Chapter 13 Conservation of Energy and Potential Energy

So far we have analyzed the motion of point-like bodies under the action of forces using Newton's Laws of Motion. We shall now introduce the Principle of Conservation of Energy to study the changes in energy of a system between an initial state and final state. In particular we shall introduce the concept of potential energy to describe the effect of conservative internal forces acting on the constituent components of a system.

13.1 Conservation of Energy

When a system and its surroundings undergo a transition from an initial state to a final state, the total change in energy is zero,

$$\Delta E^{\text{total}} = \Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0\,. \tag{13.1.1}$$

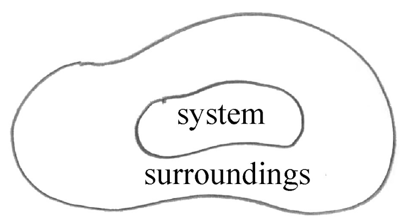


Figure 13.1 A diagram of a system and its surroundings

This conservation law is our basic assumption. In any physical application, we first identify our system and surroundings, and then attempt to quantify changes in energy. In order to do this, we need to identify every type of change of energy in every possible physical process.

If we add up all known changes in energy in the system and surroundings and do not arrive at a zero sum, we have an open scientific problem. By searching for the missing changes in energy, we may uncover some new physical phenomenon. Recently, one of the most exciting open problems in cosmology is the apparent acceleration of the expansion of the universe, which has been attributed to *dark energy* that resides in space itself, an energy type without a clearly known source.[1]

Energy can change forms inside a system, for example chemical energy stored in the molecular bonds of gasoline can be converted into kinetic energy and heat via combustion. Energy can also flow into or out of the system across a boundary. A system in which no energy flows across the boundary is called a *closed system.* Then the total change in energy of the system is zero,

$$\Delta E_{\text{system}}^{\text{closed}} = 0\,. \tag{13.1.2}$$

[1] http://www-supernova.lbl.gov/~evlinder/sci.html .

13.2 Conservative and Non-Conservative Forces

Our first type of "energy accounting" involves *mechanical energy.* There are two types of mechanical energy, *kinetic energy* and *potential energy*. Our first task is to define what we mean by the change of the potential energy of a system.

We defined the work done by a force $\vec{\mathbf{F}}$, on an object which moves along a path from an initial point $\vec{\mathbf{r}}_0$ to a final point $\vec{\mathbf{r}}_f$, as the integral of the component of the force tangent to the path with respect to the displacement of the point of contact of the force and the object,

$$W = \int_{\text{path}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} . \qquad (13.2.1)$$

Does the work done on the object by the force depend on the path taken by the object? First consider the motion of an object under the influence of a gravitational force near the surface of the earth, as was considered in Sections 7.4 and 7.7. The gravitational force always points downward, so the work done by gravity only depends on the change in the vertical position (we choose the positive y-direction upwards),

$$W_{\text{grav}} = \int_{\text{path}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{y_0}^{y_f} F_{\text{grav},y}\, dy = \int_{y_0}^{y_f} -mg\, dy = -mg(y_f - y_0) \qquad (13.2.2)$$

Therefore when an object falls, $(y_f - y_0) < 0$, and the work done by gravity is positive. When an object rises, $(y_f - y_0) > 0$, and the work done by gravity is negative. Suppose an object first rises and then falls, returning to the original starting height. The positive work done on the falling portion exactly cancels the negative work done on the rising portion, as in Figure 13.2. The total work done is zero. Thus the gravitational work done between two points will not depend on the path taken, but only on the initial and final positions.

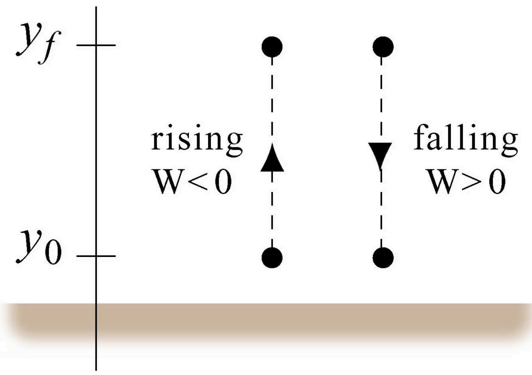


Figure 13.2 Gravitational work sums to zero in a closed loop.

This is also true for projectile motion. The displacement of the projectile has both a horizontal component and a vertical component. However the gravitational force is only in the vertical direction, so the horizontal motion does not contribute to the work done.

Now consider the motion of an object on a surface with a kinetic frictional force between the object and the surface and denote the coefficient of kinetic friction by μ_{k}. Let's compare two paths from an initial point x_0 to a final point x_f. The first path is a straight-line path. Along this path the work done is just

$$W_{\text{friction}} = \int_{\text{path 1}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\text{path 1}} F_x \, dx = -\mu_{\text{k}} N \, s_1 = -\mu_{\text{k}} N \, \Delta x < 0 \qquad (13.2.3)$$

where the length of the path is equal to the displacement, $s_1 = \Delta x$. Note that the fact that the kinetic friction force is directed opposite to the displacement is reflected in the minus sign in Equation (13.2.3). The second path goes past x_f some distance and them comes back to x_f (Figure 13.3). Since the force of friction always opposes the motion, the work done by friction is negative,

$$W_{\text{friction}} = \int_{\text{path 2}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\text{path 2}} F_x \, dx = -\mu_{\text{k}} N \, s_2 < 0 \, . \qquad (13.2.4)$$

The total work depends on the total distance traveled s_2, and is greater than the displacement $s_2 > \Delta x$. The magnitude of the work done along the second path is greater than the magnitude of the work done along the first path.

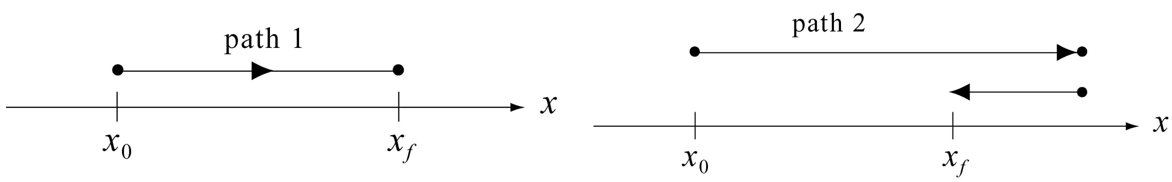


Figure 13.3 Two different paths from x_0 to x_f.

These two examples typify two fundamentally different types of forces and their contribution to work. The gravitation force near the surface of the earth does the same amount of work regardless of the path taken between the initial and final points. In the case of sliding friction, the work done depends on the path taken.

Definition: **Conservative Force**

Whenever the work done by a force in moving an object from an initial point to a final point is independent of the path, the force is called a *conservative force*.

The work done by a conservative force $\vec{\mathbf{F}}_{\mathrm{c}}$ in going around a closed path is zero. Consider the two paths shown in Figure 13.4 that form a closed path starting and ending at the point A with Cartesian coordinates $(1,0)$. The work done along path 1 (the upper path in the figure, blue if viewed in color) from point A to point B with coordinates $(0,1)$ is given by

$$W_{\text{path 1}} = \int_A^B \vec{\mathbf{F}}_{\mathrm{c}}(1) \cdot d\vec{\mathbf{r}}_1 \,. \tag{13.2.5}$$

The work done along path 2 (the lower path, green in color) from B to A is given by

$$W_{\text{path 2}} = \int_B^A \vec{\mathbf{F}}_{\mathrm{c}}(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{13.2.6}$$

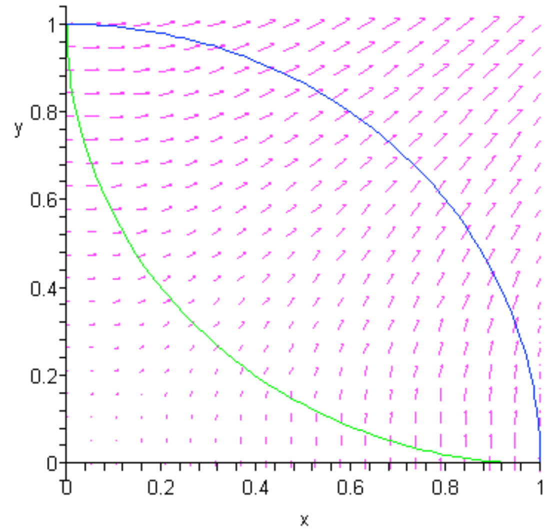


Figure 13.4 Two paths in the presence of a conservative force.

The work done around the closed path is just the sum of the work along paths 1 and 2,

$$W_{\text{closed path}} = W_{\text{path 1}} + W_{\text{path 2}} = \int_A^B \vec{\mathbf{F}}_{\text{c}}(1)\cdot d\vec{\mathbf{r}}_1 + \int_B^A \vec{\mathbf{F}}_{\text{c}}(2)\cdot d\vec{\mathbf{r}}_2 \,. \tag{13.2.7}$$

If we reverse the endpoints of path 2, then the integral changes sign,

$$W_{\text{path 2}} = \int_B^A \vec{\mathbf{F}}_{\text{c}}(2)\cdot d\vec{\mathbf{r}}_2 = -\int_A^B \vec{\mathbf{F}}_{\text{c}}(2)\cdot d\vec{\mathbf{r}}_2 \,. \tag{13.2.8}$$

We can then substitute Equation (13.2.8) into Equation (13.2.7) to find that the work done around the closed path is

$$W_{\text{closed path}} = \int_A^B \vec{\mathbf{F}}_{\text{c}}(1)\cdot d\vec{\mathbf{r}}_1 - \int_A^B \vec{\mathbf{F}}_{\text{c}}(2)\cdot d\vec{\mathbf{r}}_2 \,. \tag{13.2.9}$$

Since the force is conservative, the work done between the points A to B is independent of the path, so

$$\int_A^B \vec{\mathbf{F}}_{\text{c}}(1)\cdot d\vec{\mathbf{r}}_1 = \int_A^B \vec{\mathbf{F}}_{\text{c}}(2)\cdot d\vec{\mathbf{r}}_2 \,. \tag{13.2.10}$$

We now use path independence of work for a conservative force (Equation (13.2.10) in Equation (13.2.9)) to conclude that the work done by a conservative force around a closed path is zero,

$$W_{\text{closed path}} = \oint_{\substack{\text{closed}\\ \text{path}}} \vec{\mathbf{F}}_{\text{c}} \cdot d\vec{\mathbf{r}} = 0\,. \qquad (13.2.11)$$

13.3 Changes in Potential Energies of a System

Consider an isolated body near the surface of the earth as a system that is initially given a velocity directed upwards. Once the body is released, the gravitation force, acting as an external force, does a negative amount of work on the body, and the kinetic energy decreases until the body reaches its highest point, at which its kinetic energy is zero. The gravitation then force does positive work until the body returns to its initial starting point with a velocity directed downward. If we ignore any effects of air resistance, the descending body will then have the identical kinetic energy as when it was thrown. All the kinetic energy was completely recovered.

Now consider both the earth and the body as a system and assume that there are no other external forces acting on the system. Then the gravitation force is an internal conservative force, and does work on both the body and the earth during the motion. As the body moves upward, the kinetic energy of the system decreases, primarily because the body slows down, but there is also an imperceptible increase in the kinetic energy of the earth. The change in kinetic energy of the earth must also be included because the earth is part of the system. When the body returns to its original height (vertical distance from the surface of the earth), all the kinetic energy in the system is recovered, even though a very small amount has been transferred to the Earth. If we included the air as part of the system, and the air resistance as a non-conservative internal force, then the kinetic energy lost due to the work done by the air resistance is not recoverable. This lost kinetic energy, which we have called thermal energy, is distributed as random kinetic energy in both the air molecules and the molecules that compose the body (and, to a smaller extent, the earth).

We shall define a new quantity, the change in the internal *potential energy* of the system, which measures the amount of lost kinetic energy that can be recovered during an interaction. When only internal conservative forces act on the system, the sum of the changes of the kinetic and potential energies of the system is zero.

Consider a system consisting of two bodies with masses m_1 and m_2 respectively. Assume that there is one conservative force (internal force) that is the source of the interaction between two bodies. We denote the force on body 1 due to the interaction with body 2 by $\vec{\mathbf{F}}_{1,2}$ and the force on body 2 due to the interaction with body 1 by $\vec{\mathbf{F}}_{2,1}$. From Newton's Third Law,

$$\vec{\mathbf{F}}_{1,2} = -\vec{\mathbf{F}}_{2,1}\,. \qquad (13.3.1)$$

The forces acting on the bodies are shown in Figure 13.5.

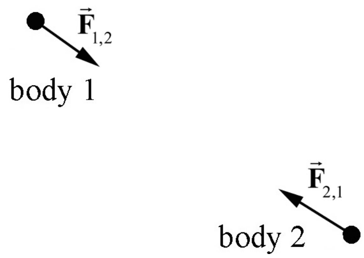


Figure 13.5 Internal forces acting on two bodies

Choose a coordinate system (Figure 13.6) in which the position vector of body 1 is given by $\vec{\mathbf{r}}_1$ and the position vector of body 2 is given by $\vec{\mathbf{r}}_2$. The relative position of body 1 with respect to body 2 is given by $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$. During the course of the interaction, body 1 is displaced by $d\,\vec{\mathbf{r}}_1$ and body 2 is displaced by $d\,\vec{\mathbf{r}}_1$, so the relative displacement of the two bodies during the interaction is given by $d\,\vec{\mathbf{r}}_{1,2} = d\,\vec{\mathbf{r}}_1 - d\,\vec{\mathbf{r}}_2$.

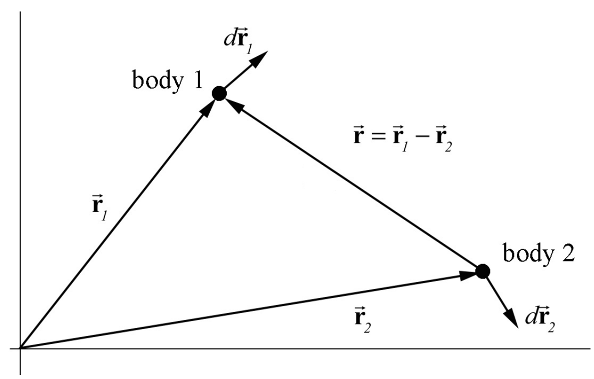


Figure 13.6 Coordinate system for two bodies with relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$

In Module 12 we observed that the change in the total kinetic energy of a body is equal to the work done by the forces in displacing the body. For two bodies displaced from an initial state A to a final state B,

$$\Delta K_{\text{system}} = \Delta K_1 + \Delta K_2 = W_{\text{c}} = \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\,\vec{\mathbf{r}}_1 + \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\,\vec{\mathbf{r}}_2 . \qquad (13.3.2)$$

(In Equation (13.3.2), the labels "A" and "B" refer to initial and final states, not paths.)

From Newton's Third Law, Equation (13.3.1), the sum in Equation (13.3.2) becomes

$$\Delta K_{\text{system}} = W_{\text{c}} = \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_1 - \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_2 = \int_A^B \vec{\mathbf{F}}_{1,2} \cdot (d\,\vec{\mathbf{r}}_1 - d\,\vec{\mathbf{r}}_2) = \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\,\vec{\mathbf{r}}_{1,2} \qquad (13.3.3)$$

where $d\,\vec{\mathbf{r}}_{1,2} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$ is the relative displacement of the two bodies. Note that since $\vec{\mathbf{F}}_{1,2} = -\vec{\mathbf{F}}_{2,1}$ and $d\,\vec{\mathbf{r}}_{1,2} = -d\vec{\mathbf{r}}_{2,1}$, $\int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\,\vec{\mathbf{r}}_{1,2} = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\,\vec{\mathbf{r}}_{2,1}$.

Definition: **Change in Potential Energy for Two Bodies**

> Consider a system consisting of two bodies interacting through a *conservative* force. Let $\vec{\mathbf{F}}_{1,2}$ denote the force on body 1 due to the interaction with body 2 and let $d\,\vec{\mathbf{r}}_{1,2} = d\,\vec{\mathbf{r}}_1 - d\,\vec{\mathbf{r}}_2$ be the relative displacement of the two bodies. The *change in internal potential energy of the system* is defined to be the negative of the work done by the conservative force when the bodies undergo a relative displacement from the initial state A to the final state B along any displacement that changes the initial state A to the final state B,

$$\Delta U_{\text{system}} = -W_{\text{c}} = -\int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_{1,2} = -\int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} . \qquad (13.3.4)$$

Our definition of potential energy only holds for conservative forces, because the work done by a conservative force does not depend on the path but only on the initial and final positions. Since the work done by the conservative force is equal to the change in kinetic energy, we have that

$$\Delta U_{\text{system}} = -\Delta K_{\text{system}} . \qquad (13.3.5)$$

Recall that the work done by a conservative force in going around a closed path is zero (Equation (13.2.11)); therefore the change in kinetic energy when a system returns to its initial state is zero. This means that the kinetic energy is completely recoverable.

In the *Appendix 12.A: Work Done on a System of Two Particles* we showed that the work done by an internal force in changing a system of two particles of masses m_1 and m_2 respectively from an initial state A to a final state B is equal to

$$W_{\text{c}} = \frac{1}{2}\mu(v_B^2 - v_A^2) \qquad (13.3.6)$$

where v_B^2 is the square of the relative velocity in state B, v_A^2 is the square of the relative velocity in state A, and $\mu = m_1 m_2 / (m_1 + m_2)$ is a quantity known as the *reduced mass* of the system.

Change in Potential Energy for Several Conservative Forces

When there are several internal conservative forces acting on the closed system we define a separate change in potential energy for the work done by each conservative force,

$$\Delta U_{\text{system},\, i} = -W_i = -\int_A^B \vec{\mathbf{F}}_{\text{c},\, i} \cdot d\vec{\mathbf{r}}_i \,. \tag{13.3.7}$$

where $\vec{\mathbf{F}}_{\text{c},\, i}$ is a conservative internal force and $d\vec{\mathbf{r}}_i$ a change in the relative positions of the bodies on which $\vec{\mathbf{F}}_{\text{c},\, i}$ when the system is changed from state A to state B.

The total work done is the sum of the work done by the individual conservative forces,

$$W_{\text{c}}^{\text{total}} = W_{\text{c},1} + W_{\text{c},2} + \cdots . \tag{13.3.8}$$

Hence, the sum of the changes in potential energies for the system is the sum

$$\Delta U_{\text{system}}^{\text{total}} = \Delta U_{\text{system},1} + \Delta U_{\text{system},2} + \cdots . \tag{13.3.9}$$

Therefore the total change in potential energy of the system is equal to the negative of the total work done

$$\Delta U_{\text{system}}^{\text{total}} = -W_{\text{c}}^{\text{total}} = -\sum_i \int_A^B \vec{\mathbf{F}}_{\text{c},\, i} \cdot d\vec{\mathbf{r}}_i \,. \tag{13.3.10}$$

Once again, if the external forces do no work,

$$\Delta K_{\text{system}} = -\Delta U_{\text{system}}^{\text{total}} \,. \tag{13.3.11}$$

13.4 Examples: Change in Potential Energy

In Module 12, we calculated the work done by different conservative forces: constant gravity near the surface of the earth, the spring force, and the universal gravitational force. We chose the system in each case so that the conservative force was an external force. In each case, there was no change of potential energy and the work done was equal to the change of kinetic energy,

$$W_{\text{ext}}^{\text{total}} = \Delta K_{\text{system}} . \qquad (13.4.1)$$

We now treat each of these conservative forces as internal forces and calculate the change in potential energy according to our definition

$$\Delta U_{\text{system}} = -W_{\text{c}} = -\int_A^B \vec{\mathbf{F}}_{\text{c}} \cdot d\vec{\mathbf{r}} . \qquad (13.4.2)$$

We shall also choose a *zero reference potential* for the potential energy of the system, so that we can consider all changes in potential energy relative to this reference potential.

13.4.1 Example: Change in Gravitational Potential Energy Near the Surface of the Earth

Let's consider the example of a body falling near the surface of the earth. Choose our system to consist of the earth and the body. The gravitational force is now an internal conservative force acting inside the system. The initial and final state are specified by the distance separating the body and the center of mass of the earth, and the velocities of the earth and the body.

Let's choose a coordinate system with the origin on the surface of the earth and the $+y$-direction pointing away from the center of the earth. Since the displacement of the earth is negligible, we need only consider the displacement of the body in order to calculate the change in potential energy of the system.

Suppose the body starts at an initial height y_0 above the surface of the earth and ends at final height y_f. The gravitation force on the body is given by

$$\vec{\mathbf{F}}_{\text{gravity}} = m\vec{\mathbf{g}} = F_{\text{gravity},y}\ \hat{\mathbf{j}} = -mg\ \hat{\mathbf{j}} . \qquad (13.4.3)$$

The work done by the gravitational force on the body is then

$$W_{\text{gravity}} = F_{\text{gravity},y}\ \Delta y = -mg\,\Delta y , \qquad (13.4.4)$$

which is of course the same result as found in Equation (7.4.16).

The change in potential energy is then given by

$$\Delta U_{\text{system}} = -W_{\text{gravity}} = mg\,\Delta y = mg\,y_f - mg\,y_0 . \qquad (13.4.5)$$

Choice of Zero Point Reference for Potential Energy

We introduce a potential energy function U so that

$$\Delta U_{\text{system}} \equiv U_f - U_0 . \tag{13.4.6}$$

Only differences in the function U have a physical meaning. We can choose a zero reference point for the potential energy anywhere we like, since change in potential energy only depends on the displacement, Δy (in general, the change in configuration). We have some flexibility to adapt our choice of zero for the potential energy to best fit a particular problem. In the above expression for the change of potential energy, let $y_f \equiv y$ be an arbitrary point and $y = 0$ be the origin. In addition, we choose the zero reference potential for the potential energy to be at the surface of the earth corresponding to our origin, $U(y = 0) = 0$. Then

$$\Delta U = U(y) - U(y = 0) = U(y) - 0 = mg\,y - 0 , \tag{13.4.7}$$

$$U(y) = mg\,y, \text{ with } U(y = 0) = 0 . \tag{13.4.8}$$

13.4.2 Example: Hooke's Law Spring-Body System

Consider a spring-body system lying on a frictionless horizontal surface with one end of the spring fixed to a wall and the other end attached to a body of mass m (Figure 13.6). The spring force is an internal conservative force. The wall exerts an external force on the spring-body system but since the point of contact of the wall with the spring undergoes no displacement this external force does no work.

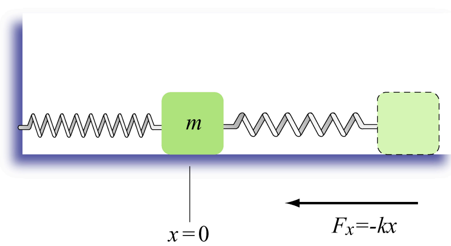


Figure 13.6 A spring-body system.

Choose the origin at the position of the center of the body when the spring is relaxed (the equilibrium position). Let x be the displacement of the body from the origin. We choose the $+\hat{\mathbf{i}}$ unit vector to point in the direction the body moves when the spring is being stretched (to the right of $x = 0$ in the figure). The spring force on a mass is then given by

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} = -kx\,\hat{\mathbf{i}} . \tag{13.4.9}$$

The work done by the spring force on the mass is

$$W_{\text{spring}} = \int_{x=x_0}^{x=x_f} (-kx)\,dx = -\frac{1}{2}k(x_f^{\,2} - x_0^{\,2})\,. \tag{13.4.10}$$

This is of course the result obtained in Module 12

We then define the change in potential energy in the spring-body system in moving the body from an initial position x_0 from equilibrium to a final position x_f from equilibrium by

$$\Delta U_{\text{spring}} \equiv U_{\text{spring}}(x_f) - U_{\text{spring}}(x_0) = -W_{\text{spring}} = \frac{1}{2}k(x_f^{\,2} - x_0^{\,2})\,. \tag{13.4.11}$$

Therefore an arbitrary stretch or compression of a spring-body system from equilibrium $x_0 = 0$ to a final position $x_f = x$ changes the potential energy by

$$\Delta U = U_{\text{spring}}(x_f) - U_{\text{spring}}(x_0) = \frac{1}{2}k\,x^2\,. \tag{13.4.12}$$

For the spring-body system, there is an obvious choice of position where the potential energy is zero, the equilibrium position of the spring- body,

$$U_{\text{spring}}(x = 0) \equiv 0\,. \tag{13.4.13}$$

Then with this choice of zero reference potential, the potential energy function is given by

$$U_{\text{spring}}(x) = \frac{1}{2}k\,x^2, \text{ with } U_{\text{spring}}(x = 0) \equiv 0\,. \tag{13.4.14}$$

13.4.3 Example: Inverse Square Gravitational Force

Consider a system consisting of two bodies of masses m_1 and m_2 that are separated by a center-to-center distance r. The internal gravitational force between the two bodies is given by

$$\vec{\mathbf{F}}_{\text{grav}} = -\frac{G\,m_1\,m_2}{r^2}\hat{\mathbf{r}}\,. \tag{13.4.15}$$

The work done by this gravitational force in moving the two bodies from an initial position in which the center of mass of the two bodies are a distance r_0 apart to a final position in which the center of mass of the two bodies are a distance r_f apart is given by

$$W = \int_{r_0}^{r_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{r_0}^{r_f} \left(-\frac{G m_1 m_2}{r^2} \right) dr \,. \tag{13.4.16}$$

Evaluating this integral, we have

$$W = \int_{r_0}^{r_f} \left(-\frac{G m_1 m_2}{r^2} \right) dr = \frac{G m_1 m_2}{r} \bigg|_{r_0}^{r_f} = G m_1 m_2 \left(\frac{1}{r_f} - \frac{1}{r_0} \right). \tag{13.4.17}$$

Therefore the change in potential energy of the system is

$$\Delta U_{\text{gravity}} = -W_{\text{gravity}} = -G m_1 m_2 \left(\frac{1}{r_f} - \frac{1}{r_0} \right) \tag{13.4.18}$$

analogous to the result given in Equation (7.6.9).

We now choose our reference point for the zero of the potential energy to be at infinity, $r_0 = \infty$, with the choice that $U_{\text{gravity}}(r_0 = \infty) \equiv 0$. By making this choice, the term $1/r$ in the expression for the change in potential energy vanishes when $r_0 = \infty$. The gravitational potential energy function for the two bodies when their center of mass to center of mass distance is $r_f = r$ becomes

$$U_{\text{gravity}}(r) = -\frac{G m_1 m_2}{r}, \text{ with } U_{\text{gravity}}(r_0 = \infty) \equiv 0 \,. \tag{13.4.19}$$

There's a subtle aspect to the above calculation. In Equation (13.4.15), our convention is that $d\vec{\mathbf{r}}$ is the differential change in the relative separation of the bodies, but we don't specify which body moves, or if they both do. From the final result, Equation (13.4.19), we might correctly infer that it doesn't matter. Also, we have used the fact that $\hat{\mathbf{r}} \cdot d\vec{\mathbf{r}} = dr$; the component of $d\vec{\mathbf{r}}$ in the radial $\hat{\mathbf{r}}$-direction is the scalar change dr in the separation. Lastly, watch all the minus signs. In Equation (13.4.16), as $r_0 \to \infty$, it might appear that the improper integral should need one more minus sign, but in this case $dr < 0$ and the calculus does the sign accounting for us.

13.5 Mechanical Energy

Definition: **Change in Mechanical Energy**

The total change in the *mechanical energy* of the system is defined to be the sum of the changes of the kinetic and the potential energies,

$$\Delta E_{\text{mechanical}} = \Delta K_{\text{system}} + \Delta U_{\text{system}} \,. \tag{13.5.1}$$

From Equation (13.3.11), for a closed system (no external forces) with only conservative internal forces, the total change in the mechanical energy is zero,

$$\Delta E_{\text{mechanical}} = \Delta K_{\text{system}} + \Delta U_{\text{system}} = 0 . \quad (13.5.2)$$

Equation (13.5.2) is the symbolic statement of what is called the *conservation of mechanical energy*. Recall that the work done by a conservative force in going around a closed path is zero (Equation (13.2.11)), therefore the both the changes in kinetic energy and potential energy are zero when a system returns to its initial state. Throughout the process, the kinetic energy may change into internal potential energy but if the system returns to its initial state, the kinetic energy is completely recoverable. We shall refer to closed system in which processes take place in which only conservative forces act as *completely reversible processes*.

13.5.1 Example: Change in Gravitational Potential Energy Near the Surface of the Earth

Let's consider the example of a body falling near the surface of the earth. Choose our system to consist of the earth and the body. The gravitational force is now an internal conservative force acting inside the system. The initial and final state are specified by the distance separating the body and the center of mass of the earth, and the velocities of the earth and the body. The change in kinetic energy between the initial and final states for the system is

$$\Delta K_{\text{system}} = \Delta K_{\text{earth}} + \Delta K_{\text{body}} , \quad (13.5.3)$$

$$\Delta K_{\text{system}} = \left(\frac{1}{2} m_{\text{e}} (v_{\text{earth},f})^2 - \frac{1}{2} m_{\text{e}} (v_{\text{earth},0})^2\right) + \left(\frac{1}{2} m_{\text{b}} (v_{\text{body},f})^2 - \frac{1}{2} m_{\text{b}} (v_{\text{body},0})^2\right) . (13.5.4)$$

The change of kinetic energy of the earth due to the gravitational interaction between the earth and the body is negligible.[2] The total change in kinetic energy of the system is approximately equal to the change in kinetic energy of the body,

$$\Delta K_{\text{system}} \cong \Delta K_{\text{body}} = \frac{1}{2} m_{\text{b}} (v_{\text{body},f})^2 - \frac{1}{2} m_{\text{b}} (v_{\text{body},0})^2 . \quad (13.5.5)$$

We now define the mechanical energy function for the system

$$E_{\text{mechanical}} = K + U = \frac{1}{2} m_{\text{b}} (v_{\text{body}})^2 + mg\, y, \text{ with } U(y = 0) = 0 , \quad (13.5.6)$$

where K is the kinetic energy and U is the potential energy. The change in mechanical energy is then

[2] A brief outline of a proof of this statement is given in Appendix 13.A.

$$\Delta E_{\text{mechanical}} \equiv E_{\text{mechanical},f} - E_{\text{mechanical},0} = (K_f + U_f) - (K_0 + U_0) . \qquad (13.5.7)$$

When the work done by the external forces is zero and there are no internal non-conservative forces, the total mechanical energy of the system is constant,

$$E_{\text{mechanical},f} = E_{\text{mechanical},0} , \qquad (13.5.8)$$

or equivalently

$$(K_f + U_f) = (K_0 + U_0) . \qquad (13.5.9)$$

The change of kinetic energy of the earth due to the gravitational interaction between the earth and the body is negligible. The total change in kinetic energy of the system is approximately equal to the change in kinetic energy of the body,

$$\Delta K_{\text{system}} \cong \Delta K_{\text{body}} = \frac{1}{2} m_{\text{b}} (v_{\text{body},f})^2 - \frac{1}{2} m_{\text{b}} (v_{\text{body},0})^2 . \qquad (13.5.10)$$

13.6 Spring Force Energy Diagram

The spring force on a body is a restoring force $\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} = -k\,x\,\hat{\mathbf{i}}$ where we choose a coordinate system with the equilibrium position at $x_0 = 0$ and x is the amount the spring has been stretched $(x > 0)$ or compressed $(x < 0)$ from its equilibrium position. The negative of the work done by the spring force in moving the body from x_0 to x defines the potential energy difference

$$U(x) - U(x_0) = -\int_{x_0}^{x} F_x\,dx = \frac{1}{2} k (x^2 - x_0{}^2) . \qquad (13.6.1)$$

The first fundamental theorem of calculus states that

$$U(x) - U(x_0) = \int_{x_0}^{x} \frac{dU}{dx} dx . \qquad (13.6.2)$$

Comparing Equation (13.6.1) with Equation (13.6.2) shows that the force is the negative derivative (with respect to position) of the potential energy,

$$F_x = -\frac{dU(x)}{dx} . \qquad (13.6.3)$$

Choose the zero reference point for the potential energy to be at the equilibrium position, $U(x_0 = 0) \equiv 0$. Then the potential energy function becomes

$$U(x) = \frac{1}{2} k x^2 . \tag{13.6.4}$$

From this, we obtain the spring force law as

$$F_x = -\frac{dU(x)}{dx} = -\frac{d}{dx}\left(\frac{1}{2} k x^2\right) = -k x . \tag{13.6.5}$$

In Figure 13.7 we plot the potential energy function for the spring force as function of x with $U(x_0 = 0) \equiv 0$ (the units are arbitrary).

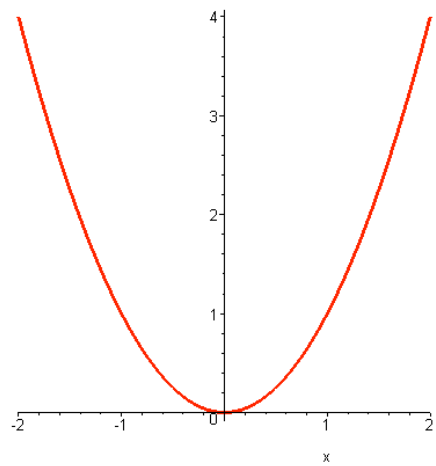


Figure 13.7 Graph of potential energy function for the spring.

The minimum of the potential energy function occurs at the point where the first derivative vanishes

$$\frac{dU(x)}{dx} = 0 . \tag{13.6.6}$$

From Equation (13.6.4), the minimum occurs at $x = 0$,

$$0 = \frac{dU(x)}{dx} = k x . \tag{13.6.7}$$

Since the force is the negative derivative of the potential energy, and this derivative vanishes at the minimum, we have that the spring force is zero at the minimum $x = 0$ agreeing with our force law, $F_x\big|_{x=0} = -k x\big|_{x=0} = 0$.

Suppose the potential energy function has positive curvature in the neighborhood of the minimum. If the body is extended a small distance $x > 0$ away from equilibrium, the slope of the potential energy function is positive, $dU(x)/dx > 0$; hence the component of the force is negative since $F_x = -dU(x)/dx < 0$. Thus the body experiences a restoring force towards the minimum point of the potential. If the body is compresses with $x < 0$ then $dU(x)/dx < 0$; the component of the force is positive, $F_x = -dU(x)/dx > 0$, and the body again experiences a force back towards the minimum of the potential energy as in Figure 13.8.

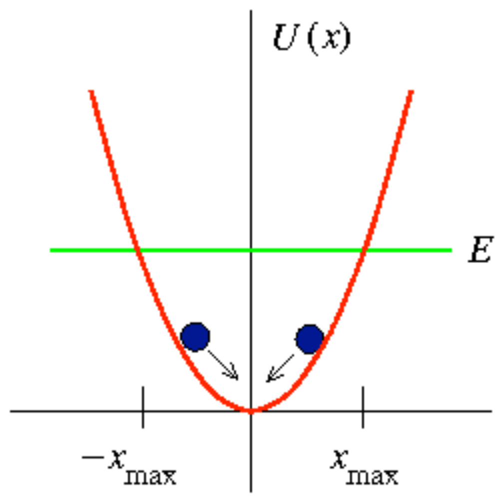


Figure 13.8 Stability diagram for the spring force.

Suppose our spring-body system has no loss of mechanical energy due to dissipative forces such as friction or air resistance. The total mechanical energy at any time will then be the sum of the kinetic energy $K(x)$ and the potential energy $U(x)$

$$E = K(x) + U(x). \tag{13.6.8}$$

Both the kinetic energy and the potential energy are functions of the position of the body with respect to equilibrium. The energy is a constant of the motion and with our choice of $U(x_0 = 0) \equiv 0$, the energy can be either a positive value or zero. When the energy is zero, the body is at rest at the equilibrium position.

In Figure 13.8, we draw a straight horizontal line corresponding to a non-zero positive value for the energy on the graph of potential energy as a function of x. The energy intersects the potential energy function at two points $\{-x_{\max}, x_{\max}\}$ with $x_{\max} > 0$. These points correspond to the maximum compression and maximum extension of the spring, which are called the *turning points*.

The kinetic energy is the difference between the energy and the potential energy,

$$K(x) = E - U(x). \tag{13.6.9}$$

At the turning points, where $E = U(x)$, the kinetic energy is zero. Regions where the kinetic energy is negative, $x < -x_{\max}$ or $x > x_{\max}$ are called the *classically forbidden regions*, which the body can never reach if subject to the laws of classical mechanics. In quantum mechanics, there is a very small probability that the body can be found in the classically forbidden regions.

Example: A particle of mass m, moving in the x-direction, is acting on by a potential

$$U(x) = -U_1\left(\left(\frac{x}{x_1}\right)^3 - \left(\frac{x}{x_1}\right)^2\right), \tag{10}$$

where U_1 and x_1 are positive constants and $U(0) = 0$.

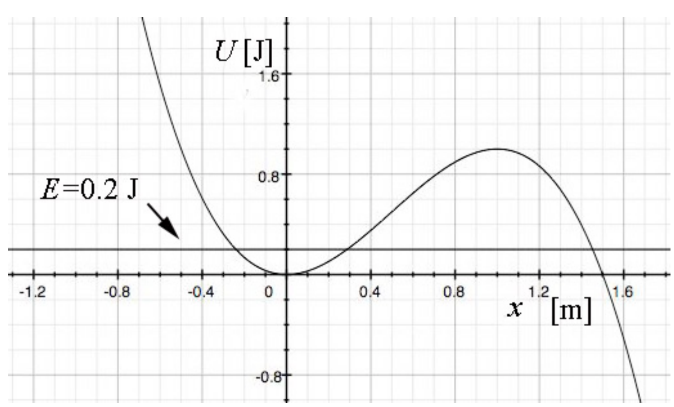
a) Sketch $U(x)/U_1$ as a function of x/x_1.

b) Find the points where the force on the particle is zero. Classify them as stable or unstable. Calculate the value of $U(x)/U_1$ at these equilibrium points.

c) For energies E that lies in $0 < E < (4/27)U_1$ find an equation whose solution yields the turning points along the x-axis about which the particle will undergo periodic motion.

d) Suppose $E = (4/27)U_1$ and that the particle starts at $x = 0$ with speed v_0. Find v_0.

Solution: a) The figure below shows a graph of $U(x)$ vs. x, with the choice of values $x_1 = 1.5\text{ m}$, $U_1 = 27/4\text{ J}$, and $E = 0.2\text{ J}$.



b) The force on the particle is zero at the minimum of the potential which occurs at

$$F_x(x) = -\frac{dU}{dx}(x) = U_1\left(\left(\frac{3}{x_1^{\,3}}\right)x^2 - \left(\frac{2}{x_1^{\,2}}\right)x\right) = 0 \tag{11}$$

which becomes

$$x^2 = (2x_1/3)x\,. \tag{12}$$

We can solve Eq. (12) for the extrema. This has two solutions

$$x = (2x_1/3) \quad \text{and} \quad x = 0\,. \tag{13}$$

The second derivative is given by

$$\frac{d^2U}{dx^2}(x) = -U_1\left(\left(\frac{6}{x_1^{\,3}}\right)x - \left(\frac{2}{x_1^{\,2}}\right)\right). \tag{14}$$

Evaluating the second derivative at $x = (2x_1/3)$ yields a negative quantity

$$\frac{d^2U}{dx^2}(x = (2x_1/3)) = -U_1\left(\left(\frac{6}{x_1^{\,3}}\right)\frac{2x_1}{3} - \left(\frac{2}{x_1^{\,2}}\right)\right) = -\frac{2U_1}{x_1^{\,2}} < 0 \tag{15}$$

indicating the solution $x = (2x_1/3)$ represents a local maximum and hence is an unstable point. At $x = (2x_1/3)$, the potential energy is given by the value $U((2x_1/3)) = (4/27)U_1$.

Evaluating the second derivative at $x = 0$ yields a positive quantity

$$\frac{d^2U}{dx^2}(x = 0) = -U_1\left(\left(\frac{6}{x_1^{\,3}}\right)0 - \left(\frac{2}{x_1^{\,2}}\right)\right) = \frac{2U_1}{x_1^{\,2}} > 0 \tag{16}$$

indicating the solution $x = 0$ represents a local minimum and is a stable point. At the local minimum, $x = 0$, the potential energy $U(0) = 0$.

c) Because the kinetic energy $K(x) = E - U(x) > 0$ must be always be positive, for energies in the range of

$$U(0) = 0 < E < U(2x_1/3) = \frac{4U_1}{27}\ . \tag{17}$$

the particle will undergo periodic motion, between the values $x_a < x < x_b < 2x_1/3$, where x_a and x_b are the turning points and are solutions to the equation

$$E = U(x) = -U_1\left(\left(\frac{x}{x_1}\right)^3 - \left(\frac{x}{x_1}\right)^2\right) . \qquad (18)$$

For $E > U(2x_1/3) = \dfrac{4U_1}{27}$, Eq. (18) has only one solution x_a and for all values of $x > x_a$ the kinetic energy $K(x) = E - U(x) > 0$ which means that the particle can "escape" to infinity but can never enter the region $x < x_a$.

For $E < U(0) = 0$, the kinetic energy is negative for all values of x i.e. $K(x) = E - U(x) < 0; \quad -\infty < x < +\infty$. All regions of space are forbidden.

d) If the particle has speed v_0 at $x = 0$ where the potential energy is zero $U(0) = 0$, the energy of the particle is constant and equal to kinetic energy

$$E = K(0) = \frac{1}{2}mv_0^2 . \qquad (19)$$

Therefore

$$(4/27)U_1 = \frac{1}{2}mv_0^2 \qquad (20)$$

which we can solve for the speed v_0,

$$v_0 = \sqrt{8U_1/27m} . \qquad (21)$$

Appendix 13.A: Energy Changes Near the Surface of the Earth

Consider Equation (13.5.4) from the text,

$$\Delta K_{\text{system}} = \left(\frac{1}{2}m_{\text{e}}(v_{\text{earth},f})^2 - \frac{1}{2}m_{\text{e}}(v_{\text{earth},0})^2\right) + \left(\frac{1}{2}m_{\text{b}}(v_{\text{body},f})^2 - \frac{1}{2}m_{\text{b}}(v_{\text{body},0})^2\right) .(13.A.1)$$

Re-express this relation in terms of the momenta,

$$
\begin{aligned}
\Delta K_{\text{system}} &= \frac{1}{2} m_{\text{e}} ((v_{\text{earth},f})^2 - (v_{\text{earth},0})^2) + \frac{1}{2} m_{\text{b}} ((v_{\text{body},f})^2 - (v_{\text{body},0})^2) \\
&= \frac{1}{2} [m_{\text{e}} (v_{\text{earth},f}) - (v_{\text{earth},0})][(v_{\text{earth},f}) + (v_{\text{earth},0})] \\
&\quad + \frac{1}{2} [m_{\text{b}} (v_{\text{body},f}) - (v_{\text{body},0})][(v_{\text{body},f}) + (v_{\text{body},0})] \\
&= \frac{1}{2} \Delta p_{\text{earth}} [(v_{\text{earth},f}) + (v_{\text{earth},0})] + \frac{1}{2} \Delta p_{\text{body}} [(v_{\text{body},f}) + (v_{\text{body},0})].
\end{aligned}
\tag{13.A.2}
$$

From conservation of momentum, we have $\left|\Delta p_{\text{earth}}\right| = \left|\Delta p_{\text{body}}\right|$. In the example in the text, the observations were made in the frame of the earth, and so the speed of the earth, both initial and final, are vanishingly small compared to the speed of the falling body. Therefore the first term in square brackets in the last line of (13.A.2) is negligible compared to the second, and the change in kinetic energy (but not momentum!) of the earth may be ignored for the purposes of this example.