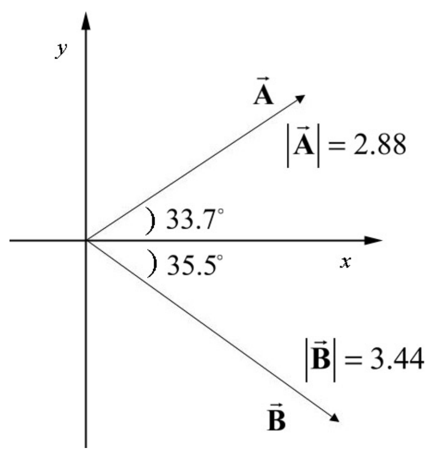
Problem Solving Vectors
Challenge Problem Solutions

Problem 1: Vector Addition

1.1 Consider the two vectors shown in the figure below. The magnitude of $\left|\vec{\mathbf{A}}\right| = 2.88$ and the vector $\vec{\mathbf{A}}$ makes an angle 33.7° with the positive x-axis. The magnitude of $\left|\vec{\mathbf{B}}\right| = 3.44$ and the vector $\vec{\mathbf{B}}$ makes an angle 35.5° with the positive x-axis pointing down to the right as shown in the figure below. Find the x and y components of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.



Problem 1.1 Solution: We need to use $\theta_A = 33.7^\circ$ in order to determine the x and y components of the vector $\vec{\mathbf{A}}$:

$$A_x = \left|\mathbf{A}\right|\cos\theta_A = (2.88)(\cos(33.7^\circ) = 2.40\,,$$
$$A_y = \left|\mathbf{A}\right|\sin\theta_A = (2.88)(\sin(33.7^\circ) = 1.60\,.$$

Thus

$$\vec{\mathbf{A}} = 2.40\,\hat{\mathbf{i}} + 1.60\,\hat{\mathbf{j}}\ .$$

We need to use $\theta_B = -35.5^\circ$ in order to determine the x and y components of the vector $\vec{\mathbf{B}}$:

$$B_x = \left|\mathbf{B}\right|\cos\theta_B = (3.44)(\cos(-35.5^\circ) = 2.80\,,$$

$$B_y = |\mathbf{B}|\cos\theta_B = (3.44)(\sin(-35.5^\circ)) = -2.00 .$$

Thus

$$\vec{\mathbf{B}} = 2.80\ \hat{\mathbf{i}} - 2.00\,\hat{\mathbf{j}} .$$

1.2 Runner

At 2 am one morning you decide to take a jogging run through the MIT buildings. You take 54 seconds to run 250 m along the infinite corridor at MIT from Mass Ave to the end of Building 8, you turn right at the end of the corridor and take 42 seconds to run 178 m to the end of Building 2, and then you turn right, run down the hall for 9 seconds covering 30 m until you to stop.

a) Construct a vector diagram that represents your motion. Indicate your choice of unit vectors.

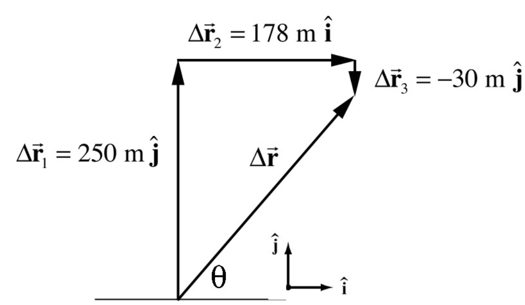
b) What are the directions and magnitudes of your average velocity for each leg of your trip?

c) What is the direction and magnitude your average velocity for the entire trip?

d) Is your average speed for the entire trip greater than, equal to, or less than your average speed for each individual leg? Explain your answer.

Problem 1.2 Solution:

a) The figure below shows the vector diagram with a convenient choice of unit vectors for your motion with the overall displacement vector denoted by $\Delta\vec{\mathbf{r}}$.



b) Your three separate displacements are given by the vectors $\Delta\vec{\mathbf{r}}_1 = 250\text{ m }\hat{\mathbf{j}}$, $\Delta\vec{\mathbf{r}}_2 = 178\text{ m }\hat{\mathbf{i}}$, and $\Delta\vec{\mathbf{r}}_3 = -30\text{ m }\hat{\mathbf{j}}$. Your average velocities for each leg of the trip are

$$\vec{\mathbf{v}}_{\text{ave},1} = \frac{\Delta\vec{\mathbf{r}}_1}{\Delta t_1} = \frac{250\text{ m }\hat{\mathbf{j}}}{54\text{ s}} = 4.6\text{ m}\cdot\text{s}^{-1}\ \hat{\mathbf{j}},$$

$$\vec{\mathbf{v}}_{\text{ave},2} = \frac{\Delta\vec{\mathbf{r}}_2}{\Delta t_2} = \frac{178\text{ m }\hat{\mathbf{i}}}{42\text{ s}} = 4.2\text{ m}\cdot\text{s}^{-1}\,\hat{\mathbf{i}},$$

$$\vec{\mathbf{v}}_{\text{ave},3} = \frac{\Delta\vec{\mathbf{r}}_3}{\Delta t_3} = \frac{-30\text{ m }\hat{\mathbf{j}}}{9\text{ s}} = 3.3\text{ m}\cdot\text{s}^{-1}\,\hat{\mathbf{i}}$$

c) The displacement vector for your entire run is given by

$$\Delta\vec{\mathbf{r}} = \Delta\vec{\mathbf{r}}_1 + \Delta\vec{\mathbf{r}}_2 + \Delta\vec{\mathbf{r}}_3 = 250\text{ m }\hat{\mathbf{j}} + 178\text{ m }\hat{\mathbf{i}} + -30\text{ m }\hat{\mathbf{j}} = 178\text{ m }\hat{\mathbf{i}} + 220\text{ m}\,\hat{\mathbf{j}}.$$

The magnitude of the your displacement from start to finish is

$$\left|\Delta\vec{\mathbf{r}}\right| = \left((178\text{ m})^2 + (220\text{ m})^2\right)^{1/2} = 283\text{ m}.$$

The direction with respect to the $+\hat{\mathbf{i}}$ - direction is given by the angle

$$\theta = \tan^{-1}\left[\frac{220\text{ m}}{178\text{ m}}\right] = 51.0°.$$

The length of time for the trip is

$$\Delta t = \Delta t_1 + \Delta t_2 + \Delta t_3 = 54\text{ s} + 42\text{ s} + 9\text{ s} = 105\text{ s}.$$

The average velocity for your trip is therefore

$$\vec{\mathbf{v}}_{\text{ave}} = \frac{\Delta\vec{\mathbf{r}}}{\Delta t} = \frac{178\text{ m }\hat{\mathbf{i}} + 220\text{ m}\,\hat{\mathbf{j}}}{105\text{ s}} = 1.7\text{ m}\cdot\text{s}^{-1}\,\hat{\mathbf{i}} + 2.1\text{ m}\,\hat{\mathbf{j}}.$$

Your average speed is

$$v_{\text{ave}} = \left|\vec{\mathbf{v}}_{\text{ave}}\right| = \frac{\left|\Delta\vec{\mathbf{r}}\right|}{\Delta t} = \frac{283\text{ m}}{105\text{ s}} = 2.7\text{ m}\cdot\text{s}^{-1}\ .$$

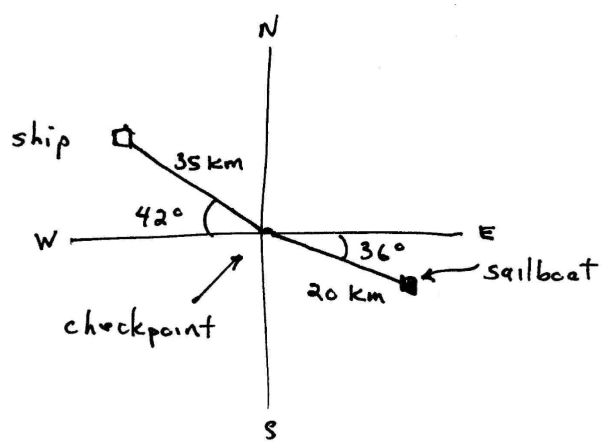
The direction of your average velocity is the same as the direction of the displacement for the entire trip, $è = 51.0°$.

d) Your average speed for the entire trip is the less than the average speed for each individual leg. The reason is that you are not running in a straight line therefore the length of your displacement vector for the entire trip is significantly less than the distance you ran. If you were running at different speeds for each leg and the turns were not as sharp, it may be possible that the average speed for the trip was greater than the average speed on one of the legs.

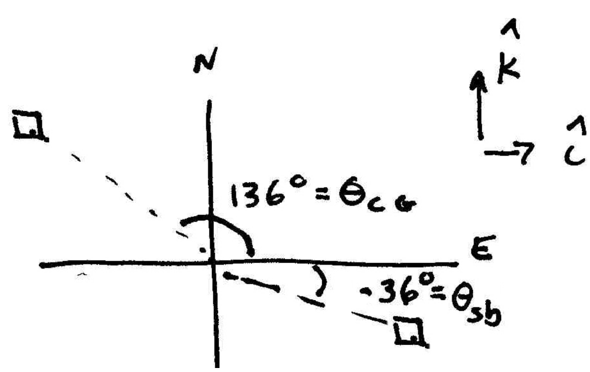
1.3 Sinking Sailboat

A Coast Guard ship is located 35 km away from a checkpoint in a direction 42° north of west. A distressed sailboat located in still water 20 km from the same checkpoint in a direction 36° south of east is about to sink. Draw a diagram indicating the position of both ships. In what direction and how far must the Coast Guard ship travel to reach the sailboat?

Problem 1.3 Solution: The diagram of the set-up is shown below.



Choose the checkpoint as the origin, with North as the positive $\hat{\mathbf{k}}$-direction and East as the positive $\hat{\mathbf{i}}$-direction (see figure below). The Coast Guard ship is then at an angle $\theta_{CG} = 180^\circ - 42^\circ = 138^\circ$ from the checkpoint, and the sailboat is at an angle $\theta_{sb} = -36^\circ$ from the checkpoint.



The position of the Coast Guard ship is then

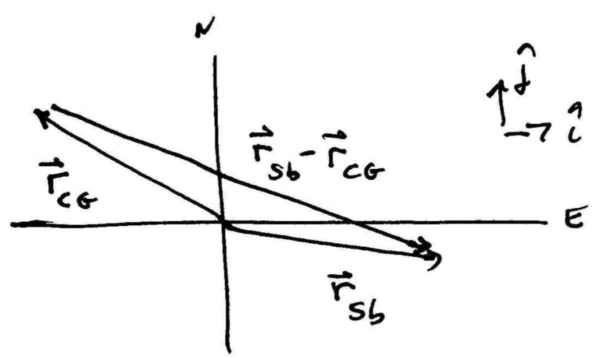
$$\vec{\mathbf{r}}_{CG} = (r_{CG})\left(\cos\theta_{CG}\,\hat{\mathbf{i}} + \sin\theta_{CG}\hat{\mathbf{k}}\right)$$
$$= -26.0\,\text{km}\,\hat{\mathbf{i}} + 23.4\,\text{km}\,\hat{\mathbf{k}}$$

and the position of the sailboat is

$$\vec{\mathbf{r}}_{sb} = (r_{sb})\left(\cos\theta_{sb}\,\hat{\mathbf{i}} + \sin\theta_{sb}\hat{\mathbf{k}}\right)$$
$$= 16.2\,\text{km}\,\hat{\mathbf{i}} - 11.8\,\text{km}\,\hat{\mathbf{k}}.$$

Note that an extra significant figure has been kept for the intermediate calculations. The position vector from the Coast Guard ship to the sailboat is

$$\vec{\mathbf{r}}_{sb} - \vec{\mathbf{r}}_{CG} = \left(16.2\,\text{km}\,\hat{\mathbf{i}} - 11.8\,\text{km}\,\hat{\mathbf{k}}\right) - \left(-26.0\,\text{km}\,\hat{\mathbf{i}} + 23.4\,\text{km}\,\hat{\mathbf{k}}\right)$$
$$= 42.2\,\text{km}\,\hat{\mathbf{i}} - 35.2\,\text{km}\,\hat{\mathbf{k}}.$$



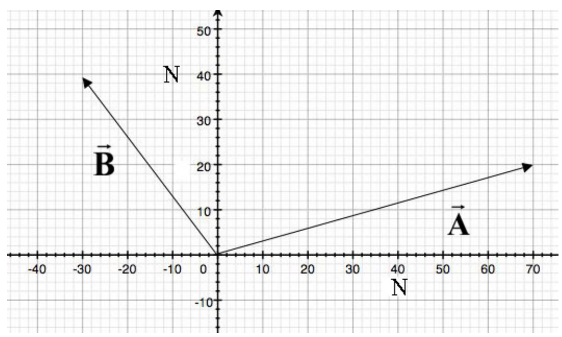
The rescue ship's heading would be the inverse tangent of the ratio of the North and East components of the relative position,

$$\theta_{\text{rescue}} = \tan^{-1}\left(-35.2/42.2\right) = -39.8^\circ,$$

roughly 40° South of East.

1.4 Balancing Forces on a Post

Two horizontal ropes are attached to a post that is stuck in the ground. The ropes pull the post producing the vector forces $\vec{\mathbf{A}} = 70\text{ N}\,\hat{\mathbf{i}} + 20\text{ N}\,\hat{\mathbf{j}}$ and $\vec{\mathbf{B}} = -30\text{ N}\,\hat{\mathbf{i}} + 40\text{ N}\,\hat{\mathbf{j}}$ as shown in the figure. Find the direction and magnitude of the horizontal component of a third force on the post that will make the vector sum of forces on the post equal to zero.



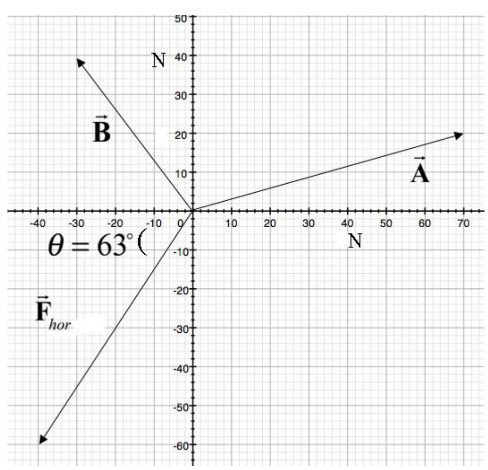
Problem 1.4 Solution: Since the ropes are pulling the post horizontally, the third force must also have a horizontal component that is equal to the negative of the sum of the two horizontal forces exerted by the rope on the post. Since there are additional vertical forces acting on the post due to its contact with the ground and the gravitational force exerted on the post by the earth, we will restrict our attention to the horizontal component of the third force.

Let $\vec{\mathbf{C}}$ denote the sum of the forces due to the ropes. Then we can write the vector $\vec{\mathbf{C}}$ as

$$\vec{\mathbf{C}} = (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\,\hat{\mathbf{j}} = (70\text{ N} + -30\text{ N})\,\hat{\mathbf{i}} + (20\text{ N} + 40\text{ N})\,\hat{\mathbf{j}}$$
$$= (40\text{ N})\,\hat{\mathbf{i}} + (60\text{ N})\,\hat{\mathbf{j}}$$

Therefore the horizontal component of the third force of the post must be equal to

$$\vec{\mathbf{F}}_{hor} = -\vec{\mathbf{C}} = -(\vec{\mathbf{A}} + \vec{\mathbf{B}}) = (-40\text{ N})\,\hat{\mathbf{i}} + (-60\text{ N})\,\hat{\mathbf{j}}.$$



The magnitude is $\left|\vec{\mathbf{F}}_{hor}\right| = \sqrt{(-40\text{ N})^2(-60\text{ N})^2} = 72\text{ N}$. The horizontal component of the force makes an angle

$$\theta = \tan^{-1}\left[\frac{60\ \text{N}}{40\ \text{N}}\right] = 63°$$

as shown in the figure above.

Problem 2:

Problem 2.1 In the methane molecule, CH_4, each hydrogen atom is at the corner of a tetrahedron with the carbon atom at the center. In a coordinate system centered on the carbon atom, if the direction of one of the C—H bonds is described by the vector $\vec{\mathbf{A}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and the direction of an adjacent C—H is described by the vector $\vec{\mathbf{B}} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$, what is the angle between these two bonds.

Problem 2.1 Solution:

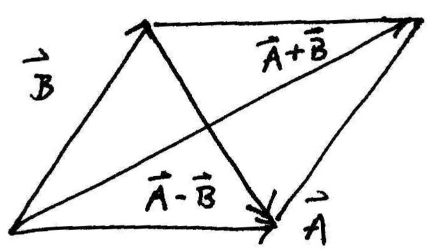
The angle between two vectors is given by $\theta = \cos^{-1}\left(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} / \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\right)$. Therefore

$$\begin{aligned}
\theta &= \cos^{-1}\left(\mathbf{A} \cdot \vec{\mathbf{B}} / \left|\mathbf{A}\right|\left|\vec{\mathbf{B}}\right|\right) \\
&= \cos^{-1}\left((1)(1) + (1)(-1) + (1)(-1)) / ((1)^2 + (1)^2 + (1)^2)^{1/2}((1)^2 + (-1)^2 + (-1)^2)^{1/2}\right) \\
&= \cos^{-1}\left((-1)/2(3)^{1/2}\right) = 106.7^\circ
\end{aligned}$$

Problem 2.2 Show that the diagonals of an equilateral parallelogram are perpendicular.

Problem 2.2 Solution:

Make an equilateral parallelogram using two equal length vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. The vectors $\vec{\mathbf{A}} + \vec{\mathbf{B}}$ and $\vec{\mathbf{A}} - \vec{\mathbf{B}}$ form the diagonals. The dot product $\left(\vec{\mathbf{A}} + \vec{\mathbf{B}}\right) \cdot \left(\vec{\mathbf{A}} - \vec{\mathbf{B}}\right) = \vec{\mathbf{A}} \cdot \vec{\mathbf{A}} - \vec{\mathbf{B}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|^2 - \left|\vec{\mathbf{B}}\right|^2$ **.** Since the vector are equal in length, $\left|\vec{\mathbf{A}}\right|^2 = \left|\vec{\mathbf{B}}\right|^2$. So $\left(\vec{\mathbf{A}} + \vec{\mathbf{B}}\right) \cdot \left(\vec{\mathbf{A}} - \vec{\mathbf{B}}\right) = 0$. Therefore the diagonals are perpendicular.

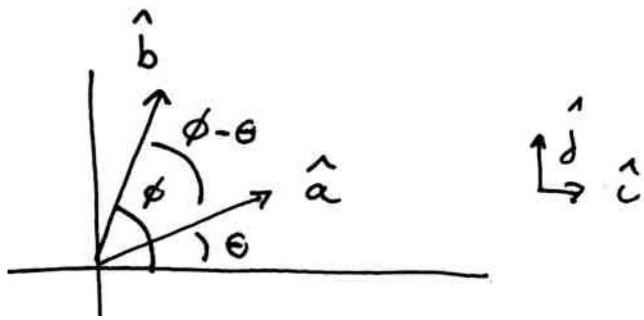


Problem 2.3 Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be unit vectors in the xy plane making angles θ and ϕ with the x axis, respectively. Show that $\hat{\mathbf{a}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}$, $\hat{\mathbf{b}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}$, and using vector algebra prove that $\cos(\phi - \theta) = \cos\theta\cos\phi + \sin\theta\sin\phi$.

Problem 2.3 Solution:
Since $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors, vector decomposition yields that

$$\hat{\mathbf{a}} = |\hat{\mathbf{a}}|\cos\theta\hat{\mathbf{i}} + |\hat{\mathbf{a}}|\sin\theta\hat{\mathbf{j}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}},$$

and similarly for $\hat{\mathbf{b}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}$ (see figure).



Since the angle between the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is equal to $\theta - \phi$, the dot product is given by $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = |\hat{\mathbf{a}}||\hat{\mathbf{b}}|\cos(\phi-\theta) = \cos(\phi-\theta)$. Equivalently in Cartesian coordinates the dot product is also equal to $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)\cdot\left(\cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}\right) = \cos\theta\cos\phi + \sin\theta\sin\phi$.
Therefore $\cos(\phi-\theta) = \cos\theta\cos\phi + \sin\theta\sin\phi$.

Problem 3: Problem Solving Cross Product

Problem 3.1 Find a unit vector perpendicular to $\vec{\mathbf{A}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.

Problem 3.1 Solution: The cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Therefore the unit vectors

$\hat{\mathbf{n}} = \pm\vec{\mathbf{A}} \times \vec{\mathbf{B}} / \left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right|$ are perpendicular to both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. We first calculate

$$\begin{aligned}\vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (A_y B_z - A_z B_y)\,\hat{\mathbf{i}} + (A_z B_x - A_x B_z)\,\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\,\hat{\mathbf{k}} \\ &= ((1)(3) - (-1)(-1))\,\hat{\mathbf{i}} + ((-1)(2) - (1)(3))\,\hat{\mathbf{j}} + ((1)(-1) - (1)(2))\,\hat{\mathbf{k}}. \\ &= 2\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} - 3\,\hat{\mathbf{k}}\end{aligned}$$

We now calculate the magnitude

$$\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = \left(2^2 + 5^2 + 3^2\right)^{1/2} = (38)^{1/2} \quad .$$

Therefore the perpendicular unit vectors are

$$\hat{\mathbf{n}} = \pm\vec{\mathbf{A}} \times \vec{\mathbf{B}} / \left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = \pm\left(2\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} - 3\,\hat{\mathbf{k}}\right)/(38)^{1/2}.$$

Problem 3.2 Let $\vec{\mathbf{A}}$ be an arbitrary vector and let $\hat{\mathbf{n}}$ be a unit vector in some fixed direction. Show that $\vec{\mathbf{A}} = \left(\vec{\mathbf{A}} \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}} + \left(\hat{\mathbf{n}} \times \vec{\mathbf{A}}\right) \times \hat{\mathbf{n}}$.

Problem 3.2 Solution: Let $\vec{\mathbf{A}} = A_n \hat{\mathbf{n}} + A_\perp \hat{\mathbf{e}}$ where A_n is the component $\vec{\mathbf{A}}$ in the direction of $\hat{\mathbf{n}}$, $\hat{\mathbf{e}}$ is the direction of the projection of $\vec{\mathbf{A}}$ in a plane perpendicular to $\hat{\mathbf{n}}$, and $A_\perp$ is the component of $\vec{\mathbf{A}}$ in the direction of $\hat{\mathbf{e}}$. Because $\hat{\mathbf{e}} \cdot \hat{\mathbf{n}} = 0$, we have that $\vec{\mathbf{A}} \cdot \hat{\mathbf{n}} = A_n$. Note that

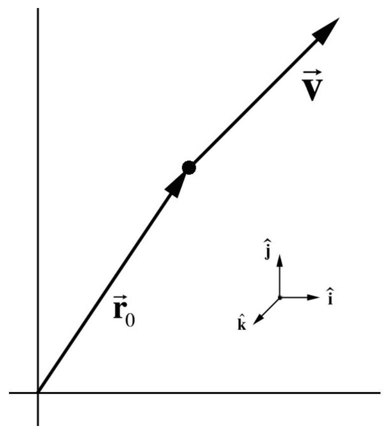
$$\hat{\mathbf{n}} \times \vec{\mathbf{A}} = \hat{\mathbf{n}} \times \left(A_n \hat{\mathbf{n}} + A_\perp \hat{\mathbf{e}}\right) = \hat{\mathbf{n}} \times A_\perp \hat{\mathbf{e}} = A_\perp (\hat{\mathbf{n}} \times \hat{\mathbf{e}}).$$

The unit vector $\hat{\mathbf{n}} \times \hat{\mathbf{e}}$ lies in the plane perpendicular to $\hat{\mathbf{n}}$ and is also perpendicular to $\hat{\mathbf{e}}$. Therefore $(\hat{\mathbf{n}} \times \hat{\mathbf{e}}) \times \hat{\mathbf{n}}$ is also a unit vector that is must be parallel to $\hat{\mathbf{e}}$ (by the right hand rule. So $\left(\hat{\mathbf{n}} \times \vec{\mathbf{A}}\right) \times \hat{\mathbf{n}} = A_\perp \hat{\mathbf{e}}$. Thus

$$\vec{\mathbf{A}} = \left(\vec{\mathbf{A}} \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}} + \left(\hat{\mathbf{n}} \times \vec{\mathbf{A}}\right) \times \hat{\mathbf{n}} = A_n \hat{\mathbf{n}} + A_\perp \hat{\mathbf{e}}.$$

Problem 3.3 Angular Momentum of a Point-like Particle

A particle of mass $m = 2.0\text{ kg}$ moves as shown in the sketch with a uniform velocity $\vec{\mathbf{v}} = 3.0\text{ m}\cdot\text{s}^{-1}\,\hat{\mathbf{i}} + 3.0\text{ m}\cdot\text{s}^{-1}\,\hat{\mathbf{j}}$. At time t, the particle passes through the point $\vec{\mathbf{r}}_{0,m} = 2.0\text{ m}\,\hat{\mathbf{i}} + 3.0\text{ m}\,\hat{\mathbf{j}}$. Find the direction and the magnitude of the angular momentum about the origin at time t.



Problem 3.3 Solution: Choose Cartesian coordinates with unit vectors shown in the figure above. The angular momentum vector $\vec{\mathbf{L}}_0$ of the particle about the origin is given by:

$$\begin{aligned}
\vec{\mathbf{L}}_0 &= \vec{\mathbf{r}}_{0,m} \times \vec{\mathbf{p}} = \vec{\mathbf{r}}_{0,m} \times m\vec{\mathbf{v}} \\
&= \left(2.0\text{m}\,\hat{\mathbf{i}} + 3.0\text{m}\,\hat{\mathbf{j}}\right) \times (2\text{kg})\left(3.0\text{m}\cdot\text{s}^{-1}\hat{\mathbf{i}} + 3.0\text{m}\cdot\text{s}^{-1}\hat{\mathbf{j}}\right) \\
&= 0 + 12\,\text{kg}\cdot\text{m}^2\cdot\text{s}^{-1}\,\hat{\mathbf{k}} - 18\,\text{kg}\cdot\text{m}^2\cdot\text{s}^{-1}(-\hat{\mathbf{k}}) + \vec{\mathbf{0}} \\
&= -6\,\text{kg}\cdot\text{m}^2\cdot\text{s}^{-1}\,\hat{\mathbf{k}}.
\end{aligned}$$

In the above, the relations

$$\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}},\ \ \vec{\mathbf{j}} \times \vec{\mathbf{i}} = -\vec{\mathbf{k}},\ \ \vec{\mathbf{i}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}} \times \vec{\mathbf{j}} = \vec{\mathbf{0}}$$

were used.