Module 12: Work and the Scalar Product

12.1 Scalar Product (Dot Product)

We shall introduce a vector operation, called the "dot product" or "scalar product" that takes any two vectors and generates a scalar quantity (a number). We shall see that the physical concept of work can be mathematically described by the dot product between the force and the displacement vectors.

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Because any two non-collinear vectors form a plane, we define the angle θ to be the angle between the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as shown in Figure 12.1. Note that θ can vary from 0 to π.

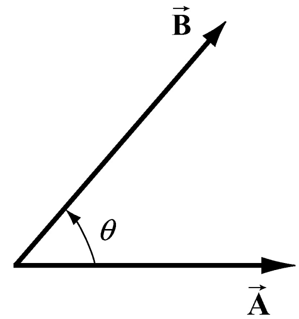


Figure 12.1 Dot product geometry.

Definition: **Dot Product**

The dot product $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the cosine of the angle θ between the two vectors:

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos(\theta) \qquad (12.1.1)$$

Where $A = |\vec{\mathbf{A}}|$ and $B = |\vec{\mathbf{B}}|$ represent the magnitude of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ respectively. The dot product can be positive, zero, or negative, depending on the value of $\cos\theta$. The dot product is always a scalar quantity.

The angle formed by two vectors is therefore

$$\theta = \cos^{-1}\left(\frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{|\vec{\mathbf{A}}||\vec{\mathbf{B}}|}\right) \qquad (12.1.2)$$

The magnitude of a vector $\vec{\mathbf{A}}$ is given by the square root of the dot product of the vector $\vec{\mathbf{A}}$ with itself.

$$\left|\vec{\mathbf{A}}\right| = (\vec{\mathbf{A}} \cdot \vec{\mathbf{A}})^{1/2} \qquad (12.1.3)$$

We can give a geometric interpretation to the dot product by writing the definition as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (A\cos(\theta))\, B \qquad (12.1.4)$$

In this formulation, the term $A\cos\theta$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction of the vector $\vec{\mathbf{B}}$. This projection is shown in Figure 12.2(a). So the dot product is the product of the projection of the length of $\vec{\mathbf{A}}$ in the direction of $\vec{\mathbf{B}}$ with the length of $\vec{\mathbf{B}}$. Note that we could also write the dot product as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A(B\cos(\theta)) \qquad (12.1.5)$$

Now the term $B\cos(\theta)$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction of the vector $\vec{\mathbf{A}}$ as shown in Figure 12.2(b). From this perspective, the dot product is the product of the projection of the length of $\vec{\mathbf{B}}$ in the direction of $\vec{\mathbf{A}}$ with the length of $\vec{\mathbf{A}}$.

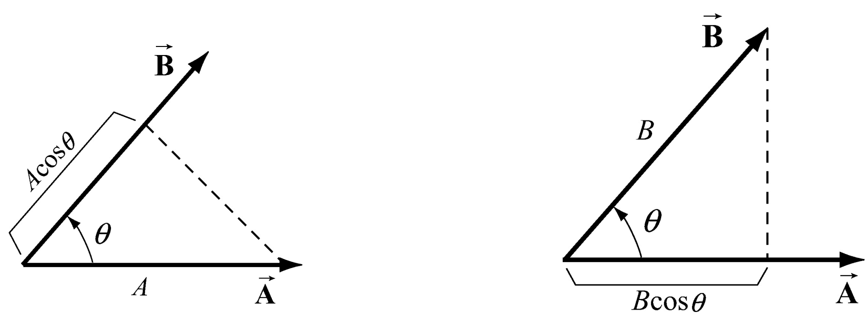


Figure 12.2(a) **and** 12.2(b) **Projection of vectors and the dot product.**

From our definition of the dot product we see that the dot product of two vectors that are perpendicular to each other is zero since the angle between the vectors is $\pi / 2$ and $\cos(\pi / 2) = 0$.

We can calculate the dot product between two vectors in a Cartesian coordinates system as follows. Consider two vectors $\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\,\hat{\mathbf{k}}$ Recall that

$$\begin{aligned} \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \\ \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = 0 \end{aligned} . \qquad (12.1.6)$$

The dot product between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is then

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z \tag{12.1.7}$$

The time derivative of the dot product of two vectors is given by

$$\begin{aligned} \frac{d}{dt}(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}) &= \frac{d}{dt}(A_x B_x + A_y B_y + A_z B_z) \\ &= \frac{d}{dt}(A_x)B_x + \frac{d}{dt}(A_y)B_y + \frac{d}{dt}(A_z)B_z + A_x \frac{d}{dt}(B_x) + A_y \frac{d}{dt}(B_y) + A_z \frac{d}{dt}(B_z), \\ &= \left(\frac{d}{dt}\vec{\mathbf{A}}\right) \cdot \vec{\mathbf{B}} + \vec{\mathbf{A}} \cdot \left(\frac{d}{dt}\vec{\mathbf{B}}\right) \end{aligned} \tag{12.1.8}$$

In particular when $\vec{\mathbf{A}} = \vec{\mathbf{B}}$, then the time derivative of the square of the magnitude of the vector $\vec{\mathbf{A}}$ is given by

$$\frac{d}{dt}A^2 = \frac{d}{dt}(\vec{\mathbf{A}} \cdot \vec{\mathbf{A}}) = \left(\frac{d}{dt}\vec{\mathbf{A}}\right) \cdot \vec{\mathbf{A}} + \vec{\mathbf{A}} \cdot \left(\frac{d}{dt}\vec{\mathbf{A}}\right) = 2\left(\frac{d}{dt}\vec{\mathbf{A}}\right) \cdot \vec{\mathbf{A}}, \tag{12.1.9}$$

12.2 Kinetic Energy and the Dot Product

For an object undergoing three-dimensional motion, the velocity of the object in Cartesian components is given by $\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$. Recall that the magnitude of a vector is given by the square root fo the dot product of the vector with itself,

$$A \equiv \left|\vec{\mathbf{A}}\right| \equiv (\vec{\mathbf{A}} \cdot \vec{\mathbf{A}})^{1/2} = (A_x^2 + A_y^2 + A_z^2)^{1/2} \tag{12.2.1}$$

Therefore the square of the magnitude of the velocity is given by the expression

$$v^2 \equiv (\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) = v_x^2 + v_y^2 + v_z^2 \tag{12.2.2}$$

Hence the kinetic energy of the object is given by

$$K = \frac{1}{2}m(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) \tag{12.2.3}$$

12.3 Work and the Dot Product

Work is an important physical example of the mathematical operation of taking the dot product between two vectors. Recall that when a constant force acts on a body and the point of application of the force undergoes a displacement along the x-axis, only the component of the force along that direction contributes to the work,

$$W = F_x \Delta x. \tag{12.3.1}$$

Suppose we are pulling a body along a horizontal surface with a force $\vec{\mathbf{F}}$. Choose coordinates such that horizontal direction is the x-axis and the force $\vec{\mathbf{F}}$ forms an angle β with the positive x-direction. In Figure 12.3 we show the force vector $\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}$ and the displacement vector of the point of application of the force $\Delta\vec{\mathbf{x}} = \Delta x\,\hat{\mathbf{i}}$. Note that $\Delta\vec{\mathbf{x}} = \Delta x\,\hat{\mathbf{i}}$ is the component of the displacement and hence can be greater, equal, or less than zero (but is shown as greater than zero in the figure for clarity).

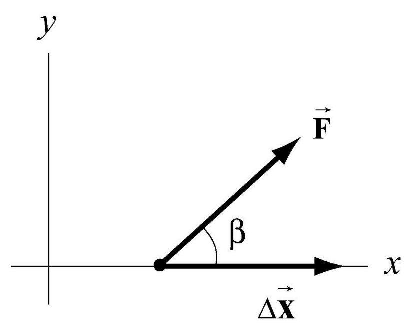


Figure 12.3 Force and displacement vectors

The dot product between the force vector $\vec{\mathbf{F}}$ and the displacement vector $\Delta\vec{\mathbf{x}}$ is

$$\vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}} = (F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}) \cdot (\Delta x\,\hat{\mathbf{i}}) = F_x\,\Delta x. \tag{12.3.2}$$

The work done by the force is then

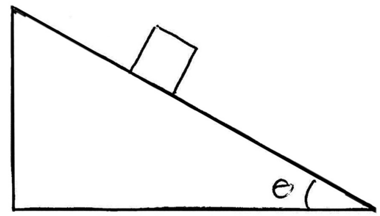
$$\Delta W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}}. \tag{12.3.3}$$

In general, the angle β takes values within the range $-\pi \le \beta \le \pi$ (in Figure 2.3, $0 \le \beta \le \pi/2$). Since the x-component of the force is $F_x = F\cos(\beta)$ where $F = |\vec{\mathbf{F}}|$ denotes the magnitude of $\vec{\mathbf{F}}$, the work done by the force is

$$W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{x}} = (F\cos(\beta))\Delta x. \tag{12.3.4}$$

12.3.1 Worked Example

An object of mass $m = 4.0\,kg$, starting from rest, slides down an inclined plane of length $l = 3.0\,m$. The plane is inclined by an angle of $\theta = 30^0$ to the ground. The coefficient of kinetic friction is $\mu_k = 0.2$.

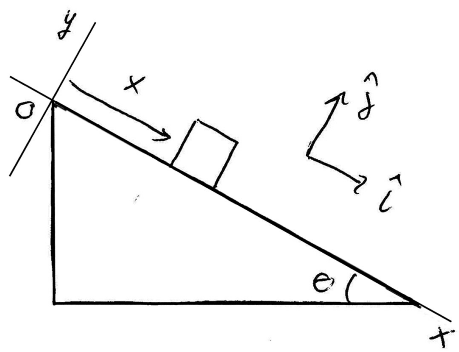


a) What is the work done by each of the three forces while the object is sliding down the inclined plane?

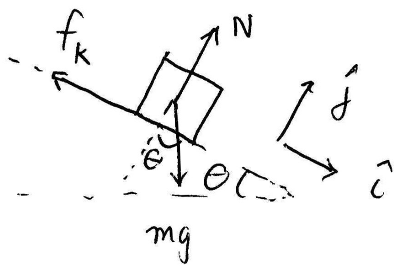
b) For each force, is the work done by the force positive or negative?

c) What is the sum of the work done by the three forces? Is this positive or negative?

Solution: Choose a coordinate system with the origin at the top of the inclined plane and the positive x-direction pointing down the inclined plane, and the positive y-direction pointing towards the upper right as shown in the figure.



While the object is sliding down the inclined plane, three uniform forces act on the object, the gravitational force which points downward and has magnitude $F_g = mg$, the normal force N which is perpendicular to the surface of the inclined plane, and the friction force which opposes the motion and is equal in magnitude to $f_k = \mu_k N$. A force diagram on the object is shown below.

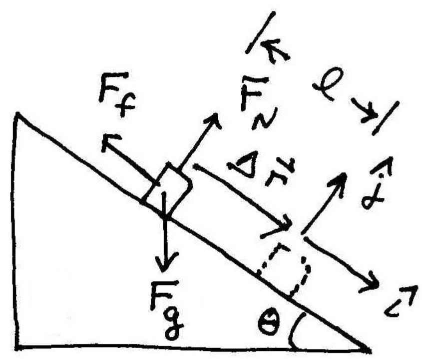


In order to calculate the work we need to determine which forces have a component in the direction of the displacement. Only the component of the gravitational force along the positive x-direction $F_{gx} = mg\sin\theta$ and the friction force are directed along the displacement and therefore contribute to the work. We need to use Newton's Second Law to determine the magnitudes of the normal force. Because the object is constrained to move along the positive x-direction, $a_y = 0$, Newton's Second Law in the $\hat{\mathbf{j}}$-direction

$$N - mg\cos\theta = 0$$

Therefore $N = mg\cos\theta$ and the magnitude of the friction force is $f_k = \mu_k mg\cos\theta$.

With our choice of coordinate system with the origin at the top of the inclined plane and the positive x-direction pointing down the inclined plane, the displacement of the object is given by the vector $\Delta\vec{\mathbf{r}} = \Delta x\,\hat{\mathbf{i}}$.



The vector decomposition of the three forces are $\vec{\mathbf{F}}_g = mg\sin\theta\,\hat{\mathbf{i}} - mg\cos\theta\hat{\mathbf{j}}$, $\vec{\mathbf{F}}_f = -\mu_k mg\cos\theta\hat{\mathbf{i}}$, and $\vec{\mathbf{F}}_N = mg\cos\theta\hat{\mathbf{j}}$. Then the work done by the friction force is negative and given by

$$W_f = \vec{\mathbf{F}}_f \cdot \Delta\vec{\mathbf{r}} = -\mu_k mg\cos\theta\hat{\mathbf{i}} \cdot l\,\hat{\mathbf{i}} = -\mu_k mg\cos\theta l < 0\,.$$

Substituting in the appropriate values yields

$$W_f = -\mu_k mg\cos\theta l = -(0.2)(4.0\,\mathrm{kg})(9.8\,\mathrm{m}\cdot\mathrm{s}^{-2})(3.0\,\mathrm{m})(\cos(30^o)(3.0\,\mathrm{m}) = -20.4\ \mathrm{J}$$

The work done by the gravitational force is positive and given by

$$W_g = \vec{\mathbf{F}}_g \cdot \Delta\vec{\mathbf{r}} = \left(mg\sin\theta\,\hat{\mathbf{i}} - mg\cos\theta\hat{\mathbf{j}}\right) \cdot l\,\hat{\mathbf{i}} = mgl\sin\theta > 0\,.$$

Substituting in the appropriate values yields

$$W_g = mgl\sin\theta = (4.0\,\mathrm{kg})(9.8\,\mathrm{m}\cdot\mathrm{s}^{-2})(3.0\,\mathrm{m})(\sin(30^o)) = 58.8\,\mathrm{J}$$

The work done by the normal force is zero because the normal force is perpendicular the displacement

$$W_N = \vec{\mathbf{F}}_N \cdot \Delta\vec{\mathbf{r}} = mg\cos\theta\hat{\mathbf{j}}\cdot l\,\hat{\mathbf{i}} = 0\,.$$

The scalar sum of the work done by the three forces is then

$$W = W_g + W_f = mgl(\sin\theta - \mu_k\cos\theta)$$

$$W = mgl(\sin\theta - \mu_k\cos\theta) = (4.0\,\mathrm{kg})(9.8\mathrm{m}\cdot\mathrm{s}^{-2})(3.0\,\mathrm{m})(\sin(30^o) - (0.2)(\cos(30^o)) = 38.4\,\mathrm{J}$$

12.4 Work done by a Non-Constant Force Along an Arbitrary Path

Now suppose that a non-constant force $\vec{\mathbf{F}}$ acts on a point-like body of mass m while the body is moving on a three dimensional curved path. The position vector of the body at time t with respect to a choice of origin is $\vec{\mathbf{r}}(t)$. In Figure 12.4 we show the orbit of the body for a time interval $[t_0,t_f]$ moving from an initial position $\vec{\mathbf{r}}_0 \equiv \vec{\mathbf{r}}(t=t_0)$ at time $t=t_0$ to a final position $\vec{\mathbf{r}}_f \equiv \vec{\mathbf{r}}(t=t_f)$ at time $t=t_f$.

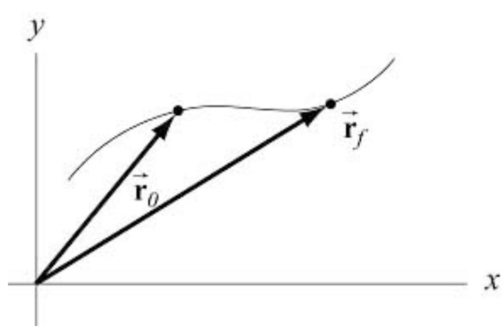


Figure 12.4 Path traced by the motion of a body.

We divide the time interval $[t_0,t_f]$ into N smaller intervals with $[t_{j-1},t_j]$, $j = 1\cdots N$ with $t_N = t_f$. Consider two position vectors $\vec{\mathbf{r}}_j \equiv \vec{\mathbf{r}}(t=t_j)$ and $\vec{\mathbf{r}}_{j-1} \equiv \vec{\mathbf{r}}(t=t_{j-1})$ the displacement vector during the corresponding time interval as $\Delta\vec{\mathbf{r}}_j = \vec{\mathbf{r}}_j - \vec{\mathbf{r}}_{j-1}$.

Let $\vec{\mathbf{F}}$ denote the force acting on the body during the interval $[t_{j-1},t_j]$. The average force in this interval is $(\vec{\mathbf{F}}_j)_{\mathrm{ave}}$ and the average work ΔW_j done by the force during the time interval $[t_{j-1}, t_j]$ is the dot product between the average force vector and the displacement vector,

$$\Delta W_j = (\vec{\mathbf{F}}_j)_{\text{ave}} \cdot \Delta\vec{\mathbf{r}}_j \,. \tag{12.4.1}$$

The force and the displacement vectors for the time interval $[t_{j-1}, t_j]$ are shown in Figure 12.5 (note that the figure uses "i" as index instead of "j" and the subscript "ave" on $(\vec{\mathbf{F}}_j)_{\text{ave}}$ has been suppressed).

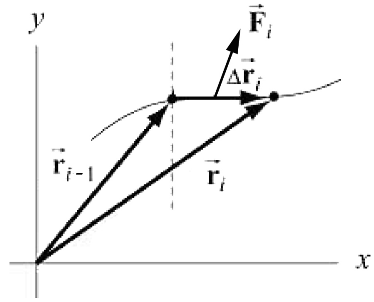


Figure 12.5 An infinitesimal work element.

We calculate the work by adding these scalar contributions to the work for each interval $[t_{j-1}, t_j]$, for $j = 1$ to N,

$$W_N = \sum_{j=1}^{j=N} \Delta W_j = \sum_{j=1}^{j=N} (\vec{\mathbf{F}}_j)_{\text{ave}} \cdot \Delta\vec{\mathbf{r}}_j \,. \tag{12.4.2}$$

We would like to define work in a manner that is independent of the way we divide the interval, so we take the limit as $N \to \infty$ *and* $\left|\Delta\vec{\mathbf{r}}_j\right| \to 0$ for all j. In this limit, as the intervals become smaller and smaller, the distinction between the average force and the actual force vanishes. Thus if this limit exists and is well defined, then the work done by the force is

$$W = \lim_{\substack{N\to\infty \\ \left|\Delta\vec{\mathbf{r}}_j\right|\to 0}} \sum_{j=1}^{j=N} (\vec{\mathbf{F}}_j)_{\text{ave}} \cdot \Delta\vec{\mathbf{r}}_j = \int_{r_0}^{r_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \,. \tag{12.4.3}$$

Notice that this summation involves adding scalar quantities. This limit is called the *line integral* of the force $\vec{\mathbf{F}}$. The symbol $d\vec{\mathbf{r}}$ is called the *infinitesimal vector line element*. At time t, $d\vec{\mathbf{r}}$ is tangent to the orbit of the body and is the limit of the displacement vector $\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)$ as Δt approaches zero. In this limit, the parameter t does not appear in the expression in Equation (12.4.3).

In general this line integral depends on the particular path the body takes between the initial position $\vec{\mathbf{r}}_0$ and the final position $\vec{\mathbf{r}}_f$, which matters when the force $\vec{\mathbf{F}}$ is non-constant in space, and when the contribution to the work can vary over different paths in space. An example is given in the Problems at the end of this Review Module.

We can represent the integral in Equation (12.4.3) explicitly in a coordinate system by specifying the infinitesimal vector line element $d\vec{\mathbf{r}}$ and then explicitly computing the dot product.

Work Integral in Cartesian Coordinates:

In Cartesian coordinates the line element is

$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}\,, \tag{12.4.4}$$

where dx, dy, and dz represent arbitrary displacements in the $\hat{\mathbf{i}}$-, $\hat{\mathbf{j}}$-, and $\hat{\mathbf{k}}$-directions respectively as seen in Figure 12.6a.

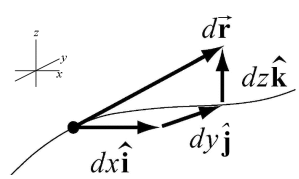


Figure 12.6a A line element in Cartesian coordinates.

The force vector can be represented in vector notation by

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}}\,. \tag{12.4.5}$$

So the infinitesimal work is the sum of the work done by the component of the force times the component of the displacement in each direction

$$dW = F_x dx + F_y dy + F_z dz\,. \tag{12.4.6}$$

Eq. (12.4.6) is just the dot product

$$\begin{aligned} dW &= \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = (F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} + F_z\,\hat{\mathbf{k}}) \cdot (dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}) \\ &= F_x dx + F_y dy + F_z dz \end{aligned}, \tag{12.4.7}$$

The total work is

$$W = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} (F_x dx + F_y dy + F_z dz) = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_x dx + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_y dy + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_z dz \quad (12.4.8)$$

Work Integral in Cylindrical Coordinates:

In cylindrical coordinates the line element is

$$d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\mathbf{k}}, \quad (12.4.9)$$

where dr, $rd\theta$, and dz represent arbitrary displacements in the $\hat{\mathbf{r}}$-, $\hat{\boldsymbol{\theta}}$-, and $\hat{\mathbf{k}}$-directions respectively as seen in Figure 12.6b.

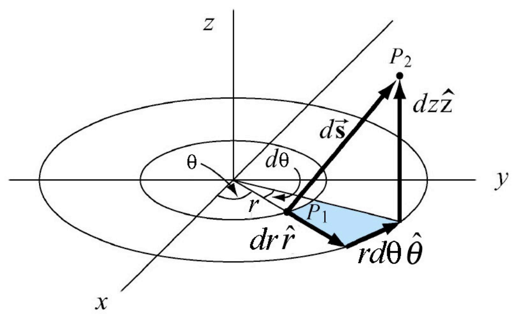


Figure 12.6b displacement vector $d\vec{\mathbf{s}}$ between two points

The force vector can be represented in vector notation by

$$\vec{\mathbf{F}} = F_r\,\hat{\mathbf{r}} + F_\theta\,\hat{\boldsymbol{\theta}} + F_z\,\hat{\mathbf{k}}. \quad (12.4.10)$$

So the infinitesimal work is the dot product

$$\begin{aligned} dW &= \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = (F_r\,\hat{\mathbf{r}} + F_\theta\,\hat{\boldsymbol{\theta}} + F_z\,\hat{\mathbf{k}}) \cdot (dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\mathbf{k}}) \\ &= F_r dr + F_\theta r d\theta + F_z dz \end{aligned}, \quad (12.4.11)$$

The total work is

$$W = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} (F_r dr + F_\theta r d\theta + F_z dz) = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_r dr + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_\theta r d\theta + \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} F_z dz \quad (12.4.12)$$

12.5 Worked Examples

Example 12.5.1: Work Done in a Constant Gravitation Field

The work done in a uniform gravitation field is a fairly straightforward calculation when the body moves in the direction of the field. Suppose the body is moving under the influence of gravity, $\vec{\mathbf{F}} = -mg\,\hat{\mathbf{j}}$ along a parabolic curve

The body begins at the point (x_0, y_0) and ends at the point (x_f, y_f). What is the work done by the gravitation force on the body?

Answer: The infinitesimal line element $d\vec{\mathbf{r}}$ is therefore

$$d\vec{\mathbf{r}} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}. \tag{12.5.1}$$

So the dot product that appears in the line integral can now be calculated,

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -mg\,\hat{\mathbf{j}} \cdot [dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}}] = -mgdy. \tag{12.5.2}$$

This result is not surprising since the force is only in the y-direction. Therefore the only non-zero contribution to the work integral is in the y-direction, with the result that

$$W = \int_{\mathrm{r}_0}^{\mathrm{r}_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{y=y_0}^{y=y_f} F_y dy = \int_{y=y_0}^{y=y_f} -mgdy = -mg(y_f - y_0) \tag{12.5.3}$$

In this case of a constant force, the work integral is independent of path.

12.5.2 Example: Hooke's Law Spring-Body System

Consider a spring-body system lying on a frictionless horizontal surface with one end of the spring fixed to a wall and the other end attached to a body of mass m (Figure 12.7). Calculate the work done by the spring force on body as the body moves from some initial position to some final position.

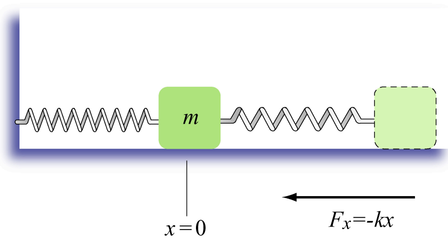


Figure 12.7 A spring-body system.

Solution: Choose the origin at the position of the center of the body when the spring is relaxed (the equilibrium position). Let x be the displacement of the body from the origin. We choose the $+\hat{\mathbf{i}}$ unit vector to point in the direction the body moves when the spring is being stretched (to the right of $x = 0$ in the figure). The spring force on a mass is then given by

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} = -kx\,\hat{\mathbf{i}}\,. \tag{12.5.4}$$

The work done by the spring force on the mass is

$$W_{\text{spring}} = \int_{x=x_0}^{x=x_f} (-kx)\,dx = -\frac{1}{2}k(x_f^{\ 2} - x_0^{\ 2})\,. \tag{12.5.5}$$

Example 12.5.3: Work done by the Inverse Square Gravitation Force

Consider a body of mass m in moving in a fixed orbital plane about the sun. The mass of the sun is m_s. How much work does the gravitation interaction between the sun and the body do on the body during this motion?

Solution: Let's assume that the sun is fixed (it is not fixed but also moves in a very small ellipse in the same orbital plane). and choose a polar coordinate system with the origin at the center of the sun. Initially the body is at a distance r_0 from the center of the sun. In the final configuration the body has moved to a distance $r_f < r_0$ from the center of the sun. The infinitesimal displacement of the body is given by $d\vec{\mathbf{r}} = dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}}$. The gravitation force between the sun and the body is given by

$$\vec{\mathbf{F}}_{grav} = F_{grav}\;\hat{\mathbf{r}} = -\frac{Gm_s m}{r^2}\,\hat{\mathbf{r}}\,. \tag{12.5.6}$$

The infinitesimal work done work done by this gravitation force on the body is given by $dW = \vec{\mathbf{F}}_{grav}\cdot d\vec{\mathbf{r}} = (F_{grav,r}\;\hat{\mathbf{r}})\cdot(dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}}) = F_{grav,r}dr$. Therefore the work done on the object ahs the object moves from r_0 to r_f is given by the integral

$$W = \int_{r_0}^{r_f} \vec{\mathbf{F}}_{grav}\cdot d\vec{\mathbf{r}} = \int_{r_0}^{r_f} F_{grav,r}dr = \int_{r_0}^{r_f}\left(-\frac{Gm_{\text{sun}}m}{r^2}\right)dr\,. \tag{12.5.7}$$

Upon evaluation of this integral, we have for the work

$$W = \int_{r_0}^{r_f}\left(-\frac{Gm_{\text{sun}}m}{r^2}\right)dr = \frac{Gm_{\text{sun}}m}{r}\bigg|_{r_0}^{r_f} = Gm_{\text{sun}}m\left(\frac{1}{r_f} - \frac{1}{r_0}\right) \tag{12.5.8}$$

Since the body has moved closer to the sun, $r_f < r_0$, hence $1/r_f > 1/r_0$. Thus the work done by gravitation force between the sun and the body on the body is positive,

$$W = Gm_{\text{sun}}m\left(\frac{1}{r_f} - \frac{1}{r_0}\right) > 0 \tag{12.5.9}$$

We expect this result because the gravitation force points along the inward radial direction, so the dot product and hence work of the force and the displacement is positive when the body moves closer to the sun. Also we expect that the sign of the work is the same for a body moving closer to the sun as a body falling towards the earth in a constant gravitation field, as seen in Example 4.7.1 above.

Example 12.5.4: Work Done by the Inverse Square Electrical Force

Let's consider two point-like bodies, body 1 and body 2, with charges q_1 and q_2 respectively interacting via the electric force alone. Body 1 is fixed in place while body 2 is free to move in an orbital plane. How much work does the electric force do on the body 2 during this motion? If the charges of the bodies are of the same sign they will repel and $r_f > r_0$. If the charges of the bodies are of opposite signs, the bodies will attract and $r_f < r_0$.

Solution: The calculation in nearly identical to the calculation of work done by the gravitational inverse square force in Example 4.7.3. The most significant difference is that the electric force can be either attractive or repulsive while the gravitation force is always attractive. Once again we choose polar coordinates centered on body 2 in the plane of the orbit. Initially a distance r_0 separates the bodies and in the final state a distance r_f separates the bodies. The electric force between the bodies is given by

$$\vec{\mathbf{F}}_{\text{elec}} = F_{\text{elec}}\,\hat{\mathbf{r}} = F_{elec,r}\,\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2}\,\hat{\mathbf{r}}\,. \tag{12.5.10}$$

The work done by this electric force on the body 2 is given by the integral

$$W = \int_{r_0}^{r_f} \vec{\mathbf{F}}_{elec} \cdot d\vec{\mathbf{r}} = \int_{r_0}^{r_f} F_{elec,r}\,dr = \frac{1}{4\pi\varepsilon_0}\int_{r_0}^{r_f}\frac{q_1q_2}{r^2}\,dr \tag{12.5.11}$$

Evaluating this integral, we have for the work done by the electric force

$$W = \int_{r_0}^{r_f} \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\,dr = -\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\bigg|_{r_0}^{r_f} = -\frac{1}{4\pi\varepsilon_0} q_1 q_2 \left(\frac{1}{r_f} - \frac{1}{r_0}\right). \tag{12.5.12}$$

If the bodies have the opposite signs, $q_1 q_2 < 0$, we expect that the body 2 will move closer to body 1 so $r_f < r_0$, and $1/r_f > 1/r_0$. From our result for the work, the work done by electrical force in moving body 2 is positive,

$$W = -\frac{1}{4\pi\varepsilon_0} q_1 q_2 (\frac{1}{r_f} - \frac{1}{r_0}) > 0. \tag{12.5.13}$$

Once again we see that bodies under the influence of electric forces only will naturally move in the directions in which the force does positive work.

If the bodies have the same sign, then $q_1 q_2 > 0$. They will repel with $r_f > r_0$ and $1/r_f < 1/r_0$. Thus the work is once again positive:

$$W = -\frac{1}{4\pi\varepsilon_0} q_1 q_2 \left(\frac{1}{r_f} - \frac{1}{r_0}\right) > 0. \tag{12.5.14}$$

12.6 Work-Kinetic Energy Theorem in Three Dimensions

Recall our mathematical result that for one-dimensional motion

$$m\int_{\text{initial}}^{\text{final}} a_x dx = m\int_{\text{initial}}^{\text{final}} \frac{dv_x}{dt} dx = m\int_{\text{initial}}^{\text{final}} dv_x \frac{dx}{dt} = m\int_{\text{initial}}^{\text{final}} (dv_x) v_x = \frac{1}{2} m v_{xf}^2 - \frac{1}{2} m v_{xi}^2 . \tag{12.6.1}$$

When we introduce Newton's Second Law in the form $F_x^{\text{total}} = m a_x$, then

$$\int_{\text{initial}}^{\text{final}} F_x^{\text{total}}\, dx = \frac{1}{2} m v_{xf}^2 - \frac{1}{2} m v_{xi}^2 \tag{12.6.2}$$

Eq. (12.6.2) generalizes to the $y-$ and $z-$directions:

$$\int_{\text{initial}}^{\text{final}} F_y^{\text{total}}\, dy = \frac{1}{2} m v_{yf}^2 - \frac{1}{2} m v_{yi}^2 \tag{12.6.3}$$

$$\int_{\text{initial}}^{\text{final}} F_z^{\text{total}}\, dz = \frac{1}{2} m v_{zf}^2 - \frac{1}{2} m v_{zi}^2 \tag{12.6.4}$$

Adding Eqs. (12.6.2), (12.6.3), (12.6.4) and yield

$$\int_{\text{initial}}^{\text{final}} (F_x^{\text{total}}\, dx + F_y^{\text{total}}\, dy + F_z^{\text{total}}\, dz) = \frac{1}{2} m(v_{xf}^2 + v_{yf}^2 + v_{zf}^2) - \frac{1}{2} m(v_{xi}^2 + v_{yi}^2 + v_{zi}^2) \quad (12.6.5)$$

Recall (Eq. (12.4.8)) that the left hand side of Eq. (12.6.5) is the work done by the total force $\vec{\mathbf{F}}^{\text{total}}$ on the object

$$W^{\text{total}} = \int_{\text{initial}}^{\text{final}} dW^{\text{total}} = \int_{\text{initial}}^{\text{final}} (F_x^{\text{total}}\, dx + F_y^{\text{total}}\, dy + F_z^{\text{total}}\, dz) \quad (12.6.6)$$

The right hand side of Eq. (12.6.5) is the change in kinetic energy of the object

$$\Delta K \equiv K_f - K_i = \frac{1}{2} m v_f^{\,2} - \frac{1}{2} m v_0^{\,2} = \frac{1}{2} m(v_{xf}^2 + v_{yf}^2 + v_{zf}^2) - \frac{1}{2} m(v_{xi}^2 + v_{yi}^2 + v_{zi}^2) \quad (12.6.7)$$

Therefore Eq. (12.6.5) is the three dimensional generalization of the work-kinetic energy theorem

$$W^{\text{total}} = \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_f} \vec{\mathbf{F}}^{\text{total}} \cdot d\vec{\mathbf{r}} = K_f - K_i. \quad (12.6.8)$$

When the total work done on an object is positive, the object will increase its speed, and negative work done on a object causes a decrease in speed. When the total work done is zero, the object will maintain a constant speed.

12.7 Instantaneous Power Applied by a Non-Constant Force for Three Dimensional Motion

Recall that for one-dimensional motion, the *instantaneous power* at time t is defined to be the limit of the average power as the time interval $[t, t + \Delta t]$ approaches zero,

$$P = F_{\text{applied},x}\, v_x . \quad (12.7.1)$$

A more general result for the instantaneous power is found by using the expression for dW as given in Equation (12.4.7),

$$P = \frac{dW}{dt} = \frac{\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} . \quad (12.7.2)$$

Time Rate of Change of Kinetic Energy and Power

The time rate of change of the kinetic energy for a body of mass m

$$\frac{dK}{dt} = \frac{1}{2} m \frac{d}{dt}\left(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \right), \tag{12.7.3}$$

Using Eq. (12.1.9) the time rate of change of the kinetic energy for a body of mass m

$$\frac{dK}{dt} = \frac{1}{2} m \frac{d}{dt}\left(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \right) = m \left(\frac{d}{dt} \vec{\mathbf{v}} \right) \cdot \vec{\mathbf{v}} = m \vec{\mathbf{a}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} = P, \tag{12.7.4}$$

consistent with Equation (12.7.2).

Appendix 12.A: Work Done on a System of Two Particles

We shall show that the work done by an internal force in changing a system of two particles of masses m_1 and m_2 respectively from an initial state A to a final state B is equal to

$$W_{\mathrm{c}} = \frac{1}{2} \mu (v_B^2 - v_A^2) \tag{12.A.1}$$

where v_B^2 is the square of the relative velocity in state B, v_A^2 is the square of the relative velocity in state A, and $\mu = m_1 m_2 / (m_1 + m_2)$ is a quantity known as the *reduced mass* of the system.

Proof:

Newton's Second Law applied to body 1 is given expressed as

$$\vec{\mathbf{F}}_{1,2} = m_1 \frac{d^2 \vec{\mathbf{r}}_1}{dt^2} \tag{12.A.2}$$

and to body 2 as

$$\vec{\mathbf{F}}_{2,1} = m_2 \frac{d^2 \vec{\mathbf{r}}_2}{dt^2}. \tag{12.A.3}$$

Divide each side of Equation (12.A.2) by m_1,

$$\frac{\vec{\mathbf{F}}_{1,2}}{m_1} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} \tag{12.A.4}$$

and divide each side of Equation (12.A.3) by m_2,

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_2} = \frac{d^2\vec{\mathbf{r}}_2}{dt^2}. \tag{12.A.5}$$

Subtract Equation (12.A.5) from Equation (12.A.4) yielding

$$\frac{\vec{\mathbf{F}}_{1,2}}{m_1} - \frac{\vec{\mathbf{F}}_{2,1}}{m_2} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} - \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \frac{d^2\vec{\mathbf{r}}_{1,2}}{dt^2} \tag{12.A.6}$$

where as in Section 8.4 of the text, $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$.

Use Newton's Third Law, $\vec{\mathbf{F}}_{1,2} = -\vec{\mathbf{F}}_{2,1}$ on the left hand side of Equation (12.A.6) to obtain

$$\vec{\mathbf{F}}_{1,2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} - \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \frac{d^2\vec{\mathbf{r}}_{1,2}}{dt^2}. \tag{12.A.7}$$

The quantity $d^2\vec{\mathbf{r}}_{1,2} / dt^2$ is the *relative acceleration* of body 1 with respect to body 2.

Define

$$\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2}; \tag{12.A.8}$$

as stated above, the quantity μ is known as the *reduced mass* of the system. Equation (12.A.7) now takes the form

$$\vec{\mathbf{F}}_{1,2} = \mu \frac{d^2\vec{\mathbf{r}}_{1,2}}{dt^2}. \tag{12.A.9}$$

The total work done in the system in displacing the two masses from an initial state A to a final state B is given by

$$W = \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_1 + \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_2, \tag{12.A.10}$$

as shown in Equation (8.4.3) of the text. Recall that in this usage, the labels A and B denote states of the system, not paths. Except for trivial cases, the two bodies will not follow the same path.

From Newton's Third Law, the sum in Equation (12.A.10) becomes

$$W = \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_1 - \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_2 = \int_A^B \vec{\mathbf{F}}_{1,2} \cdot \left(d\,\vec{\mathbf{r}}_1 - d\,\vec{\mathbf{r}}_2\right) = \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\,\vec{\mathbf{r}}_{1,2} \tag{12.A.11}$$

where $d\,\vec{\mathbf{r}}_{1,2}$ is the relative displacement of the two bodies. We can now substitute Newton's Second Law, Equation (12.A.9) for the relative acceleration into Equation (12.A.11),

$$W = \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_{1,2} = \int_A^B \mu \frac{d^2\vec{\mathbf{r}}_{1,2}}{dt^2} \cdot d\,\vec{\mathbf{r}}_{1,2} = \mu \int_A^B \left(\frac{d^2\vec{\mathbf{r}}_{1,2}}{dt^2} \cdot \frac{d\vec{\mathbf{r}}_{1,2}}{dt} \right) dt\,, \tag{12.A.12}$$

where we have used the relation between the differential elements $d\vec{\mathbf{r}}_{1,2} = \dfrac{d\vec{\mathbf{r}}_{1,2}}{dt}dt$.

The product rule for derivatives of the dot product of a vector with itself is given for this case by

$$\frac{1}{2}\frac{d}{dt}\left(\frac{d\vec{\mathbf{r}}_{1,2}}{dt} \cdot \frac{d\vec{\mathbf{r}}_{1,2}}{dt} \right) = \frac{d^2\vec{\mathbf{r}}_{1,2}}{dt^2} \cdot \frac{d\vec{\mathbf{r}}_{1,2}}{dt} \tag{12.A.13}$$

Substitute Equation (12.A.13) into Equation (12.A.12), which then becomes

$$W = \mu \int_A^B \frac{1}{2}\frac{d}{dt}\left(\frac{d\vec{\mathbf{r}}_{1,2}}{dt} \cdot \frac{d\vec{\mathbf{r}}_{1,2}}{dt} \right) dt\,. \tag{12.A.14}$$

Equation (12.A.14) is now the integral of an exact derivative, yielding

$$W = \frac{1}{2}\mu \left(\frac{d\vec{\mathbf{r}}_{1,2}}{dt} \cdot \frac{d\vec{\mathbf{r}}_{1,2}}{dt} \right)\Bigg|_A^B = \frac{1}{2}\mu(\vec{\mathbf{v}}_{1,2} \cdot \vec{\mathbf{v}}_{1,2})\Big|_A^B = \frac{1}{2}\mu(v_B^2 - v_A^2)\,, \tag{12.A.15}$$

where $\vec{\mathbf{v}}_{1,2}$ is the *relative velocity* between the two bodies.

It's important to note that if in the above derivation we had used

$$\begin{aligned}\vec{\mathbf{F}}_{2,1} &= -\vec{\mathbf{F}}_{1,2} \\ \vec{\mathbf{r}}_{2,1} &= \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = -\vec{\mathbf{r}}_{1,2} \\ d\,\vec{\mathbf{r}}_{2,1} &= d\left(\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1\right) = -d\,\vec{\mathbf{r}}_{1,2} \\ \vec{\mathbf{v}}_{1,2} &= -\vec{\mathbf{v}}_{2,1}\end{aligned} \tag{12.A.16}$$

at any point we would have obtained the same result.

Equation (12.A.15) implies that the work done is the change in the kinetic energy of the system,

$$\Delta K = \frac{1}{2}\mu(v_B^2 - v_A^2)\,. \tag{12.A.17}$$

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