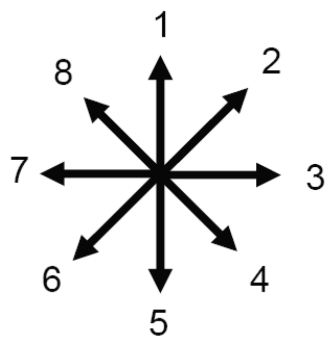
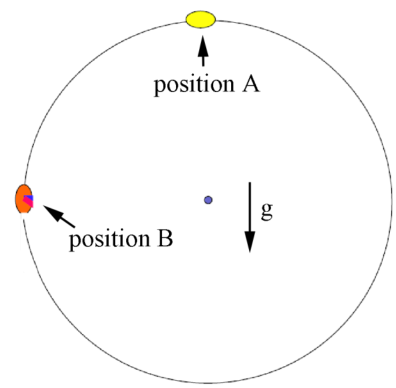
Problem Solving Circular Motion Kinematics
Challenge Problem Solutions

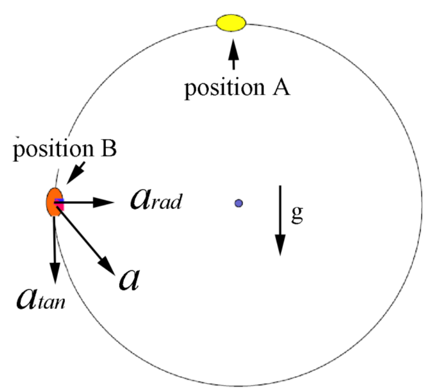
Problem 1

A bead is given a small push at the top of a hoop (position A) and is constrained to slide around a frictionless circular wire (in a vertical plane). Circle the arrow that best describes the direction of the acceleration when the bead is at the position B.





Problem 1 Solution: The bead is speeding up at position B therefore it has a tangential component of the acceleration (pointing downward) and it is traveling in a circular trajectory so it has a radial component of the acceleration pointing towards the center of the circle. Direction 4 best describes the sum of these two components.



Problem 2

The earth is spinning about its axis with a period of 23 hours 56 min and 4 sec. The equatorial radius of the earth is $6.38\times10^{6}\ \text{m}$. The latitude of Cambridge, Mass is $42^{o}\ 22'$.

a) Find the velocity of a person at MIT as they undergo circular motion about the earth's axis of rotation.

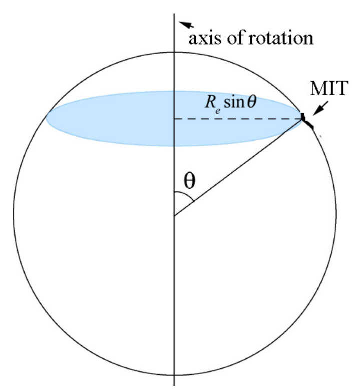
b) Find the person's centripetal acceleration.

Problem 2 Solutions:

a) The rotational period of the earth is given by

$$T_{\text{earth, 2004}} = (23\ \text{hr})(3600\ \text{s}\cdot\text{hr}^{-1}) + (56\ \text{min})(60\ \text{s}\cdot\text{min}^{-1}) + 4\,\text{s} = 86164\ \text{s} \quad (2.1)$$

Note that this period is less than $24\,\text{hr}$. Twenty-four hours is one *solar* day (noon to noon), while the above period is one *sidereal* day; sidereal means with respect to the "fixed" stars, and you should be able to see why the two are different.



A person at MIT undergoes circular motion about the axis of the earth. The radius of the orbit is given by $R = R_e \sin\theta$ where θ is the angle between MIT and the axis of rotation. Since the latitude λ is measured form the equator, $\sin\theta = \sin(\pi/2-\lambda) = \cos\lambda$ (the angle θ is sometimes called the "colatitude"). Hence $\theta = \pi/2-\lambda$, and the radius of the orbit of a person at MIT is

$$R = R_e \cos\lambda = (6.38\times10^{6}\,\text{m})(\cos 42.36) = 4.71\times10^{6}\,\text{m} \qquad (2.2)$$

Since the motion is uniform, during one period of rotation the person travels a distance

$$s = 2\pi R = vT \tag{2.3}$$

where v is the magnitude of the velocity and $s = 2\pi R$ is the circumference.

The magnitude of the velocity (speed) is then given by

$$v = \frac{2\pi R}{T} \tag{2.4}$$

So for a person at MIT, the magnitude of the velocity is

$$v = \frac{2\pi R}{T} = \frac{2\pi R_e \cos\lambda}{T} = \frac{(2\pi)(4.714\times 10^6\,\mathrm{m})}{(86164\,\mathrm{s})} = 3.44\times 10^2\,\mathrm{m}\cdot\mathrm{s}^{-1}. \tag{2.5}$$

b) The centripetal acceleration is given by

$$\left|a_r\right| = \frac{v^2}{R} = \frac{v^2}{R_e \cos\lambda} = \frac{(3.44\times 10^2\,\mathrm{m}\cdot\mathrm{s}^{-1})^2}{(4.71\times 10^6\,\mathrm{m})} = 2.51\times 10^{-2}\,\mathrm{m}\cdot\mathrm{s}^{-2}. \tag{2.6}$$

Problem 3: Earth and a Geostationary Satellite

The earth is spinning about its axis with a period of 23 hours 56 minutes and 4 seconds. The equatorial radius of the earth is $6.38 \times 10^{6}\ \mathrm{m}$. The latitude of Cambridge, Mass is $42° \, 22'$.

Find the magnitude of the velocity and the centripetal acceleration with respect to the earth's axis of a person at MIT as they undergo circular motion about the earth's axis of rotation.

Problem 3 Solution:

For recent values of the physical constants used in this problem, see
http://pdg.lbl.gov/2007/reviews/contents_sports.html - constantsetc .

The rotational period of the earth is given by

$$T_{\text{earth, 2005}} = (23\ \mathrm{hr})\left(3600\ \mathrm{s} \cdot \mathrm{hr}^{-1}\right) + (56\ \mathrm{min})\left(60\ \mathrm{s} \cdot \mathrm{min}^{-1}\right) + 4\,\mathrm{s} = 86164\ \mathrm{s}\,. \quad (3.1)$$

Note that this period is less than $24\,\mathrm{hr}$. Twenty-four hours is one solar day (noon to noon), while the above period is one sidereal day; sidereal means with respect to the "fixed" stars, and you should be able to see why the two are different.

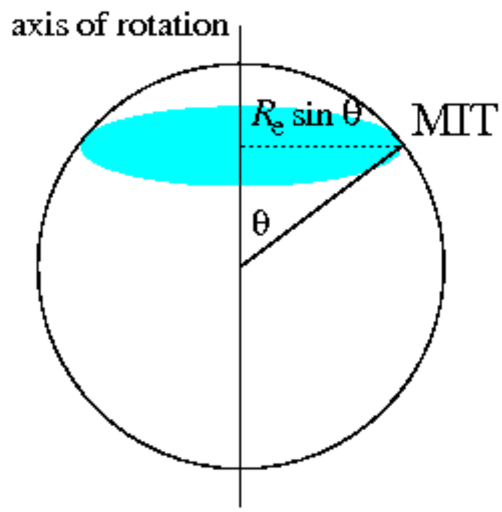
A person at MIT undergoes circular motion about the axis of the earth. The radius of this circle is $R = R_{\mathrm{e}} \sin\theta$ where θ is the angle between MIT and the axis of rotation. Since the latitude λ is measured form the equator, $\sin\theta = \sin(\pi/2 - \lambda) = \cos\lambda$ (the angle θ is sometimes called the "colatitude"). Hence the radius of the orbit of a person at MIT is

$$R = R_{\mathrm{e}} \sin\theta = R_{\mathrm{e}} \cos\lambda = (6.38 \times 10^{6}\,\mathrm{m})\left(\cos 42.36°\right) = 4.71 \times 10^{6}\,\mathrm{m}\,. \quad (3.2)$$

Since the motion is uniform, during one period of rotation the person travels a distance

$$s = 2\pi R = vT \quad (3.3)$$

where v is the magnitude of the velocity and $s = 2\pi R$ is the circumference.



The magnitude of the velocity (the speed) is then given by

$$v = \frac{2\pi R}{T} \tag{3.4}$$

and so for a person at MIT, the magnitude of the velocity is

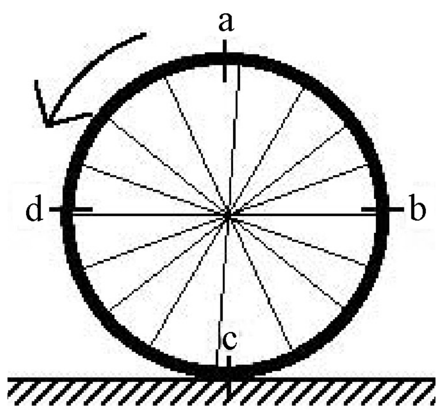
$$v = \frac{2\pi R}{T} = \frac{(2\pi)\left(4.71\times10^{6}\,\mathrm{m}\right)}{86164\ \mathrm{s}} = 3.44\times10^{2}\,\mathrm{m}\cdot\mathrm{s}^{-1}\,. \tag{3.5}$$

The magnitude $|a_r|$ of the centripetal acceleration is given by

$$|a_r| = \frac{v^2}{R} = \frac{\left(3.44\times10^{2}\,\mathrm{m}\cdot\mathrm{s}^{-1}\right)^2}{4.71\times10^{6}\,\mathrm{m}} = 2.51\times10^{-2}\,\mathrm{m}\cdot\mathrm{s}^{-2}\,. \tag{3.6}$$

Problem 4:

A bicycle tire rolls without slipping along the ground with its center of mass moving with speed v_0. You may neglect any rolling resistance. What is the speed with respect to the ground of each of the four marked points on the tire shown in the figure below? Explain each of your answers.



Problem 4 Solutions:

As an intermediate calculation (introducing an "auxiliary parameter"), let the wheel radius be R_{wheel}. The angular speed of the wheel about its center is then $\omega_0 = v_0 / R_{\text{wheel}}$ if the wheel is to roll without slipping. The speed of any point on the rim of the wheel *with respect to the center* is then $v = \omega_0 R_{\text{wheel}} = v_0$.

This should be obvious, and such a detailed calculation may seem pointless. However, for more complicated systems, or, if we needed to consider points other than those on the rim, we have to include this intermediate step.

Speed of point ***a*****:** This point is moving parallel to the velocity of the center of the wheel, so the speed is the sum $v_0 + v_0 = 2v_0$, a perhaps apparent result.

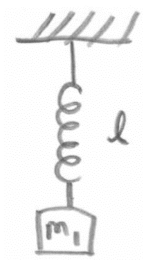
Speed of point ***b*****:** This point has a horizontal velocity component v_0 and a vertical velocity component, also v_0, and so the speed is $\sqrt{2}\,v_0$.

Speed of point ***c*****:** For this point, the velocity with respect to the center of the wheel is directed opposite to the velocity of the center of the wheel, and so the speed with respect to the ground is $v_0 - v_0 = 0$. Or, we could say quite correctly that if the wheel is rolling without slipping, the contact point, being in contact with the ground, cannot be moving with respect to the ground.

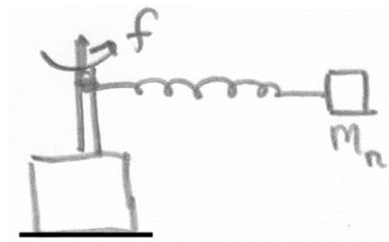
Speed of point *d*: This is the same speed as that of point *b*, $\sqrt{2}\,v_0$. The vertical component of velocity is of course down, but the magnitude is the same.

Problem 5: Uniform Circular Motion

Consider a spring with negligible mass that has an unstretched length $l_0 = 8.8\times10^{-2}\,\text{m}$. A body of mass $m_1 = 1.5\times10^{-1}\,\text{kg}$ is suspended from one end of the spring. The other end of the spring is fixed. After a series of oscillations has died down, the new stretched length of the spring is $l = 9.8\times10^{-2}\,\text{m}$.



part a)



parts b) - d)

a) Assume that the spring satisfies Hooke's Law when it is stretched. What is the spring constant?

The body is then removed and one end of the spring is attached to the central axis of a motor. The axis of the motor is the vertical direction. A small ball of mass $m_n = 3.0\times10^{-3}\,\text{kg}$ is then attached to the other end of the spring. The motor rotates at a constant frequency $f = 2.0\times10^{1}\,\text{Hz}$.

b) How long does it take the ball to complete one rotation?

c) What is the angular frequency of the ball in radians per second?

Problem 5 Solutions:

a) Hooke's Law states that the extension of the spring is proportional to the applied force (in Latin, "Ut Tensio Sic Vis"); the spring constant is the proportionality constant,

$$k = \frac{\Delta F}{\Delta l} = \frac{m_1 g}{l - l_0} = \frac{(1.5\times10^{-1}\,\text{kg})(9.8\,\text{m}\cdot\text{s}^{-2})}{9.8\times10^{-2}\,\text{m} - 8.8\times10^{-2}\,\text{m}} = 1.5\times10^{2}\,\text{N}\cdot\text{m}^{-1} \qquad (5.1)$$

to two significant figures.

b) The time T to complete one revolution is the reciprocal of the frequency,

$$T = \frac{1}{f} = \frac{1}{2.0\times10^{1}\ \mathrm{Hz}} = 0.050\,\mathrm{s}. \qquad (5.2)$$

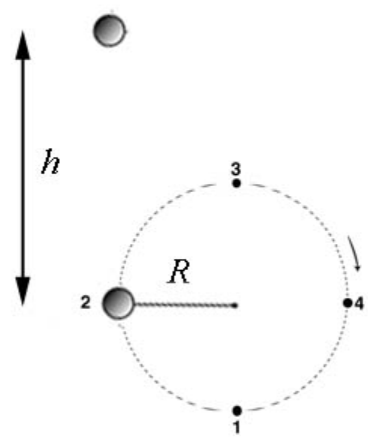
c) The angular frequency ω is the total radian measure of one cycle, 2π , divided by the time taken to complete one revolution,

$$\omega = \frac{2\pi}{T} = 2\pi\, f = 2\pi\left(2.0\times10^{1}\ \mathrm{Hz}\right) = 1.3\times10^{2}\ \mathrm{radians}\cdot\mathrm{s}^{-1} \qquad (5.3)$$

to two significant figures.

Problem 6: Whirling Stone

A stone (or a ball in the demo), attached to a wheel and held in place by a string, is whirled in circular orbit of radius R in a vertical plane. Suppose the string is cut when the stone is at position 2 in the figure, and the stone then rises to a height h above the point at position 2. What was the angular velocity of the stone when the string was cut? Give your answer in terms of R, h and g.



Problem 6 Solution:

There are two distinct stages of motion. The first is circular motion in which the stone is being whirled at a speed v_0 when at position 2. Once the string is cut, the stone is moving vertically upwards, interacting gravitationally with the earth. For the vertical motion, set $t = 0$ when the string is just cut. Let's choose a coordinate system with the origin at point 2 in the figure and the positive y-direction upwards. Thus $y_0 = 0$. The key point to note is that for the vertical motion the initial speed when the string is cut is the speed of the uniform circular motion,

$$v_{y,0} = v_0 \tag{6.1}$$

Newton's Second Law becomes

$$-m\,g = m\,a_y\,. \tag{6.2}$$

The y - component of the acceleration is then

$$a_y = -g\,. \tag{6.3}$$

The y -component of the velocity is given by

$$v_y = v_0 - g\,t\,. \tag{6.4}$$

The stone reaches its highest point at time $t = t_1$ when the y-component of the velocity is zero,

$$v_y(t = t_1) = v_0 - g\,t_1 = 0 \tag{6.5}$$

We can solve Equation (6.5) for the time it takes for the stone to reach its highest point,

$$t_1 = \frac{v_0}{g}\,. \tag{6.6}$$

The y-component of the position at the highest point is

$$y(t = t_1) = h = v_0\,t_1 - \frac{1}{2}g\,t_1^2\,. \tag{6.7}$$

Substitute Equation (6.6) into Equation (6.7) to obtain

$$h = \frac{1}{2}\frac{v_0^2}{g}\,. \tag{6.8}$$

We can solve Equation (6.8) for the speed of the circular motion,

$$v_0 = \sqrt{2gh}\,. \tag{6.9}$$

The angular velocity ω is related to the speed by $v_0 = \omega R$, so

$$\omega = \frac{v_0}{R} = \frac{\sqrt{2g\,h}}{R} = \sqrt{\frac{2g\,h}{R^2}}\,. \tag{6.10}$$

The last expression in Equation (6.10) is included to make checking the dimensions slightly easier;

$$\dim\left[\frac{2g\,h}{R^2}\right] = \frac{\left[\mathrm{L}\cdot\mathrm{T}^{-2}\right]\mathrm{L}}{\mathrm{L}^2} = \mathrm{T}^{-2}\,, \tag{6.11}$$

and so the expression for ω in Equation (6.10) has dimensions of inverse time, as it should.