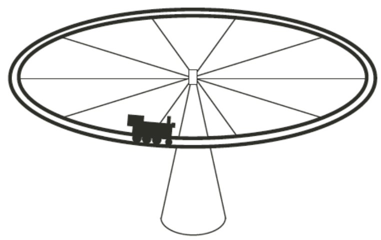
Angular Momentum Problems
Challenge Problems

Problem 1:

Toy Locomotive A toy locomotive of mass m_L runs on a horizontal circular track of radius R and total mass m_T. The track forms the rim of an otherwise massless wheel which is free to rotate without friction about a vertical axis. The locomotive is started from rest and accelerated without slipping to a final speed of v relative to the track. What is the locomotive's final speed, v_f, relative to the floor?



Problem 1 Solution:

We begin by choosing our system to consist of the locomotive and the track. Because there are no external torques about the central axis, the angular momentum of the system consisting of the engine and the track is constant about that axis. The initial angular momentum of the system is zero because the locomotive and the track are at rest.

Let $\omega_{T,f}$ denote the final angular speed of the track. In the figure above the locomotive is rotating counterclockwise as seen from above so the track must spin in the clockwise direction. If we choose the positive z-direction to point upward then the final angular momentum of the track is given by

$$\vec{L}_{T,f} = -I_{T,z}\omega_{T,f}\hat{k} = -m_T R^2 \omega_{T,f}\hat{k} \qquad \textbf{(1.1)}$$

where we have assumed that the moment of inertia of the track about an axis passing perpendicularly through the center of the circle forms by the track is $I_{T,z} = m_T R^2$.

The locomotive is moving tangentially with respect to the ground so we can choose polar coordinates and then the final velocity of the locomotive with respect to the ground is

$$\vec{v}_{L,f} = v_f\hat{\theta}\,. \qquad \textbf{(1.2)}$$

A point on the rim of the track has final velocity

$$\vec{v}_{T,f} = -R\omega_{T,f}\hat{\theta}\,. \quad \textbf{(1.3)}$$

Therefore the relative velocity $\vec{v}_{rel} = v\,\hat{\theta}$ of the locomotive to the track is given by the difference of the velocity of the locomotive and the appoint on the rim of the track,

$$\vec{v}_{rel} = \vec{v}_{L,f} - \vec{v}_{T,f} = v_f\hat{\theta} - (-R\omega_{T,f}\hat{\theta}) = (v_f + R\omega_{T,f})\hat{\theta} = v\hat{\theta}\,, \quad \textbf{(1.4)}$$

therefore

$$v = v_f + R\omega_{T,f}\,. \quad \textbf{(1.5)}$$

We can solve the above equation for the final angular speed $\omega_{T,f}$ in terms of the given relative speed and the final speed of the locomotive with respect to the ground yielding

$$\omega_{T,f} = \frac{v - v_f}{R}\,. \quad \textbf{(1.6)}$$

The final angular momentum of the locomotive with respect to the center of the circle formed by the track when it is moving with speed v_f, relative to the floor is

$$\vec{L}_{L,f} = m_L R v_f \hat{k}\,. \quad \textbf{(1.7)}$$

Because the angular momentum of the system is constant we have that

$$\vec{0} = \vec{L}_{T,f} + \vec{L}_{L,f} = (-m_T R^2 \omega_{T,f} + m_L R v_f)\hat{k}\,. \quad \textbf{(1.8)}$$

Now substitute Eq. (1.6) into the z-component of the above equation yielding

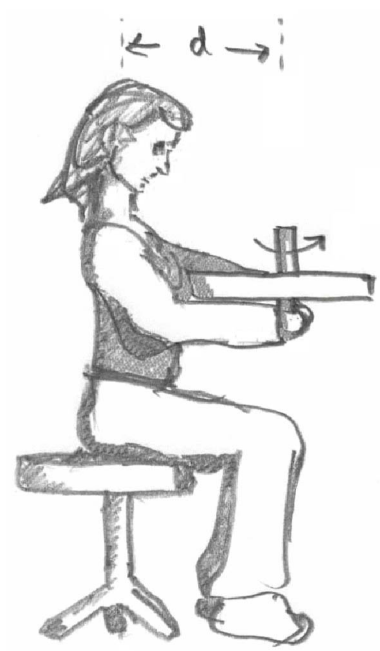
$$0 = -m_T R(v - v_f) + m_L R v_f\,. \quad \textbf{(1.9)}$$

We now solve the above equation to find the final speed of the locomotive relative to the floor

$$v_f = \frac{m_T}{m_T + m_L} v\,. \quad \textbf{(1.10)}$$

Problem 2: Spinning Chair and Wheel Solution

A person is sitting on a stool that is initially not rotating and is holding a spinning wheel. The moment of inertia of the person and the stool about a vertical axis passing through the center of the stool is $I_{S,p}$. The moment of inertia of the wheel about an axis, perpendicular to the plane of the wheel, passing through the center of mass of the wheel is $I_w = (1/4)I_{S,p}$. The mass of wheel is m_w. Suppose that the person holds the wheel as shown in the sketch such that the distance of an axis passing through the center of mass of the wheel to the axis of rotation of the stool is d and that $m_w d^2 = (1/3)I_w$. Suppose the wheel is spinning initially at an angular speed ω_s. The person then turns the spinning wheel upside down. You may ignore any frictional torque in the bearings of the stool. **What is the angular speed of the person and stool after the spinning wheel is turned upside down?**



Problem 2 Solution:

There are no external torques acting on the system so angular momentum is conserved. When the person is spinning on the chair, the angular momentum about a vertical axis (z-axis) passing through the center of the stool is the sum of two contributions. The first contribution is due to the angular momentum of the person and the stool,

$$\vec{\mathbf{L}}_{S,p} = I_{S,p}\omega\hat{\mathbf{k}}$$

where $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$ is the angular velocity of the person and the stool about a vertical axis passing through the center of the stool.

The second contribution to the angular momentum about a vertical axis passing through the center of the stool is due to the rotational motion of the wheel. From the parallel axis theorem, the moment of inertia of the wheel about a vertical axis passing through the center of the stool is

$$I_{S,w} = I_w + m_w d^2 .$$

Since the wheel is rotating about this axis with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\mathbf{k}}$, the angular momentum is given by

$$\vec{\mathbf{L}}_{S,w}^{\text{rot}} = \left(I_w + m_w d^2\right)\omega\hat{\mathbf{k}} .$$

Note the first term represent s the fact that when the wheel rotates once about the axis, it is also rotating once about its center of mass. The second term can be thought of as the angular momentum of all the mass of the wheel located at its center of mass undergoing uniform circular motion.

Since the wheel is also spinning about its center of mass with angular velocity $\vec{\boldsymbol{\omega}}_{\text{spin}} = \omega_s\hat{\mathbf{k}}$, there is the 'spin angular momentum' due entirely to the spinning of the wheel about it's center of mass,

$$\vec{\mathbf{L}}_{w}^{\text{spin}} = I_w\omega_s\hat{\mathbf{k}} .$$

So the total angular momentum of the wheel about the central axis is

$$\vec{\mathbf{L}}_{S,w}^{\text{total}} = \vec{\mathbf{L}}_{S,w}^{\text{rot}} + \vec{\mathbf{L}}_{cm,w}^{\text{spin}} = \left((I_w + m_w d^2)\omega + I_w\omega_s\right)\hat{\mathbf{k}}$$

The total angular momentum of the person, stool, and wheel, is then the sum of these three terms.

$$\vec{\mathbf{L}}_{S}^{\text{total}} = \vec{\mathbf{L}}_{S,p} + \vec{\mathbf{L}}_{S,w}^{\text{rot}} + \vec{\mathbf{L}}_{w}^{\text{spin}} = \left((I_{s,p} + I_w + m_w d^2)\omega + I_w\omega_s\right)\hat{\mathbf{k}}$$

Initially, $\omega_i = 0$, therefore the initial angular momentum is only due to the 'spin angular momentum of the wheel',

$$\vec{\mathbf{L}}_{S,i}^{\text{total}} = I_w\omega_s\hat{\mathbf{k}}$$

When the person turns the wheel over, the spin angular momentum of the wheel has reversed direction,

$$\vec{\mathbf{L}}_{w,f}^{\text{spin}} = -I_w \omega_s \hat{\mathbf{k}} .$$

The system begins to rotate with angular velocity ω_f, so the final angular momentum is

$$\vec{\mathbf{L}}_{S,f}^{\text{total}} = (I_{s,p} + I_w + m_w d^2)\omega_f \,\hat{\mathbf{k}} - I_w \omega_s \hat{\mathbf{k}} .$$

Notice that the sign of the spin angular momentum of the wheel has changed. Since angular momentum is conserved,

$$\vec{\mathbf{L}}_{S,i}^{\text{total}} = \vec{\mathbf{L}}_{S,f}^{\text{total}} ,$$

we have that (for the z-component of angular momentum)

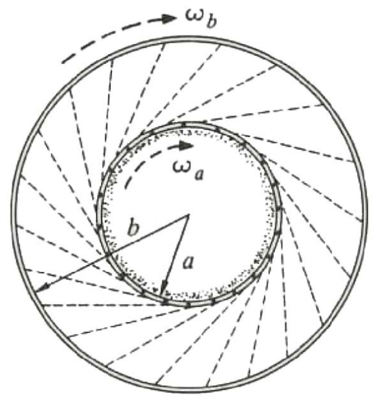
$$I_w \omega_s = (I_{s,p} + I_w + m_w d^2)\omega_f - I_w \omega_s$$

We can solve for ω_f:

$$\omega_f = \frac{2I_w \omega_s}{(I_{s,p} + I_w + m_w d^2)} = \frac{2I_w \omega_s}{(I_{s,p} + I_w + m_w d^2)} = \frac{3}{8}\omega_s$$

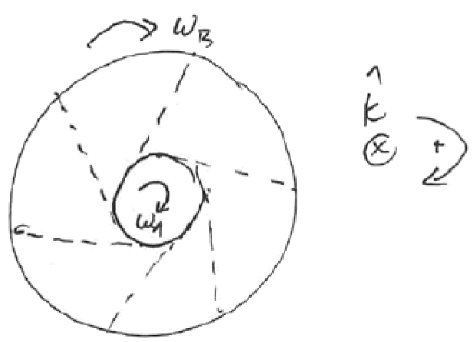
Problem 3:

A drum of mass m_A and radius a rotates freely with initial angular velocity $\omega_{A,0}$. A second drum with mass m_B and radius $b > a$ is mounted on the same axle and is at rest, although it is free to rotate. A thin layer of sand with mass m_S is distributed on the inner surface of the smaller drum. At $t = 0$, small perforations in the inner drum are opened. The sand starts to fly out at a constant rate $\lambda\ \mathrm{kg \cdot s^{-1}}$ and sticks to the outer drum. Find the subsequent angular velocities of the two drums ω_A and ω_B. Ignore the transit time of the sand.



Problem 3 Solution:

In this problem, we choose as our system the outside drum with accumulated sand and the small amount of sand m_S that leaves the inner drum and settles against the outer drum, with coordinate system shown below.



The angular impulse equation states that

$$\int \vec{\tau}_S dt = \Delta \vec{L}_S . \tag{3.1}$$

As the sand leaves the inner drum it does not exert any force on the inner drum, it flies through the open whole, hence the inner drum does not exert any force or torque about the center S of the drums on the system consisting of the sand and outer drum. Therefore the angular momentum about the center S of the drums is constant.

$$\vec{L}_{S,i} = \vec{L}_{S,f} . \qquad \textbf{(3.2)}$$

The initial angular momentum about S is

$$\vec{L}_{S,i} = m_s a^2 \omega_{A,0} \hat{k} . \qquad \textbf{(3.3)}$$

When all the sand has reached the outer drum, the final angular momentum about S is

$$\vec{L}_{S,f} = (m_s + m_B) b^2 \omega_{B,f} \hat{k} . \qquad \textbf{(3.4)}$$

The z-component of the angular momentum equation (Eq. (3.2)) becomes

$$m_s a^2 \omega_{A,0} = (m_s + m_B) b^2 \omega_{B,f} . \qquad \textbf{(3.5)}$$

Therefore the final angular speed of the outer drum is given by

$$\omega_{B,f} = \frac{m_s a^2 \omega_{A,0}}{(m_s + m_B) b^2} . \qquad \textbf{(3.6)}$$

The inner drum does not change its angular speed.

Note that at time t, an amount of sand $m_s(t) = \lambda t$ has been transferred so the angular momentum at time t is given by

$$\vec{L}_S(t) = (m_s - \lambda t) a^2 \omega_{A,0} \hat{k} + (m_B + \lambda t) b^2 \omega_B(t) \hat{k} . \qquad \textbf{(3.7)}$$

So the angular momentum equation becomes

$$m_s a^2 \omega_{A,0} = (m_s - \lambda t) a^2 \omega_{A,0} + (m_B + \lambda t) b^2 \omega_B(t) . \qquad \textbf{(3.8)}$$

which we can solve for the angular speed of the outer drum as a function of time

$$\omega_B(t) = \frac{\lambda t a^2 \omega_{A,0}}{\left(\lambda t + m_B\right) b^2} . \qquad \textbf{(3.9)}$$

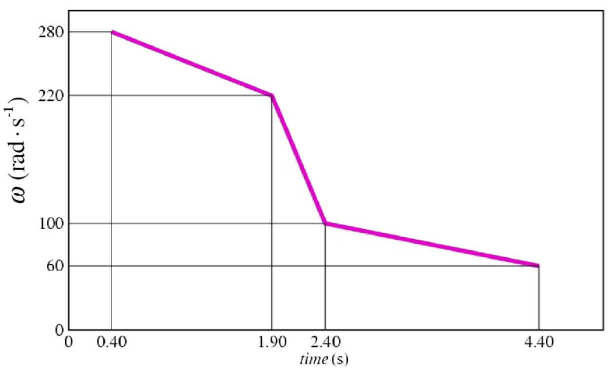
Problem 4: Measuring Moment of Inertia

In the angular momentum experiment, shown to the right, a washer is dropped smooth side down onto the spinning rotor.

The graph below shows the rotor angular velocity $\omega\left(\text{rad}\cdot\text{s}^{-1}\right)$ as a function of time.

Assume the following:

- The rotor and washer have the same moment of inertia I.
- The friction torque $\vec{\tau}_f$ on the rotor is constant during the measurement.



a) Find an expression for the magnitude $\left|\vec{\tau}_f\right|$ in terms of I and numbers you obtain from the graph.

b) What torque does the rotor exert on the washer during the collision (between $t = 1.90\text{ s}$ and $t = 2.40\text{ s}$)? Express your answer in terms of I and numbers you obtain from the graph.

c) How many radians does the rotor rotate during the collision (between $t = 1.90\text{ s}$ and $t = 2.40\text{ s}$)? Give a numerical answer.

d) How many radians does the washer rotate during the collision (between $t = 1.90\text{ s}$ and $t = 2.40\text{ s}$)? Give a numerical answer.

Note: express all of your answers in terms of I and numbers you obtain from the graph. Be sure to give an analytic expression prior to substituting the numbers from the graph.

Problem 4 Solutions:

a) First, make sure that the problem makes sense. Between times $t = 0.40\,\mathrm{s}$ and $t = 1.90\,\mathrm{s}$, the magnitude of the angular acceleration is $\Delta\omega / \Delta t = 40\,\mathrm{rad \cdot s^{-2}}$ and between times $t = 2.40\,\mathrm{s}$ and $t = 4.40\,\mathrm{s}$ the magnitude of the angular acceleration is $\Delta\omega / \Delta t = 20\,\mathrm{rad \cdot s^{-2}}$. During these two time intervals, the only torque is the friction torque, assumed constant, and doubling the net moment of inertia halves the angular acceleration.
We then have $\left|\vec{\tau}_f\right| = I\left(40\,\mathrm{rad \cdot s^{-2}}\right)$. This is also $\left|\vec{\tau}_f\right| = 2I\left(20\,\mathrm{rad \cdot s^{-2}}\right)$, but that's not part of this problem, just a consistency check.

b) The main point to recognize in this part of the problem is that the washer accelerates from an initial angular velocity of zero to the final angular velocity shown on the graph. The only torque on the washer is the torque that the rotor exerts on the washer, and the magnitude of this torque is

$$\tau_{\text{rotor-washer}} = I\frac{\Delta\omega_{\text{washer}}}{\Delta t} = I\frac{100\,\mathrm{rad \cdot s^{-1}}}{0.50\,\mathrm{s}} = I\left(200\,\mathrm{rad \cdot s^{-2}}\right) \qquad \textbf{(4.1)}$$

c) Let's go the simple way, and say that the angle through which the rotor rotates is the product of the average angular velocity and the time interval,

$$\Delta\theta = \omega_{\text{ave}}\Delta t = \left(160\,\mathrm{rad \cdot s^{-1}}\right)\left(0.50\,\mathrm{s}\right) = 80\,\mathrm{rad}\,, \qquad (4.2)$$

about $50\,\mathrm{revolutions}$.

d) We've done part c) simply, so let's do the same for the washer. Here, the average angular velocity is $50\,\mathrm{rad \cdot s^{-1}}$ over the same time interval, or $25\,\mathrm{rad}$.

Problem 5

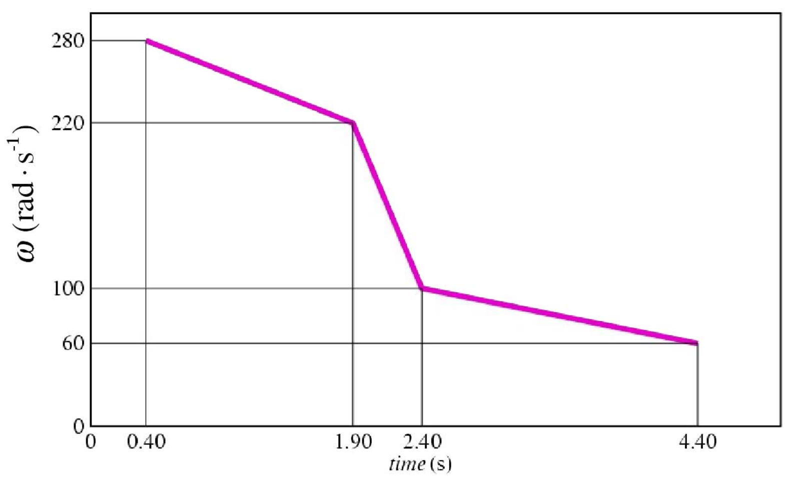
In the angular momentum experiment, shown to the right, a washer is dropped smooth side down onto the spinning rotor.

The graph below shows the rotor angular velocity $\omega\left(\mathrm{rad}\cdot\mathrm{s}^{-1}\right)$ as a function of time.

Assume the following:

- The rotor and washer have the same moment of inertia I.
- The friction torque $\vec{\tau}_f$ on the rotor is constant during the measurement.

Note: express all of your answers in terms of I and numbers you obtain from the graph. Be sure to give an analytic expression prior to substituting the numbers from the graph.



a. Find an expression for the magnitude $\left|\vec{\tau}_f\right|$ in terms of I and numbers you obtain from the graph.

b. How much mechanical energy is lost to bearing friction during the collision (between $t = 1.90\ \mathrm{s}$ and $t = 2.40\ \mathrm{s}$)?

c. How much mechanical energy is lost to friction *between the rotor and the washer* during the collision (between $t = 1.90\ \mathrm{s}$ and $t = 2.40\ \mathrm{s}$)?

Problem 5 Solutions:

a) First, make sure that the problem makes sense. Between times $t = 0.40\,\mathrm{s}$ and $t = 1.90\,\mathrm{s}$, the magnitude of the angular acceleration is $\Delta\omega/\Delta t = 40\,\mathrm{rad\cdot s^{-2}}$ and between times $t = 2.40\,\mathrm{s}$ and $t = 4.40\,\mathrm{s}$ the magnitude of the angular acceleration is $\Delta\omega/\Delta t = 20\,\mathrm{rad\cdot s^{-2}}$. During these two time intervals, the only torque is the friction torque, assumed constant, and doubling the net moment of inertia halves the angular acceleration.
We then have $\left|\vec{\tau}_f\right| = I\left(40\,\mathrm{rad\cdot s^{-2}}\right)$. This is also $\left|\vec{\tau}_f\right| = 2I\left(20\,\mathrm{rad\cdot s^{-2}}\right)$, but that's not part of this problem, just a consistency check.

For parts (b) and (c), denote $\omega_{\text{initial}} = 220\,\mathrm{rad\cdot s^{-1}}$, $\omega_{\text{final}} = 100\,\mathrm{rad\cdot s^{-1}}$, so that

$$K_{\text{initial}} = \frac{1}{2}I\omega_{\text{initial}}^2 = I\left(24{,}200\,\mathrm{rad^2\cdot s^{-2}}\right)$$
$$K_{\text{final}} = \frac{1}{2}(2I)\omega_{\text{final}}^2 = I\left(10{,}000\,\mathrm{rad^2\cdot s^{-2}}\right).$$

b) The mechanical energy lost due to the bearing friction is the product of the magnitude of the frictional torque and the total angle $\Delta\theta$ through which the bearing has turned during the collision. A quick way to calculate $\Delta\theta$ is to use

$$\Delta\theta = \omega_{\text{ave}}\Delta t = \left(160\,\mathrm{rad\cdot s^{-1}}\right)\left(0.50\,\mathrm{s}\right) = 80\,\mathrm{rad}\,,$$

so $-\Delta E_{\text{bearing}} = \left|\vec{\tau}_f\right|\Delta\theta = I\left(3200\,\mathrm{rad^2}\right)$.

c) The energy lost due to friction between the rotor and the washer is then

$$-\Delta K + -\Delta E_{\text{bearing}} = K_{\text{initial}} - K_{\text{final}} - I\left(3200\,\mathrm{rad^2}\right) = I\left(11{,}000\,\mathrm{rad^2}\right).$$

Problem 6: Work Done by Frictional Torque

A steel washer is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is I_0. The washer is set into motion. When it reaches an initial angular velocity ω_0, at $t = 0$, the power to the motor is shut off, and the washer slows down during a time interval $\Delta t_1 = t_a$ until it reaches an angular velocity of ω_a at time t_a. At that instant, a second steel washer with a moment of inertia I_{w} is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time $\Delta t_{\mathrm{col}} = t_b - t_a$ after which the two washers and rotor rotate with angular speed ω_b. Assume the frictional torque on the axle is independent of speed, and remains the same when the second washer is dropped.

a) What angle does the rotor rotate through during the collision?

b) What is the work done by the friction torque τ_f from the bearings during the collision?

Problem 6 Solutions:

During the collision, the component of the average angular acceleration of the rotor is given by

$$\alpha_{\mathrm{rotor}} = \frac{\omega_b - \omega_a}{\Delta t_{\mathrm{int}}} < 0 .$$

The angle the rotor rotates through during the collision is (analogous to linear motion with constant acceleration)

$$\Delta\theta_{\mathrm{rotor}} = \omega_a \Delta t_{\mathrm{int}} + \frac{1}{2}\alpha_{\mathrm{rotor}} \Delta t_{\mathrm{int}}{}^2 = \omega_a \Delta t_{\mathrm{int}} + \frac{1}{2}\left(\frac{\omega_b - \omega_a}{\Delta t_{\mathrm{int}}}\right)\Delta t_{\mathrm{int}}{}^2 = \frac{1}{2}(\omega_b + \omega_2)\Delta t_{\mathrm{int}} .$$

The non-conservative work done by the bearing friction during the collision is

$$W_{nc,b} = \tau_f \Delta\theta_{rotor} = \tau_f \frac{1}{2}(\omega_a + \omega_b)\Delta t_{\mathrm{int}} .$$

The frictional torque is given by

$$\tau_f = I_0\left(\frac{\omega_0 - \omega_a}{\Delta t_1}\right) < 0.$$

So the work done by the bearing friction during the collision is

$$W_{nc,b} = \frac{1}{2} I_0 \left(\frac{\omega_0 - \omega_a}{\Delta t_1} \right) (\omega_a + \omega_b) \Delta t_{\text{int}} < 0 \,.$$

Problem 7: Experiment Angular Momentum: ***Angular Collision***

A steel washer is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is I_0. The washer is set into motion. When it reaches an initial angular velocity ω_0, at $t = 0$, the power to the motor is shut off, and the washer slows down until it reaches an angular velocity of ω_a at time t_a. At that instant, a second steel washer with a moment of inertia I_w is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time $\Delta t_{col} = t_b - t_a$. Assume the frictional torque on the axle is independent of speed, and remains the same when the second washer is dropped. The two washers continue to slow down during the time interval $\Delta t_2 = t_f - t_b$ until they stop at time $t = t_f$. Your answers should be in terms of all or any of I_0, I_w, ω_0, ω_a, Δt_1, Δt_{col}, and Δt_2 (these would be measured or observed values).

a) What is the angular deceleration α_1 while the washer and motor are slowing down during the interval $\Delta t_1 = t_a$?

b) What is the angular impulse during the collision?

c) What is the angular velocity ω_b of the two washers immediately after the collision is finished (when the washers rotate together)?

d) What is the angular deceleration α_2 after the collision?

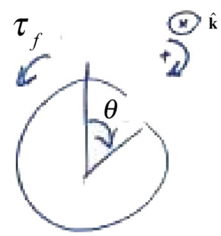
Problem 7 Solutions:

a) The angular acceleration of the motor and washer from the instant when the power is shut off until the second washer was dropped is given by

$$\alpha_1 = \frac{\omega_a - \omega_0}{\Delta t_1}. \tag{7.1}$$

b) The angular acceleration found in part a) is due to the frictional torque in the motor,

$$\tau_{friction} = I_0 \alpha_1 = \frac{I_0(\omega_a - \omega_0)}{\Delta t_1}. \tag{7.2}$$



Note that because $\omega_0 > \omega_a$ therefore $\alpha_1 < 0$, hence $\tau_{\text{friction}} = I_0\alpha_1$ is negative, as indicated in the above figure.

During the collision with the second washer, the frictional torque exerts an angular impulse (pointing along the z-axis in the figure),

$$J_z = \int_{t_a}^{t_b} \tau_{\text{friction}}\, dt = \tau_{\text{friction}}\, \Delta t_{\text{col}} = I_0(\omega_a - \omega_0)\frac{\Delta t_{\text{col}}}{\Delta t_1}. \qquad \textbf{(7.3)}$$

c) The z-component of the angular momentum about the rotation axis of the motor changes during the collision,

$$\Delta L_z = L_{f,z} - L_{0,z} = (I_0 + I_{\text{w}})\omega_b - (I_0)\omega_a. \qquad \textbf{(7.4)}$$

The change in the z-component of the angular momentum is equal to the z-component of the angular impulse

$$J_z = \Delta L_z. \qquad \textbf{(7.5)}$$

Thus,

$$I_0(\omega_a - \omega_0)\left(\frac{\Delta t_{\text{col}}}{\Delta t_1}\right) = (I_0 + I_{\text{w}})\omega_b - (I_0)\omega_a. \qquad \textbf{(7.6)}$$

Solving the equation for the angular velocity immediately after the collision,

$$\omega_b = \frac{I_0}{(I_0 + I_{\text{w}})}\left(\left(\frac{\Delta t_{\text{col}}}{\Delta t_1}\right)(\omega_a - \omega_0) + \omega_a\right). \qquad \textbf{(7.7)}$$

If there were no frictional torque, then the first term in the brackets would vanish ($\omega_a = \omega_0$), and the second term of the equation would be the only contribution to the final angular speed.

d) The final angular acceleration α_2 is given by

$$\alpha_2 = -\frac{\omega_b}{\Delta t_2} = -\frac{1}{\Delta t_2}\frac{I_0}{(I_0 + I_\mathrm{w})}\left(\left(\frac{\Delta t_\mathrm{col}}{\Delta t_1}\right)(\omega_a - \omega_0) + \omega_a\right).$$