Central Force Motion
Challenge Problems

Problem 1: Elliptic Orbit

A satellite of mass $m_s$ is in an elliptical orbit around a planet of mass $m_p$ which is located at one focus of the ellipse. The satellite has a velocity $v_a$ at the distance $r_a$ when it is furthest from the planet.

a) What is the magnitude of the velocity $v_p$ of the satellite when it is closest to the planet? Express your answer in terms of $m_s$, $m_p$, $G$, $v_a$, and $r_a$ as needed.

b) What is the distance of nearest approach $r_p$?

c) If the satellite were in a circular orbit of radius $r_0 = r_p$, is it’s velocity $v_0$ greater than, equal to, or less than the velocity $v_p$ of the original elliptic orbit? Justify your answer.

Problem 1 Solution:

The angular momentum about the origin is constant therefore

$$L \equiv L_p = L_a.$$  \hspace{1cm} (28.1.1)

Because $m_s << m_p$, the reduced mass $\mu \equiv m_s$ and so the angular momentum condition becomes

$$L = m_s r_p v_p = m_s r_a v_a.$$ \hspace{1cm} (28.1.2)

Thus

$$r_p = r_a v_a / v_p.$$ \hspace{1cm} (28.1.3)

Note: We can solve for $v_p$ in terms of the constants $G$, $m_p$, $r_a$ and $v_a$ as follows. Choose zero for the gravitational potential energy for the case where the satellite and planet are
separated by an infinite distance, \( U(r = \infty) = 0 \). The gravitational force is conservative, so the energy at closest approach is equal to the energy at the furthest distance from the planet, hence

\[
\frac{1}{2} m_s v_a^2 - \frac{G m_p m_s}{r_a} = \frac{1}{2} m_s v_p^2 - \frac{G m_p m_s}{r_p}.
\]  
(28.1.4)

Substituting Eq. (28.1.3) into Eq. (28.1.4), yields

\[
\frac{1}{2} m_s v_a^2 - \frac{G m_p m_s}{r_a} = \frac{1}{2} m_s v_p^2 - \frac{G m_p m_s v_p}{r_a v_a}.
\]  
(28.1.5)

After a little algebraic manipulation, Eq. (28.1.5) becomes

\[
\frac{2G m_p}{r_a v_a} (v_p - v_a) = (v_p + v_a)(v_p - v_a).
\]  
(28.1.6)

We can solve this equation for \( v_p \)

\[
v_p = \frac{2G m_p}{r_a v_a} - v_a.
\]  
(28.1.7)

b) We can use Eq. (28.1.3) to find the distance of nearest approach,

\[
r_p = \frac{r_a v_a}{v_p} = r_a \frac{1}{\left(\frac{2G m_p}{r_a v_a} - 1\right)}.
\]  
(28.1.8)

c) The speed of the satellite undergoing uniform circular motion can be found from the force equation,

\[
- \frac{G m_p m_s}{r_0^2} = \frac{m_s v_0^2}{r_0}.
\]  
(28.1.9)

So the speed is

\[
v_0 = \sqrt{\frac{G m_p}{r_0}}.
\]  
(28.1.10)
Note that if we assume our original orbit was circular and set $v_a = v_0$ and $r_a = r_0$ in Eq. (28.1.8) then the distance of closest approach becomes

\[ r_p = r_0 \left( \frac{1}{2Gm_p \left( \frac{p}{r_0v_0^2} - 1 \right)} \right) = r_0 \]  

(28.1.11)

providing a second check on our algebra.

If we substitute $Gm_p = r_0v_0^2$ into Eq. (28.1.7), we have that

\[ v_p = \frac{2r_0v_0^2}{r_a v_a} - v_a. \]  

(28.1.12)

Let’s compare our elliptic orbit to a circular orbit with $r_p = r_0$. How does $v_p$ compare to $v_0$? If we substitute $r_p = r_0$ into Eq. (28.1.12) we have that

\[ v_p = \frac{2r_0v_0^2}{r_a v_a} - v_a. \]  

(28.1.13)

We now use the angular momentum condition that $r_a v_a = r_p v_p$ to rewrite Eq. (28.1.13) as

\[ v_p = \frac{2v_0^2}{v_p} - v_a. \]  

(28.1.14)

Thus

\[ v_p^2 + v_p v_a = 2v_0^2. \]  

(28.1.15)

We know that $v_a < v_p$, so $v_p^2 + v_p v_a < 2v_p^2$. Thus $2v_0^2 < 2v_p^2$ or

\[ v_0 < v_p. \]  

(28.1.16)

So the circular orbit with $r_p = r_0$ has speed $v_0$ less than the speed $v_p$ of closest approach for the elliptic orbit. This is not surprising because suppose we give the satellite that is in a circular motion a small increase in velocity in the tangential direction, $v_{new} = v_0 + \Delta v$. This implies that the energy increases. The satellite will no longer travel in a circular orbit since for the same radius $r_0$, 


The satellite will move away from the planet entering into an elliptical orbit. So any velocity \( v_p \) greater than \( v_0 \) will form an elliptic orbit.

\[
\frac{m v_{new}^2}{r_0} > \frac{G m_s m_p}{r_0^2}.
\] (28.1.17)
Problem 2: \textit{Planetary Orbits}

Comet Encke was discovered in 1786 by Pierre Mechain and in 1822 Johann Encke determined that its period was 3.3 years. It was photographed in 1913 at the aphelion distance, \( r_a = 6.1 \times 10^{11} \text{m} \), (furthest distance from the sun) by the telescope at Mt. Wilson. The distance of closest approach to the sun, perihelion, is \( r_p = 5.1 \times 10^{10} \text{m} \). The universal gravitation constant \( G = 6.7 \times 10^{-11} \text{N} \cdot \text{m}^2 \cdot \text{kg}^{-2} \). The mass of the sun is \( m_s = 2.0 \times 10^{30} \text{kg} \).

a) Explain why angular momentum is conserved about the focal point and then write down an equation for the conservation of angular momentum between aphelion and perihelion.

b) Explain why mechanical energy is conserved and then write down an equation for conservation of energy between aphelion and perihelion.

c) Use conservation of energy and angular momentum to find the speeds at perihelion and aphelion.

\textbf{Problem 2 Solutions}

Since the only forces are gravitational and point toward the focal point

\[ \vec{\tau}_{\text{focal}} = \vec{r}_{\text{focal,comet}} \times \vec{F}_{\text{grav}} = 0 \]

Therefore, angular momentum is conserved. At perihelion, since \( v_p \) is tangent to the orbit

\[ L_p = \mu r_p v_p \]

At apohelion:

\[ L_a = \mu r_a v_a \]
\[ L_a = L_p \Rightarrow \mu r_a v_a = \mu r_p v_p \Rightarrow \]
\[ r_a v_a = r_p v_p \]

The gravitational force is conservative so mechanical energy is conserved.

\[ E_a = \frac{1}{2} \mu v_a^2 = \frac{G m_1 m_2}{r_a}, \quad E_p = \frac{1}{2} \mu v_p^2 = \frac{G m_1 m_2}{r_p} \]

thus

\[ E_a = E_p \Rightarrow \]
\[ \frac{1}{2} \mu v_a^2 - \frac{G m_1 m_2}{r_a} = \frac{1}{2} \mu v_p^2 - \frac{G m_1 m_2}{r_p} \]

Since the mass \( m_1 \) of the comet is much less than the mass of the sun \( m_2 \).

\[ \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{m_2} = m_1 \]

the energy conservation equation becomes:

\[ \frac{1}{2} m_1 v_a^2 - \frac{G m_1 m_2}{r_a} = \frac{1}{2} m_1 v_p^2 - \frac{G m_1 m_2}{r_p} \quad \text{or} \quad \frac{1}{2} v_a^2 = \frac{G m_2}{r_a} = \frac{1}{2} v_p^2 = \frac{G m_2}{r_p} \]

The condition \( v_p = \frac{r_a v_a}{r_p} \) from conservation of angular momentum can now be used in the energy equation.

\[ \frac{1}{2} v_a^2 - \frac{G m_2}{r_a} = \frac{1}{2} \left( \frac{r_a v_a}{r_p} \right)^2 - \frac{G m_2}{r_p} \]

Solve for \( v_a \):

\[ \frac{1}{2} \left( \frac{r_a}{r_p} \right)^2 \left( 1 - 1 \right) v_a^2 = G m_2 \left( \frac{1}{r_p} - \frac{1}{r_a} \right) \]
\[
\begin{align*}
    \nu_a &= \left( \frac{2Gm_2 \left( \frac{1}{r_p} - \frac{1}{r_a} \right)}{\left( \frac{r_a}{r_p} \right)^2 - 1} \right)^{1/2} \\
    \nu_a &= \left( \frac{2Gm_2 r_p}{r_a (r_a + r_p)} \right)^{1/2}
\end{align*}
\]

\[
\nu_a = \left( \frac{2 \left( 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2} \right) \left( 2.0 \times 10^{30} \text{ kg} \right) \left( 5.1 \times 10^{10} \text{ m} \right)}{(6.1 \times 10^{11} \text{ m}) \left( (6.1 \times 10^{11} \text{ m} + 5.1 \times 10^{10} \text{ m}) \right)} \right)^{1/2} = 5.8 \times 10^3 \text{ m} \cdot \text{s}^{-1}
\]

The velocity at perihelion is then

\[
\nu_p = r_p \nu_a = \left( \frac{6.1 \times 10^{11} \text{ m}}{5.1 \times 10^{10} \text{ m}} \right) (5.8 \times 10^3 \text{ m} \cdot \text{s}^{-1}) = 6.9 \times 10^4 \text{ m} \cdot \text{s}^{-1}
\]
Problem 3: Satellite Motion

A spherical non-rotating planet (with no atmosphere) has mass \( m_1 \) and radius \( r_1 \). A projectile of mass \( m_2 \ll m_1 \) is fired from the surface of the planet at a point A with a speed \( v_A \) at an angle \( \alpha = 30^\circ \) with respect to the radial direction. In its subsequent trajectory the projectile reaches a maximum altitude at point B on the sketch. The distance from the center of the planet to the point B is \( r_2 = (5/2) r_1 \).

In this problem you will find the initial speed \( v_A \) in terms of \( G, m_1 \) and \( r_1 \).

a) Is there a point about which the angular momentum of the projectile is constant? If so, use this point to determine a relation between the speed \( v_B \) of the projectile at the point B in terms of \( v_A \) and the angle \( \alpha = 30^\circ \).

b) Now use conservation of mechanical energy constant to find an expression to find \( v_A \) in terms of \( G, m_1 \) and \( r_1 \).

Problem 3 Solutions:

a) The only force of interest is the gravitational force, which is always directed toward the center of the planet; hence angular momentum about the center of the planet is a constant.

At point A, the component \( p_{\perp} \) of the satellite’s linear momentum perpendicular to the radius vector is

\[
p_{\perp} = m_2 v_A \sin \alpha = \frac{m_2 v_A}{2}, \tag{28.3.1}
\]
Using \( \sin 30^\circ = 1/2 \). The magnitude of the angular momentum about the center of the planet is then

\[
L_A = r_1 p_{\perp} = \frac{r_1 m_2 v_A}{2}.
\]

At point B (the \textit{apogee}), the velocity vector is perpendicular to the radius vector and the magnitude of the angular momentum is the product of the distance from the center of the planet and the speed,

\[
L_B = r_2 v_B = \frac{5}{2} r_1 v_B.
\]

There is no torque on the satellite, so \( L_B = L_A \); so equating the expressions in Equations (28.3.3) and (28.3.2) yields

\[
\frac{5}{2} r_1 v_B = \frac{r_1 m_2 v_A}{2} \Rightarrow v_B = \frac{v_A}{5}.
\]

b) The mechanical energy \( E \) of the satellite as a function of speed \( v \) and radius (distance from the center of the planet) \( r \) is

\[
E = \frac{1}{2} m_2 v^2 - G \frac{m_1 m_2}{r}.
\]

Equating the energies at points A and B, and using \( r_B = r_2 = (5/2)r_1 \), \( r_A = r_1 \) and \( v_B = v_A / 5 \) from part a) (Equation (28.3.4) above),
\[
\frac{1}{2} m_2 v_A^2 - G \frac{m_1 m_2}{r_A} = \frac{1}{2} m_2 v_B^2 - G \frac{m_1 m_2}{r_B}
\]
\[
\frac{1}{2} v_A^2 - G \frac{m_1}{r} = \frac{1}{2} \left( \frac{v_A}{5} \right)^2 - G \frac{m_1}{5r_1 / 2}
\]
\[
\frac{1}{2} v_A^2 \left( 1 - \frac{1}{25} \right) = G \frac{m_1}{r_1} \left( 1 - \frac{2}{5} \right) \quad . \tag{28.3.6}
\]
\[
v_A^2 = \frac{5}{4} G \frac{m_1}{r_1}
\]
\[
v_A = \sqrt{\frac{5}{4} G \frac{m_1}{r_1}}
\]

It’s worth noting that \( v_A \) is less than the escape velocity

\[
v_{\text{escape}} = \sqrt{\frac{2G m_1}{r_1}} \tag{28.3.7}
\]

of the planet, but not by much;

\[
\frac{v_A}{v_{\text{escape}}} = \frac{\sqrt{5}}{8} \approx 0.79 \quad . \tag{28.3.8}
\]

\[
\frac{1}{2} m_2 v_A^2 - G \frac{m_1 m_2}{r_A} = \frac{1}{2} m_2 v_B^2 - G \frac{m_1 m_2}{r_B}
\]
\[
\frac{1}{2} v_A^2 - G \frac{m_1}{r} = \frac{1}{2} \left( \frac{v_A}{5} \right)^2 - G \frac{m_1}{5r_1 / 2}
\]
\[
\frac{1}{2} v_A^2 \left( 1 - \frac{1}{25} \right) = G \frac{m_1}{r_1} \left( 1 - \frac{2}{5} \right) \quad . \tag{28.3.9}
\]
\[
v_A^2 = \frac{5}{4} G \frac{m_1}{r_1}
\]
\[
v_A = \sqrt{\frac{5}{4} G \frac{m_1}{r_1}}
\]

It’s worth noting that \( v_A \) is less than the escape velocity
\[ v_{\text{escape}} = \sqrt{\frac{2 G m}{r_i}} \] \hspace{1cm} (28.3.10)

of the planet, but not by much;

\[ \frac{v_A}{v_{\text{escape}}} = \sqrt{\frac{5}{8}} \approx 0.79. \] \hspace{1cm} (28.3.11)
Problem 4: Inverse Square Central Force: *Lowest Energy Solution*

The effective potential energy for an inverse-square restoring central force \( \mathbf{F}_{1,2} = -\frac{Gm_1m_2}{r^2} \mathbf{r} \) is given by

\[
U_{\text{effective}} = \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r}
\]

Make a graph of the effective potential energy \( U_{\text{effective}}(r) \) as a function of the relative separation \( r \). Find the radius and the energy for the motion with the lowest energy. What type of motion does this correspond to?

**Problem 4 Solution:**

For plotting purposes, the horizontal scale is the ratio \( \frac{r}{r_0} \) and the vertical scale is in units of \( U_{\text{effective}} / U_{\text{effective}}(r_0) \), as found in the second part of the problem.

![Graph](image_url)

The upper (red) curve is proportional to \( 1/r^2 \) and the lower (blue) curve is proportional to \(-1/r\). The sum is the solid (green) curve in between.

To find the minimum energy, differentiate the effective potential \( U_{\text{effective}}(r) \) with respect to the radius \( r \) and set the derivative equal to zero at \( r_0 \).
\[-\frac{L^2}{\mu} \frac{1}{r_0^3} + \frac{G m_2}{r_0^2} = 0\]

\[r_0 = \frac{L^2}{\mu G m_2}.\]

The energy at this minimum value is

\[U_{\text{effective}}(r_0) = \frac{L^2}{2\mu} \left( \frac{\mu G m_2}{L^2} \right)^2 - G m_2 \frac{\mu G m_2}{L^3} \]

\[= -\frac{\mu (G m_2)^2}{2L^2}.\]

The radius of this orbit does not change; the orbit is a circle.
Problem 5

A particle of mass $m$ moves under an attractive central force of magnitude $F = br^3$. The angular momentum is equal to $L$.

a) Find the effective potential energy and make sketch of effective potential energy as a function of $r$.

b) Indicate on a sketch of the effective potential the total energy for circular motion.

c) The radius of the particle’s orbit varies between $r_0$ and $2r_0$. Find $r_0$.

Problem 5 Solutions:

a) The potential energy is, taking the zero of potential energy to be at $r = 0$, is

$$U(r) = -\int_0^r (-br'^3)dr' = \frac{b}{4}r^4$$

and the effective potential is

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} + U(r) = \frac{L^2}{2mr^2} + \frac{b}{4}r^4.$$  

A plot is shown below, including the potential (yellow if seen in color), the term $L^2/2m$ (green) and the effective potential (blue). The minimum effective potential energy is the horizontal line (red). The horizontal scale is in units of the radius of the circular orbit and the vertical scale is in units of the minimum effective potential.

b) See the solution to part (a) above and the plot to the left below.
c) In the left plot, if we could move the red line up until it intersects the blue curve at two points whose value of the radius differ by a factor of 2, those would be the respective values of $r_0$ and $2r_0$. A graph of this construction (done by computer, of course), showing the corresponding energy as the horizontal magenta is at the right above, and is not part of this problem.

To do this algebraically, we find the value of $r_0$ such that $U_{\text{eff}}(r_0) = U_{\text{eff}}(2r_0)$. This is

$$\frac{L^2}{m r_0^2} + b r_0^4 = \frac{L^2}{m (2r_0)^2} + b (2r_0)^4.$$ 

Rearranging and combining terms, and then solving for $r_0$,

$$\frac{3 L^2}{8 m r_0^2} + \frac{1}{4} b r_0^4 = \frac{15}{4} b r_0^4$$

Thus, $r_0 = \left( \frac{1}{\sqrt{10}} \right)_{\text{circular}}$ (not part of the problem), consistent with the auxiliary figure on the right above.
Problem 6:

The effective potential corresponding to a pair of particles interacting through a central force is given by the expression

\[ U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + Cr^3 \]  \hfill (30.1)

where \( L \) is the angular momentum, \( \mu \) is the reduced mass and \( C \) is a constant. The total energy of the system is \( E \). The relationship between \( U_{\text{eff}}(r) \) and \( E \) is shown in the figure, along with an indication of the associated maximum and minimum values of \( r \) and the minimum allowed energy \( E_{\text{min}} \). In what follows, assume that the center of mass of the two particles is at rest.

a) Find an expression for the radial component \( f(r) \) of the force between the two particles. Is the force attractive or repulsive?

b) What is the radius \( r_0 \) of the circular orbit allowed in this potential? Express your answer as some combination of \( L \), \( C \), and \( \mu \).

c) When \( E \) has a value larger than \( E_{\text{min}} \), find how rapidly the separation between the particles is changing, \( \frac{dr}{dt} \), as the system passes through the point in the orbit where \( r = r_0 \). Give your answer in terms of some combination of \( E \), \( E_{\text{min}} \), \( L \), \( C \), \( \mu \) and \( r_0 \).

d) Does the relative motion between the particles stop when \( r = r_{\text{max}} \)? If not, what is the total kinetic energy at that point in terms of some combination of \( E \), \( L \), \( C \), \( \mu \), \( r_{\text{max}} \) and \( r_{\text{min}} \)?
Problem 6 Solutions:

a) The effective potential is given by

\[ U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + U(r), \]  

and \( f(r) = -\frac{dU}{dr} \) for a central force, and so

\[ f(r) = -\frac{d}{dr} U(r) = -\frac{d}{dr} Cr^\gamma = -3Cr^2. \]  

From the figure, \( C > 0 \), so \( f(r) < 0 \), a restoring force.

b) The circular orbit will correspond to the minimum effective potential; at this radius the kinetic energy will have no contribution from any radial motion. This minimum effective potential, and hence the radius of the circular orbit, is found from basic calculus and algebra,

\[ \left[ \frac{d}{dr} U_{\text{eff}}(r) \right]_{r=r_0} = -\frac{L^2}{\mu r_0^3} + 3Cr_0^2 \]

\[ r_0^5 = \frac{L^2}{3\mu C}, \quad r_0 = \left( \frac{L^2}{3\mu C} \right)^{1/5}. \]  

c) Recall that the kinetic energy is

\[ K = \frac{L^2}{2\mu r^2} + \frac{1}{2} \mu \left( \frac{dr}{dt} \right)^2. \]

The difference \( E - E_{\text{min}} \) is then found by evaluating \( U_{\text{eff}} \) at \( r = r_0 \),

\[ E - E_{\text{min}} = \frac{1}{2} \mu \left( \frac{dr}{dt} \right)^2, \]

or \( |dr/dt| = \sqrt{(2/\mu)(E - E_{\text{min}})} \).

d) No; \( dr/dt = 0 \), but the kinetic energy, from Equation (28.6.5), is
\[ K_{\text{min}} = \frac{L^2}{2\mu r_{\text{max}}^2}. \] (28.6.7)
Problem 7: Determining the Mass of a Neutron Star

A binary system known as 4U0900-40 consists of a “neutron star” and a normal “optical star.” You are given two graphs of actual data obtained from observations of this system. The top graph shows the time delays (in seconds) of X-ray pulses detected from the neutron star as a function of time (in days) throughout its orbit. These delays indicate the time of flight for an X-ray pulse (traveling with the speed of light) to cross the orbit of the binary in its trip to the earth. (Ignore the heavy dots scattered about the “x”-axis.)

The bottom graph displays the velocity component of the optical star toward (or away from) the Earth as a function of its orbital phase. The velocities (in km per second) are determined from Doppler shifts in its spectral lines. Time on this plot is given in units of orbital phase, where the time between phases 0.0 and 1.0 corresponds to one orbital period.

Assume that the orbit of 4U0900-40 is circular and that we are viewing the system edge on, i.e., the Earth lies in the plane of the binary orbit. In each graph, the solid curve is a computed fit to the individual data points.

You are to find from the data the orbital period, $T$, in days. 

For this problem, use $c = 3.0 \times 10^8 \text{ m} \cdot \text{s}^{-1}$ for the speed of light and $G = 6.7 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$ for the Universal Gravitation Constant.

Suggested procedure (you will need a ruler for the first three parts):

a. Determine the orbital period, $T$, in days.

b. Estimate the velocity, $v_0$, of the optical star in its orbit around the center of mass.

c. Use $T$ and $v_0$ to find the radius of the orbit of the optical star, $a_o$, around the center of mass.

d. Estimate the size of the orbit, $a_N$, of the neutron star around the center of mass. First express your answer in light-seconds, then in meters.

e. Find the ratio of the mass of the neutron star, $m_N$, to the mass of the optical star, $m_O$.
[Hint: Recall that in any binary $m_1 r_1 = m_2 r_2$.]

f. Use Kepler’s law to find the total mass of the binary system.

g. Find $m_N$ and $m_O$; express your answers in units of the mass of the Sun ($2 \times 10^{30} \text{ kg}$).
Doppler Curves for 4U0900-40.
Problem 7 Solutions:

a. Between the first and third zero-crossings on the upper graph, the distance is 8.98 cm. The scale of the time axis is 1.00 cm to 1 day, so the orbital period, $P$, is 8.98 days.

b. The Doppler shift velocities range from a maximum of 16.0 km·s$^{-1}$ to a minimum of -30.8 km·s$^{-1}$. (The velocities are not symmetric around zero because the binary system as a whole is moving relative to the solar system.) The optical star’s orbital velocity is just half the difference between the maximum and minimum Doppler velocities, or $v_0 = 23.4$ km·s$^{-1}$.

c. In one period $P$, moving at velocity $v_0$, the optical star travels a distance $v_0P = 2\pi a_0$. So the orbital radius, $a_0$ is just

$$a_0 = \frac{1}{2\pi} \left[(9.98)(24)(60)(60) s\right](23.4 km·s^{-1}) = 2.89 \times 10^6 km$$

d. The smallest time delay is -117 seconds, while the largest is 111 seconds. The radius of the neutron star’s orbit (in light seconds) is half the difference between these, or $a_N = 114$ lt·sec. Since the speed of light is $2.998 \times 10^5$ cm·s$^{-1}$, the distance $a_N$ is equal to $3.42 \times 10^{12}$ cm.

e. The ratio $M_N/M_0$ is just equal to $a_0/a_N$. That is

$$\frac{M_N}{M_0} = \frac{2.89 \times 10^{11} cm}{3.42 \times 10^{13} cm} = 8.45 \times 10^{-2}$$

f. The semimajor axis of the absolute orbit is $a = a_N + a_0 = 3.71 \times 10^{12}$ cm and the orbital period is $7.76 \times 10^5$s. Inserting these values into Newton’s form of Kepler’s Third Law.

$$\frac{p^2}{(2\pi)^2} = \frac{a^3}{G(M_N + M_0)}$$

we have,

$$M_N + M_0 + 4\pi^2 \left(\frac{3.71 \times 10^{12}}{7.76 \times 10^5}\right)^3 \left(6.67 \times 10^{-8}\right) = 5.02 \times 10^{34} g$$

g. We use the fact that $M_0\left(1 + \frac{M_N}{M_0}\right) = M_0 + M_N$, and both $M_N/M_0$ and $(M_0 + M_N)$ are known. Then we have

$$M_0 = \frac{5.02 \times 10^{34} g}{1 + 8.45 \times 10^{-2}} = 4.63 \times 10^{34} g$$

The remaining mass in the mass of the neutron star, $M_N = 3.9 \times 10^{33}$ g.

The mass of the sun is $1.99 \times 10^{33}$ g, so the masses of the optical star and its neutron star companion are 23.3 and 2.0 solar masses, respectively.