{24}\ \mathrm{kg}$. Suppose the meteor has an initial speed of $v_0 = 1.0 \times 10^{1}\ \mathrm{m \cdot s^{-1}}$. Assume that the meteor started very far away from the earth. Suppose the meteor just grazes the earth. You may ignore all other gravitational forces except the earth. Find the impact parameter h.

Figure 24.8 Meteor Flyby of Earth

Solution: Strategy.

a) Draw a free body force diagram for the force acting on the meteor when the meteor is very far away and when the meteor just grazes the earth.

b) Find a point about which the gravitational torque of the earth's force on the meteor is zero for the entire orbit of the meteor.

c) Find the initial angular momentum (when it is very far away) and final angular momentum (when it just grazes the earth) of the meteor.

d) Apply conservation of angular momentum to find a relationship between the meteor's final velocity and the impact parameter h.

e) Apply conservation of energy to find a relationship between the final velocity of the meteor and the initial velocity of the meteor.

f) Use your results in parts d) and e) to calculate the impact parameter h.

Solution:

a) Free body diagrams.

Very Far Away

Grazing the Earth

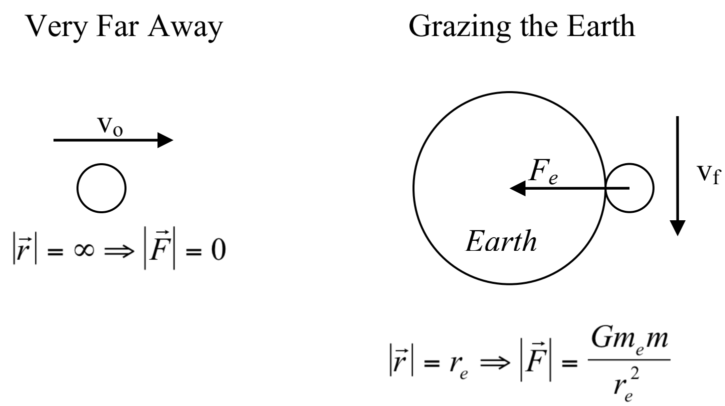


Figure 24.9 Free Body Force Diagrams for Meteor

b) Yes, the center of earth. Denote the center of the earth by S. The force on the meteor is given by

$$\vec{\mathbf{F}} = -\frac{G m_e m}{|\vec{r}|^2}\hat{\mathbf{r}} \qquad (24.3.8)$$

where $\hat{\mathbf{r}}$ is a unit vector pointing radially away from the center of the earth, and r is the distance from the center of the earth to the metoer. The torque on the meteor is given by $\vec{\tau}_S = \vec{\mathbf{r}}_{S,F} \times \vec{\mathbf{F}}$, where $\vec{\mathbf{r}}_{S,F} = r\,\hat{\mathbf{r}}$ is the vector from the point S to the position of the meteor. Since the force and the position vector are collinear, the cross product vanishes and hence the torque on the meteor vanishes about S.

c) **Initial Angular Momentum:** Choose Cartesian coordinates as shown in the figure below.

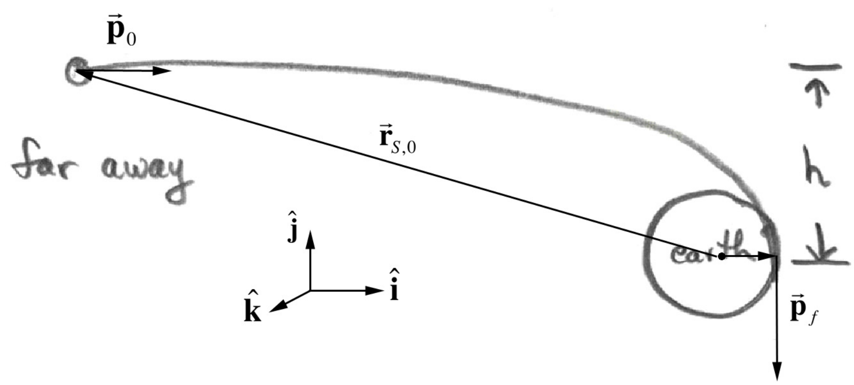


Figure 24.10 Momentum Flow Diagram for Meteor

The initial angular momentum about the center of the earth is

$$(\vec{\mathbf{L}}_S)_0 = \vec{\mathbf{r}}_{S,0} \times \vec{\mathbf{p}}_0 \qquad (24.3.9)$$

where the vector from the center of the earth to the meteor is $\vec{\mathbf{r}}_{S,0} = -x_0\,\hat{\mathbf{i}} + h\,\hat{\mathbf{j}}$ (we can choose some arbitrary x_0 for the initial distance in the x-direction), and the momentum is $\vec{\mathbf{p}}_0 = mv_0\hat{\mathbf{i}}$. Then the initial angular momentum is

$$(\vec{\mathbf{L}}_S)_0 = \vec{\mathbf{r}}_{S,0} \times \vec{\mathbf{p}}_0 = (-x_0\,\hat{\mathbf{i}} + h\,\hat{\mathbf{j}}) \times mv_0\hat{\mathbf{i}} = -mv_o h\,\hat{\mathbf{k}} \qquad (24.3.10)$$

The final angular momentum about he center of the earth is

$$(\vec{\mathbf{L}}_S)_f = \vec{\mathbf{r}}_{S,f} \times \vec{\mathbf{p}}_f \qquad (24.3.11)$$

where the vector from the center of the earth to the meteor is $\vec{\mathbf{r}}_{S,f} = r_e\,\hat{\mathbf{i}}$ since the meteor is then just grazing the surface of earth, and the momentum is $\vec{\mathbf{p}}_f = -mv_f\,\hat{\mathbf{j}}$. So the final angular momentum about the center of the earth is

$$(\vec{\mathbf{L}}_S)_f = \vec{\mathbf{r}}_{S,f} \times \vec{\mathbf{p}}_f = r_e\,\hat{\mathbf{i}} \times (-mv_f\,\hat{\mathbf{j}}) = -mr_e v_f\,\hat{\mathbf{k}} \qquad (24.3.12)$$

d) Since the angular momentum about the center of the earth is constant throughout the motion

$$(\vec{\mathbf{L}}_S)_0 = (\vec{\mathbf{L}}_S)_f \qquad (24.3.13)$$

which implies that

$$-mv_0 h\,\hat{\mathbf{k}} = -mr_e v_f\,\hat{\mathbf{k}} \qquad (24.3.14)$$

or

$$v_f = \frac{v_0 h}{r_e} \qquad (24.3.15)$$

e) The mechanical energy is constant and with our choice of zero for potential energy when the the meteor is very far away, the energy condition becomes

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 - \frac{Gm_e m}{r_e} \qquad (24.3.16)$$

So,

$$v_f^2 = v_0^2 + \frac{2Gm_e}{r_e} \qquad (24.3.17)$$

f) Substituting for v_f from part (d) and solving for h, we have

$$h = r_e \sqrt{1 + \frac{2Gm_e}{r_e v_0^2}} \qquad (24.3.18)$$

On substituting the values we have,

$$h = 1117.4\ r_e = 7.12 \times 10^9 \,\mathrm{m} \qquad (24.3.19)$$

24.4 Angular Impulse and Change in Angular Momentum

If there is a total applied torque $\vec{\boldsymbol{\tau}}_S$ about a point S over an interval of time $\Delta t = t_f - t_0$, then the torque applies an *angular impulse* about a point S, given by

$$\vec{\mathbf{J}}_S = \int_{t_0}^{t_f} \vec{\boldsymbol{\tau}}_S \, dt\,. \qquad (24.4.1)$$

Because $\vec{\boldsymbol{\tau}}_S = d\,\vec{\mathbf{L}}_S^{\text{total}} / dt$, the angular impulse about S is equal to the change in angular momentum about S,

$$\vec{\mathbf{J}}_S = \int_{t_0}^{t_f} \vec{\boldsymbol{\tau}}_S \, dt = \int_{t_0}^{t_f} \frac{d\vec{\mathbf{L}}_S}{dt} dt = \Delta \vec{\mathbf{L}}_S = \vec{\mathbf{L}}_{S,f} - \vec{\mathbf{L}}_{S,0}\,. \qquad (24.4.2)$$

This result is the rotational analog to linear impulse, which is equal to the change in momentum;

$$\vec{\mathbf{I}} = \int_{t_0}^{t_f} \vec{\mathbf{F}} \, dt = \int_{t_0}^{t_f} \frac{d\vec{\mathbf{p}}}{dt} dt = \Delta \vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_0\,. \qquad (24.4.3)$$