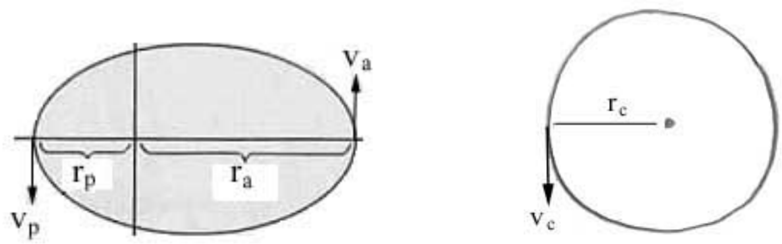
Central Force Motion
Challenge Problems

Problem 1: Elliptic Orbit

A satellite of mass m_s is in an elliptical orbit around a planet of mass m_p which is located at one focus of the ellipse. The satellite has a velocity v_a at the distance r_a when it is furthest from the planet.



a) What is the magnitude of the velocity v_p of the satellite when it is closest to the planet? Express your answer in terms of m_s, m_p, G, v_a, and r_a as needed.

b) What is the distance of nearest approach r_p?

c) If the satellite were in a circular orbit of radius $r_0 = r_p$, is it's velocity v_0 greater than, equal to, or less than the velocity v_p of the original elliptic orbit? Justify your answer.

Problem 1 Solution:

The angular momentum about the origin is constant therefore

$$L \equiv L_p = L_a. \qquad \textbf{(28.1.1)}$$

Because $m_s << m_p$, the reduced mass $\mu \cong m_s$ and so the angular momentum condition becomes

$$L = m_s r_p v_p = m_s r_a v_a \qquad \textbf{(28.1.2)}$$

Thus

$$r_p = r_a v_a / v_p. \qquad \textbf{(28.1.3)}$$

Note: We can solve for v_p in terms of the constants G, m_p, r_a and v_a as follows. Choose zero for the gravitational potential energy for the case where the satellite and planet are separated by an infinite distance, $U(r=\infty)=0$. The gravitational force is conservative, so the energy at closest approach is equal to the energy at the furthest distance from the planet, hence

$$\frac{1}{2}m_s v_a^{\,2} - \frac{Gm_s m_p}{r_a} = \frac{1}{2}m_s v_p^{\,2} - \frac{Gm_s m_p}{r_p}. \qquad \textbf{(28.1.4)}$$

Substituting Eq. (28.1.3) into Eq. (28.1.4), yields

$$\frac{1}{2}m_s v_a^{\,2} - \frac{Gm_s m_p}{r_a} = \frac{1}{2}m_s v_p^{\,2} - \frac{Gm_s m_p v_p}{r_a v_a}. \qquad \textbf{(28.1.5)}$$

After a little algebraic manipulation, Eq. (28.1.5) becomes

$$\frac{2Gm_p}{r_a v_a}(v_p - v_a) = (v_p + v_a)(v_p - v_a). \qquad (28.1.6)$$

We can solve this equation for v_p

$$v_p = \frac{2Gm_p}{r_a v_a} - v_a. \qquad \textbf{(28.1.7)}$$

b) We can use Eq. (28.1.3) to find the distance of nearest approach,

$$r_p = \frac{r_a v_a}{v_p} = r_a \frac{1}{\left(\dfrac{2Gm_p}{r_a v_a^{\,2}} - 1\right)}. \qquad \textbf{(28.1.8)}$$

c) The speed of the satellite undergoing uniform circular motion can be found from the force equation,

$$-\frac{Gm_s m_p}{r_0^{\,2}} = -\frac{m_s v_0^{\,2}}{r_0}. \qquad \textbf{(28.1.9)}$$

So the speed is

$$v_0 = \sqrt{Gm_p / r_0}. \qquad \textbf{(28.1.10)}$$

Note that if we assume our original orbit was circular and set $v_a = v_0$ and $r_a = r_0$ in Eq. (28.1.8) then the distance of closest approach becomes

$$r_p = r_0 \frac{1}{\left(\dfrac{2Gm_p}{r_0 v_0^{\,2}} - 1\right)} = r_0 \qquad \textbf{(28.1.11)}$$

providing a second check on our algebra.

If we substitute $Gm_p = r_0 v_0^{\,2}$ into Eq. (28.1.7), we have that

$$v_p = \frac{2r_0 v_0^{\,2}}{r_a v_a} - v_a . \qquad \textbf{(28.1.12)}$$

Let's compare our elliptic orbit to a circular orbit with $r_p = r_0$. How does v_p compare to v_0? If we substitute $r_p = r_0$ into Eq. (28.1.12) we have that

$$v_p = \frac{2r_p v_0^{\,2}}{r_a v_a} - v_a . \qquad \textbf{(28.1.13)}$$

We now use the angular momentum condition that $r_a v_a = r_p v_p$ to rewrite Eq. (28.1.13) as

$$v_p = \frac{2v_0^{\,2}}{v_p} - v_a . \qquad \textbf{(28.1.14)}$$

Thus

$$v_p^{\,2} + v_p v_a = 2v_0^{\,2} . \qquad \textbf{(28.1.15)}$$

We know that $v_a < v_p$, so $v_p^{\,2} + v_p v_a < 2v_p^{\,2}$. Thus $2v_0^{\,2} < 2v_p^{\,2}$ or

$$v_0 < v_p . \qquad \textbf{(28.1.16)}$$

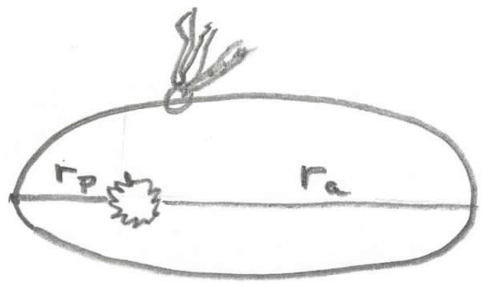
So the circular orbit with $r_p = r_0$ has speed v_0 less than the speed v_p of closest approach for the elliptic orbit. This is not surprising because suppose we give the satellite that is in a circular motion a small increase in velocity in the tangential direction, $v_{new} = v_0 + \Delta v$. This implies that the energy increases. The satellite will no longer travel in a circular orbit since for the same radius r_0 ,

$$\frac{m_s v_{new}{}^2}{r_0} > \frac{G m_s m_p}{r_0{}^2}. \quad \textbf{(28.1.17)}$$

The satellite will move away from the planet entering into an elliptical orbit. So any velocity v_p greater than v_0 will form an elliptic orbit.

Problem 2: ***Planetary Orbits***

Comet Encke was discovered in 1786 by Pierre Mechain and in 1822 Johann Encke determined that its period was 3.3 years. It was photographed in 1913 at the aphelion distance, $r_a = 6.1\times10^{11}\,m$, (furthest distance from the sun) by the telescope at Mt. Wilson. The distance of closest approach to the sun, perihelion, is $r_p = 5.1\times10^{10}\,m$. The universal gravitation constant $G = 6.7\times10^{-11}\,N\cdot m^2\cdot kg^{-2}$. The mass of the sun is $m_s = 2.0\times10^{30}\,kg$.



a) Explain why angular momentum is conserved about the focal point and then write down an equation for the conservation of angular momentum between aphelion and perihelion.

b) Explain why mechanical energy is conserved and then write down an equation for conservation of energy between aphelion and perihelion.

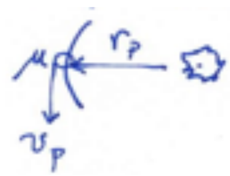
c) Use conservation of energy and angular momentum to find the speeds at perihelion and aphelion.

Problem 2 Solutions

Since the only forces are gravitational and point toward the focal point

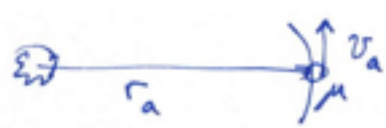
$$\vec{\tau}_{focal} = \vec{r}_{focal,comet} \times \vec{F}_{grav} = 0$$

Therefore, angular momentum is conserved. At perihelion, since v_p is tangent to the orbit



$$L_p = \mu r_p v_p$$

At apohelion:



$$L_a = \mu r_a v_a$$

$$L_a = L_p \Rightarrow \mu r_a v_a = \mu r_p v_p \Rightarrow$$

$$r_a v_a = r_p v_p$$

The gravitational force is conservative so mechanical energy is conserved.

$$E_a = \frac{1}{2}\mu v_a^2 = \frac{Gm_1 m_2}{r_a},\ E_p = \frac{1}{2}\mu v_p^2 = \frac{Gm_1 m_2}{r_p}$$

thus

$$E_a = E_p \Rightarrow$$

$$\frac{1}{2}\mu v_a^2 - \frac{Gm_1 m_2}{r_a} = \frac{1}{2}\mu v_p^2 - \frac{Gm_1 m_2}{r_p}$$

Since the mass m_1 of the comet is much less than the mass of the sun m_2.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \simeq \frac{m_1 m_2}{m_2} = m_1$$

the energy conservation equation becomes:

$$\frac{1}{2}m_1 v_a^2 - \frac{Gm_1 m_2}{r_a} = \frac{1}{2}m_1 v_p^2 - \frac{Gm_1 m_2}{r_p} \quad \text{or} \quad \frac{1}{2}v_a^2 - \frac{Gm_2}{r_a} = \frac{1}{2}v_p^2 - \frac{Gm_2}{r_p}$$

The condition $v_p = \frac{r_a v_a}{r_p}$ from conservation of angular momentum can now be used in the energy equation.

$$\frac{1}{2}v_a^2 - \frac{Gm_2}{r_a} = \frac{1}{2}\left(\frac{r_a v_a}{r_p}\right)^2 - \frac{Gm_2}{r_p}$$

Solve for v_a:

$$\frac{1}{2}\left(\left(\frac{r_a}{r_p}\right)2 - 1\right)v_a^2 = Gm_2\left(\frac{1}{r_p} - \frac{1}{r_a}\right)$$

$$v_a = \left(\frac{2Gm_2\left(\frac{1}{r_p} - \frac{1}{r_a}\right)}{\left(\frac{r_a}{r_p}\right)^2 - 1} \right)^{1/2}$$

$$v_a = \left(\frac{2Gm_2 r_p}{r_a\left(r_a + r_p\right)} \right)^{1/2}$$

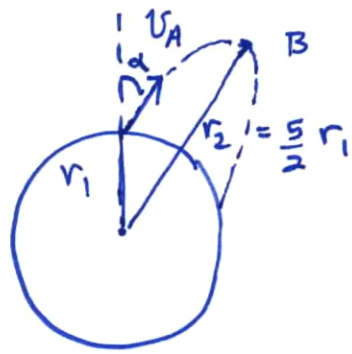
$$v_a = \left(\frac{(2)\left(6.67\times10^{-11}\,N\cdot m^2\cdot kg^{-2}\right)\left(2.0\times10^{30}\,kg\right)\left(5.1\times10^{10}\,m\right)}{\left(6.1\times10^{11}\,m\right)\left(\left(6.1\times10^{11}\,m + 5.1\times10^{10}\,m\right)\right)} \right)^{1/2} = 5.8\times10^3\,m\cdot s^{-1}$$

The velocity at perihelion is then

$$v_p = \frac{r_a v_a}{r_p} = \frac{\left(6.1\times10^{11}\,m\right)}{\left(5.1\times10^{10}\,m\right)}\left(5.8\times10^3\,m\cdot s^{-1}\right) = \boxed{6.9\times10^4\,m\cdot s^{-1}}$$

Problem 3: Satellite Motion

A spherical non-rotating planet (with no atmosphere) has mass m_1 and radius r_1. A projectile of mass $m_2 << m_1$ is fired from the surface of the planet at a point A with a speed v_A at an angle $\alpha = 30^\circ$ with respect to the radial direction. In its subsequent trajectory the projectile reaches a maximum altitude at point B on the sketch. The distance from the center of the planet to the point B is $r_2 = (5/2)r_1$.



In this problem you will find the initial speed v_A in terms of G, m_1 and r_1.

a) Is there a point about which the angular momentum of the projectile is constant? If so, use this point to determine a relation between the speed v_B of the projectile at the point B in terms of v_A and the angle $\alpha = 30^\circ$.

b) Now use conservation of mechanical energy constant to find an expression to find v_A in terms of G, m_1 and r_1.

Problem 3 Solutions:

a) The only force of interest is the gravitational force, which is always directed toward the center of the planet; hence angular momentum about the center of the planet is a constant.

At point A, the component $p_\perp$ of the satellite's linear momentum perpendicular to the radius vector is

$$p_\perp = m_2 v_A \sin\alpha = \frac{m_2 v_A}{2}, \qquad \textbf{(28.3.1)}$$

Using $\sin 30^\circ = 1/2$. The magnitude of the angular momentum about the center of the planet is then

$$L_{\mathrm{A}} = r_1\, p_{\perp} = \frac{r_1\, m_2\, v_{\mathrm{A}}}{2}. \qquad \textbf{(28.3.2)}$$

At point B (the *apogee*), the velocity vector is perpendicular to the radius vector and the magnitude of the angular momentum is the product of the distance from the center of the planet and the speed,

$$L_{\mathrm{B}} = r_2\, v_{\mathrm{B}} = \frac{5}{2} r_1\, v_{\mathrm{B}}. \qquad \textbf{(28.3.3)}$$

There is no torque on the satellite, so $L_{\mathrm{B}} = L_{\mathrm{A}}$; so equating the expressions in Equations (28.3.3) and (28.3.2) yields

$$\begin{aligned} \frac{5}{2} r_1\, v_{\mathrm{B}} &= \frac{r_1\, m_2\, v_{\mathrm{A}}}{2} \\ v_{\mathrm{B}} &= \frac{v_{\mathrm{A}}}{5}. \end{aligned} \qquad \textbf{(28.3.4)}$$

b) The mechanical energy E of the satellite as a function of speed v and radius (distance from the center of the planet) r is

$$E = \frac{1}{2} m_2\, v^2 - G\frac{m_1\, m_2}{r}. \qquad \textbf{(28.3.5)}$$

Equating the energies at points A and B, and using $r_{\mathrm{B}} = r_2 = (5/2) r_1$, $r_{\mathrm{A}} = r_1$ and $v_{\mathrm{B}} = v_{\mathrm{A}}/5$ from part a) (Equation (28.3.4) above),

$$\begin{aligned}
\frac{1}{2} m_2 v_{\mathrm{A}}^2 - G\frac{m_1 m_2}{r_{\mathrm{A}}} &= \frac{1}{2} m_2 v_{\mathrm{B}}^2 - G\frac{m_1 m_2}{r_{\mathrm{B}}} \\
\frac{1}{2} v_{\mathrm{A}}^2 - G\frac{m_1}{r_1} &= \frac{1}{2}\left(\frac{v_{\mathrm{A}}}{5}\right)^2 - G\frac{m_1}{5r_1/2} \\
\frac{1}{2} v_{\mathrm{A}}^2 \left(1-\frac{1}{25}\right) &= G\frac{m_1}{r_1}\left(1-\frac{2}{5}\right) \\
v_{\mathrm{A}}^2 &= \frac{5}{4} G\frac{m_1}{r_1} \\
v_{\mathrm{A}} &= \sqrt{\frac{5}{4}\frac{G m_1}{r_1}}
\end{aligned} \quad . \qquad \textbf{(28.3.6)}$$

It's worth noting that v_{A} is less than the escape velocity

$$v_{\mathrm{escape}} = \sqrt{\frac{2 G m_1}{r_1}} \qquad \textbf{(28.3.7)}$$

of the planet, but not by much;

$$\frac{v_{\mathrm{A}}}{v_{\mathrm{escape}}} = \sqrt{\frac{5}{8}} \approx 0.79 \, . \qquad \textbf{(28.3.8)}$$

$$\begin{aligned}
\frac{1}{2} m_2 v_{\mathrm{A}}^2 - G\frac{m_1 m_2}{r_{\mathrm{A}}} &= \frac{1}{2} m_2 v_{\mathrm{B}}^2 - G\frac{m_1 m_2}{r_{\mathrm{B}}} \\
\frac{1}{2} v_{\mathrm{A}}^2 - G\frac{m_1}{r_1} &= \frac{1}{2}\left(\frac{v_{\mathrm{A}}}{5}\right)^2 - G\frac{m_1}{5r_1/2} \\
\frac{1}{2} v_{\mathrm{A}}^2 \left(1-\frac{1}{25}\right) &= G\frac{m_1}{r_1}\left(1-\frac{2}{5}\right) \\
v_{\mathrm{A}}^2 &= \frac{5}{4} G\frac{m_1}{r_1} \\
v_{\mathrm{A}} &= \sqrt{\frac{5}{4}\frac{G m_1}{r_1}}
\end{aligned} \qquad (28.3.9)$$

It's worth noting that v_{A} is less than the escape velocity

$$v_{\text{escape}} = \sqrt{\frac{2\,G\,m_1}{r_1}} \tag{28.3.10}$$

of the planet, but not by much;

$$\frac{v_{\text{A}}}{v_{\text{escape}}} = \sqrt{\frac{5}{8}} \approx 0.79\,. \tag{28.3.11}$$

Problem 4: Inverse Square Central Force: *Lowest Energy Solution*

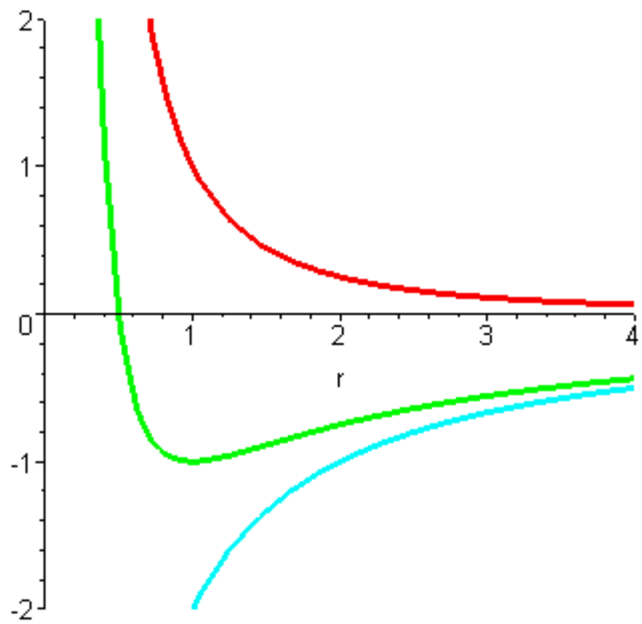
The effective potential energy for an inverse-square restoring central force $\vec{\mathbf{F}}_{1,2} = -\frac{Gm_1 m_2}{r^2}\hat{\mathbf{r}}$ is given by

$$U_{\text{effective}} = \frac{L^2}{2\mu r^2} - \frac{Gm_1 m_2}{r}$$

Make a graph of the effective potential energy $U_{\text{effective}}(r)$ as a function of the relative separation r. Find the radius and the energy for the motion with the lowest energy. What type of motion does this correspond to?

Problem 4 Solution:

For plotting purposes, the horizontal scale is the ratio r/r_0 and the vertical scale is in units of $U_{\text{effective}}/U_{\text{effective}}(r_0)$, as found in the second part of the problem.



The upper (red) curve is proportional to $1/r^2$ and the lower (blue) curve is proportional to $-1/r$. The sum is the solid (green) curve in between.

To find the minimum energy, differentiate the effective potential $U_{\text{effective}}(r)$ with respect to the radius r and set the derivative equal to zero at r_0,

$$-\frac{L^2}{\mu}\frac{1}{r_0^3}+\frac{Gm_1m_2}{r_0^2}=0$$

$$r_0=\frac{L^2}{\mu Gm_1m_2}.$$

The energy at this minimum value is

$$U_{\text{effective}}\left(r_0\right)=\frac{L^2}{2\mu}\left(\frac{\mu\, Gm_1m_2}{L^2}\right)^2-Gm_1m_2\frac{\mu\, Gm_1m_2}{L^2}$$

$$=-\frac{\mu\left(Gm_1m_2\right)^2}{2L^2}.$$

The radius of this orbit does not change; the orbit is a circle.

Problem 5

A particle of mass m moves under an attractive central force of magnitude $F = br^3$. The angular momentum is equal to L.

a) Find the effective potential energy and make sketch of effective potential energy as a function of r.

b) Indicate on a sketch of the effective potential the total energy for circular motion.

c) The radius of the particle's orbit varies between r_0 and $2r_0$. Find r_0.

Problem 5 Solutions:

a) The potential energy is, taking the zero of potential energy to be at $r = 0$, is

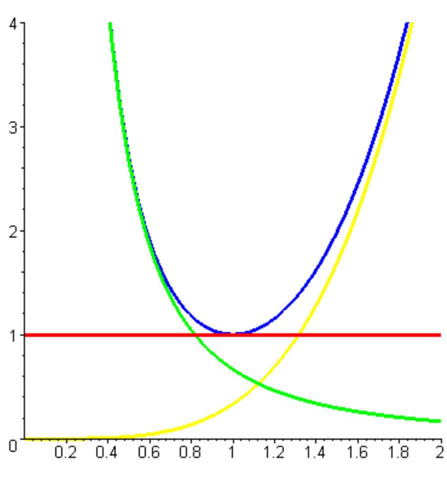
$$U(r) = -\int_0^r \left(-br'^3\right) dr' = \frac{b}{4} r^4$$

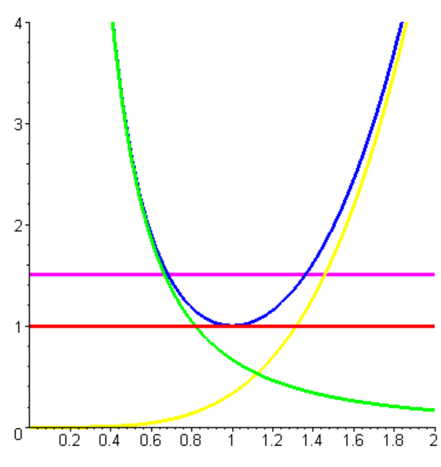
and the effective potential is

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} + U(r) = \frac{L^2}{2mr^2} + \frac{b}{4} r^4 .$$

A plot is shown below, including the potential (yellow if seen in color), the term $L^2/2m$ (green) and the effective potential (blue). The minimum effective potential energy is the horizontal line (red). The horizontal scale is in units of the radius of the circular orbit and the vertical scale is in units of the minimum effective potential.

b) See the solution to part (a) above and the plot to the left below.





c) In the left plot, if we could move the red line up until it intersects the blue curve at two point whose value of the radius differ by a factor of 2, those would be the respective values for r_0 and $2r_0$. A graph of this construction (done by computer, of course), showing the corresponding energy as the horizontal magenta is at the right above, and is not part of this problem.

To do this algebraically, we find the value of r_0 such that $U_{\text{eff}}(r_0) = U_{\text{eff}} 2(r_0)$. This is

$$\frac{L^2}{mr_0^2} + \frac{b}{4} r_0^4 = \frac{L^2}{m(2r_0)^2} + \frac{b}{4}(2r_0)^4.$$

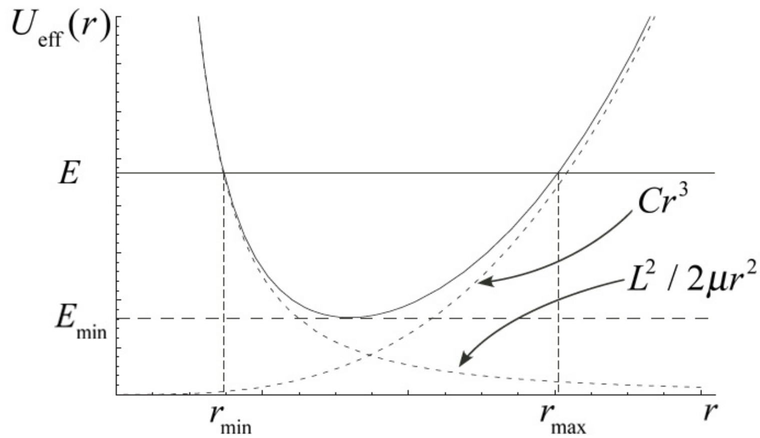
Rearranging and combining terms, and then solving for r_0,

$$\frac{3}{8}\frac{L^2}{m}\frac{1}{r_0^2} = \frac{15}{4} b r_0^4$$

$$r_0^6 = \frac{1}{10}\frac{L^2}{mb}.$$

Thus, $r_0 = \left(1/\sqrt{10}\right) r_{\text{circular}}$ (not part of the problem), consistent with the auxiliary figure on the right above.

Problem 6:



The effective potential corresponding to a pair of particles interacting through a central force is given by the expression

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + Cr^3 \tag{30.1}$$

where L is the angular momentum, μ is the reduced mass and C is a constant. The total energy of the system is E. The relationship between $U_{\text{eff}}(r)$ and E is shown in the figure, along with an indication of the associated maximum and minimum values of r and the minimum allowed energy $E_{\min}$. In what follows, assume that the center of mass of the two particles is at rest.

a) Find an expression for the radial component $f(r)$ of the force between the two particles. Is the force attractive or repulsive?

b) What is the radius r_0 of the circular orbit allowed in this potential? Express your answer as some combination of L, C, and μ.

c) When E has a value larger than $E_{\min}$, find how rapidly the separation between the particles is changing, dr/dt, as the system passes through the point in the orbit where $r = r_0$. Give your answer in terms of some combination of E, $E_{\min}$, L, C, μ and r_0.

d) Does the relative motion between the particles stop when $r = r_{\max}$? If not, what is the total kinetic energy at that point in terms of some combination of E, L, C, μ, $r_{\max}$ and $r_{\min}$?

Problem 6 Solutions:

a) The effective potential is given by

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + U(r)\,, \tag{28.6.2}$$

and $f(r) = -dU/dr$ for a central force, and so

$$f(r) = -\frac{d}{dr}U(r) = -\frac{d}{dr}Cr^3 = -3Cr^2\,. \tag{28.6.3}$$

From the figure, $C > 0$, so $f(r) < 0$, a restoring force.

b) The circular orbit will correspond to the minimum effective potential; at this radius the kinetic energy will have no contribution from any radial motion. This minimum effective potential, and hence the radius of the circular orbit, is found from basic calculus and algebra,

$$\begin{aligned} \left[\frac{d}{dr}U_{\text{eff}}(r)\right]_{r=r_0} &= -\frac{L^2}{\mu r_0^3} + 3Cr_0^2 \\ r_0^5 = \frac{L^2}{3\mu C},\quad r_0 &= \left(\frac{L^2}{3\mu C}\right)^{1/5}. \end{aligned} \tag{28.6.4}$$

c) Recall that the kinetic energy is

$$K = \frac{L^2}{2\mu r^2} + \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2. \tag{28.6.5}$$

The difference $E - E_{\min}$ is then found by evaluating U_{eff} at $r = r_0$,

$$E - E_{\min} = \frac{1}{2}\mu\left(\left.\frac{dr}{dt}\right|_{r=r_0}\right)^2, \tag{28.6.6}$$

or $\left|dr/dt\right| = \sqrt{(2/\mu)(E - E_{\min})}$.

d) No; $dr/dt = 0$, but the kinetic energy, from Equation (28.6.5), is

$$K_{\min} = \frac{L^2}{2\mu r_{\max}^2}\,. \tag{28.6.7}$$

Problem 7: Determining the Mass of a Neutron Star

A binary system known as 4U0900-40 consists of a "neutron star" and a normal "optical star." You are given two graphs of actual data obtained from observations of this system. The top graph shows the time delays (in seconds) of X-ray pulses detected from the neutron star as a function of time (in days) throughout its orbit. These delays indicate the time of flight for an X-ray pulse (traveling with the speed of light) to cross the orbit of the binary in its trip to the earth. (Ignore the heavy dots scattered about the "x"-axis.)

The bottom graph displays the velocity component of the optical star toward (or away from) the Earth as a function of its orbital phase. The velocities (in km per second) are determined from Doppler shifts in its spectral lines. Time on this plot is given in units of orbital phase, where the time between phases 0.0 and 1.0 corresponds to one orbital period.

Assume that the orbit of 4U0900-40 is circular and that we are viewing the system edge on, i.e., the Earth lies in the plane of the binary orbit. In each graph, the solid curve is a computed fit to the individual data points.

You are to find from the data the orbital period, orbital radii and the masses of the neutron star and the optical star. You need only give answers to two significant figures.

For this problem, use $c = 3.0 \times 10^8\,\mathrm{m \cdot s^{-1}}$ for the speed of light and $G = 6.7 \times 10^{-11}\,\mathrm{N \cdot m^2 \cdot kg^{-2}}$ for the Universal Gravitation Constant.

Suggested procedure (you will need a ruler for the first three parts):

a. Determine the orbital period, T, in days.

b. Estimate the velocity, v_0 of the optical star in its orbit around the center of mass.

c. Use T and v_0 to find the radius of the orbit of the optical star, a_0, around the center of mass.

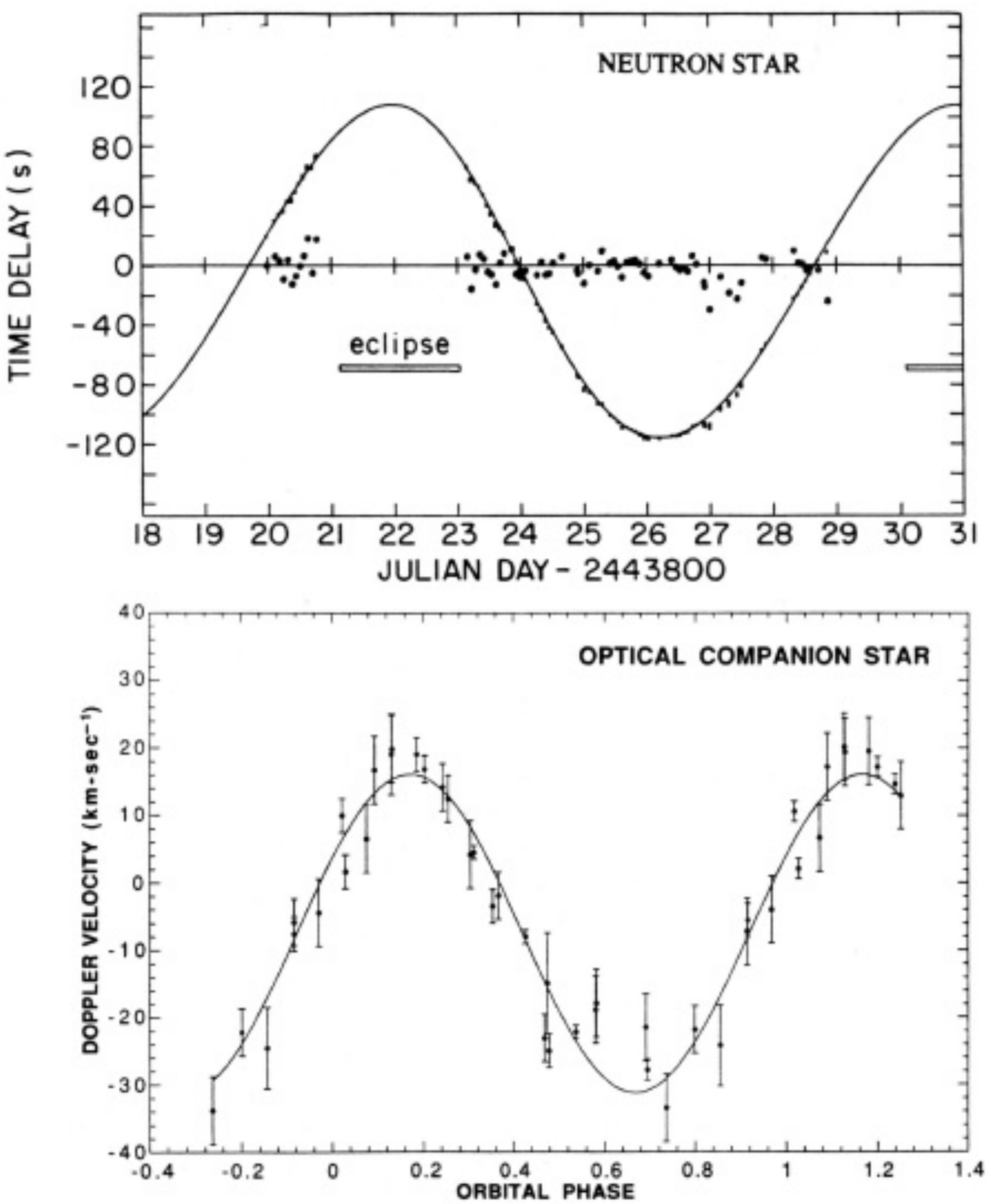
d. Estimate the size of the orbit, a_N, of the neutron star around the center of mass. First express your answer in light$\cdot$seconds, then in meters.

e. Find the ratio of the mass of the neutron star, m_N, to the mass of the optical star, m_O. [Hint: Recall that in any binary $m_1 r_1 = m_2 r_2$.]

f. Use Kepler's law to find the total mass of the binary system.

g. Find m_N and m_O; express your answers in units of the mass of the Sun (2×10^{30} kg).

BINARY X-RAY PULSAR SYSTEM



Doppler Curves for 4U0900-40.

Problem 7 Solutions:

a. Between the first and third zero-crossings on the upper graph, the distance is 8.98 cm. The scale of the time axis is 1.00 cm to 1 day, so the orbital period, *P*, is 8.98 days.

b. The Doppler shift velocities range from a maximum of 16.0 km·s^{-1} to a minimum of -30.8 km·s^{-1}. (The velocities are not symmetric around zero because the binary system as a whole is moving relative to the solar system.) The optical star's orbital velocity is just half the difference between the maximum and minimum Doppler velocities, or r_0=23.4 km·s^{-1}.

c. In one period *P*, moving at velocity v_0, the optical star travels a distance $v_0 p = 2\pi a_0$. So the orbital radius, a_0 is just

$$a_0 = \frac{1}{2\pi}\left[(9.98)(24)(60)(60)s\right]\left(23.4 km\cdot \mathrm{s}^{-1}\right) = 2.89\times 10^6\, km$$

d. The smallest time delay is -117 seconds, while the largest is 111 seconds. The radius of the neutron star's orbit (in light seconds) is half the difference between these, or $a_N = 114\mathrm{lt}\cdot\mathrm{sec}$. Since the speed of light is 2.998 x 10^{10} cm·s^{-1}, the distance a_N is equal to 3.42 x 10^{12}cm.

e. The ratio M_N/M_0 is just equal to a_0/a_N. That is

$$\frac{M_N}{M_0} = \frac{2.89\times 10^{11} cm}{3.42\times 10^{12} cm} = 8.45\times 10^{-2}$$

f. The semimajor axis of the absolute orbit is $a = a_N + a_0 = 3.71\times 10^{12}\, cm$ and the orbital period is 7.76 x 10^{5}s. Inserting these values into Newton's form of Kepler's Third Law.

$$\frac{p^2}{(2\pi)^2} = \frac{a^3}{G\left(M_N + M_0\right)}$$

we have,

$$M_N + M_0 + 4\pi^2 \frac{\left(3.71\times 10^{12}\right)^3}{\left(7.76\times 10^5\right)^2\left(6.67\times 10^{-8}\right)} = 5.02\times 10^{34}\, g$$

g. We use the fa(28.7.1)ct that $M_0\left(1+\frac{M_N}{M_0}\right) = M_0 + M_N$, and both $\frac{M_N}{M_0}$ and $\left(M_0 + M_N\right)$ are known. Then we have

$$M_0 = \frac{5.02\times 10^{34}\, g}{1+8.45\times 10^{-2}} = 4.63\times 10^{34}\, g$$

The remaining mass in the mass of the neutron star, $M_N = 3.9\times 10^{33}\, g$.

The mass of the sun is 1.99 x 10^{33} g, so the masses of the optical star and its neutron star companion are 23.3 and 2.0 solar masses, respectively.