Module 16: Momentum

Law II: The change of motion is proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force is impressed altogether and at once or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

Isaac Newton *Principia*

16.1 Introduction

When we apply a force to an object, through pushing, pulling, hitting or otherwise, we are applying that force over a discrete interval of time, Δt. During this time interval, the applied force may be constant, or it may vary in magnitude or direction. Forces may also be applied continuously without interruption, such as the gravitational interaction between the earth and the moon. In this chapter we will investigate the relationship between forces and the time during which they are applied, and in the process learn about the quantity of momentum, the principle of conservation of momentum, and its use in solving a new set of problems in mechanics: collisions.

16.2 Momentum (Quantity of Motion)

Newton defined the quantity of motion or the *momentum*, $\vec{\mathbf{p}}$, to be the product of the mass and the velocity

$$\vec{\mathbf{p}} = m\,\vec{\mathbf{v}}\,. \tag{16.2.1}$$

Momentum is a vector quantity, with direction and magnitude. The direction of momentum is the same as the direction of the velocity. The magnitude of the momentum is the product of the mass and the instantaneous speed.

Units: In the SI system of units, momentum has units of $[\mathrm{kg \cdot m \cdot s^{-1}}]$. There is no special name for this combination of units.

Average Force, Momentum, and Impulse

Suppose you are pushing a cart with a force that is non-uniform, but has an average value $\vec{\mathbf{F}}_{\text{ave}}$ during the time interval Δt. We can find the average acceleration according to Newton's Second Law,

$$\vec{\mathbf{F}}_{\text{ave}} = m\vec{\mathbf{a}}_{\text{ave}}. \tag{16.2.2}$$

Recall that the average acceleration is equal to the change in velocity $\Delta\vec{\mathbf{v}}$ divided by the time interval Δt,

$$\vec{\mathbf{a}}_{\text{ave}} = \frac{\Delta\vec{\mathbf{v}}}{\Delta t}. \tag{16.2.3}$$

Therefore Newton's Second Law can be recast as

$$\vec{\mathbf{F}}_{\text{ave}} = m\vec{\mathbf{a}}_{\text{ave}} = \frac{m\Delta\vec{\mathbf{v}}}{\Delta t}. \tag{16.2.4}$$

The change in momentum is the product of the mass and the change in velocity,

$$\Delta\vec{\mathbf{p}} = m\Delta\vec{\mathbf{v}}. \tag{16.2.5}$$

Newton's Second Law can be restated as follows: the product of the average force acting on an object and the time interval over which the force acts will produce a change in momentum of the object,

$$\vec{\mathbf{F}}_{\text{ave}}\,\Delta t = \Delta\vec{\mathbf{p}}. \tag{16.2.6}$$

This change in momentum is called the *impulse*,

$$\vec{\mathbf{I}} = \vec{\mathbf{F}}_{\text{ave}}\,\Delta t = \Delta\vec{\mathbf{p}}. \tag{16.2.7}$$

Force is a vector quantity; impulse is obtained by multiplying a vector by a scalar, and so impulse is also a vector quantity. The SI units for impulse are $[\text{N}\cdot\text{s}] = [\text{kg}\cdot\text{m}\cdot\text{s}^{-1}]$, which are the same units as momentum.

Non-Constant Force and Impulse

Suppose you now let a cart roll down an inclined plane and bounce against a spring mounted at the bottom of the inclined plane (Figure 16.1). The spring is attached to a force sensor. The force between the spring and the cart is a non-constant force, $\vec{\mathbf{F}}(t)$, applied between times t_0 and t_f.

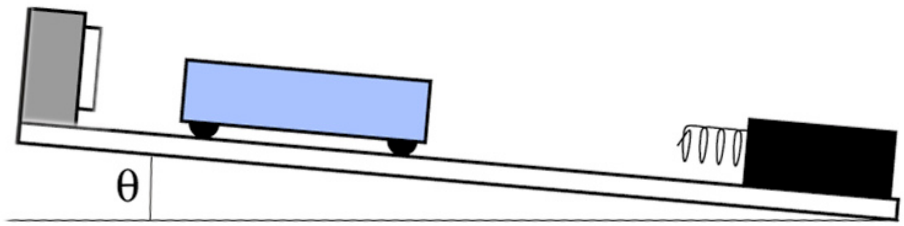

Figure 16.1: Cart sliding down inclined plane and colliding with a spring at the base and reversing motion

In Figure 16.2, we show a sample graph of force vs. time for the cart-spring system as measured by the force sensor during the time the spring is compressed by colliding with the cart.

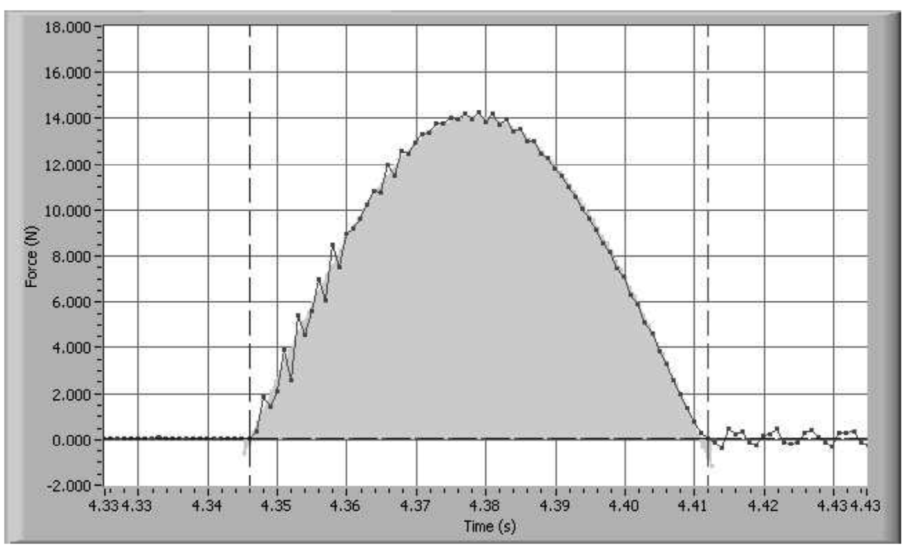

Figure 16.2 A graph of a non-constant force with respect to time.

If we divide up the time interval into N parts, then the impulse is approximately the vector sum of the impulse for each interval,

$$\vec{\mathbf{I}} \cong \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i \, \Delta t_i. \tag{16.2.8}$$

The total impulse $\vec{\mathbf{I}}$ is the limit of this sum as we take smaller and smaller intervals. This limit corresponds to the area under the force vs. time curve,

$$\vec{\mathbf{I}} = \lim_{\Delta t_i \to 0} \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i \, \Delta t_i \equiv \int_{t=t_0}^{t=t_f} \vec{\mathbf{F}}(t)\, dt. \qquad (16.2.9)$$

Since force is a vector quantity, the integral in Equation (16.2.9) is actually three integrals, one for each component of the force.

Using Equation (16.2.7) in Equation (16.2.9) we see that

$$\Delta \vec{\mathbf{p}} = \vec{\mathbf{p}}\left(t_f\right) - \vec{\mathbf{p}}\left(t_0\right) = \int_{t=t_0}^{t=t_f} \vec{\mathbf{F}}(t)\, dt. \qquad (16.2.10)$$

The Fundamental Theorem of Calculus, applied to vectors, then gives

$$\frac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{F}}\left(t\right) \qquad (16.2.11)$$

for both constant and non-constant forces. Equation (16.2.11) is also obtained by taking the limit $\Delta t \to 0$ in Equation (16.2.4). In using either expression, it must be assumed that the mass of the object in question does not change during the interval Δt.

16.2.1 Example: Impulse for a Non-Constant Force

Suppose you push an object for a total time $\Delta t = 1.0\ \mathrm{s}$ in the $+x$-direction. For the first half of the interval, you push with a force that increases linearly with time according to

$$\vec{\mathbf{F}}(t) = \left(b\,t\right)\hat{\mathbf{i}}, \quad 0 \le t \le 0.5\,\mathrm{s} \ \text{ with } \ b = 2.0 \times 10^{1}\ \mathrm{N \cdot s^{-1}}. \qquad (16.2.12)$$

Then for the second half of the interval, you push with a linearly decreasing force,

$$\vec{\mathbf{F}}(t) = \left(d - bt\right)\hat{\mathbf{i}}, \quad 0.5\,\mathrm{s} \le t \le 1.0\,\mathrm{s} \ \text{ with } d = 2.0 \times 10^{1}\ \mathrm{N} \qquad (16.2.13)$$

The force vs. time graph is shown in Figure 16.3. What is the total impulse applied to the object?

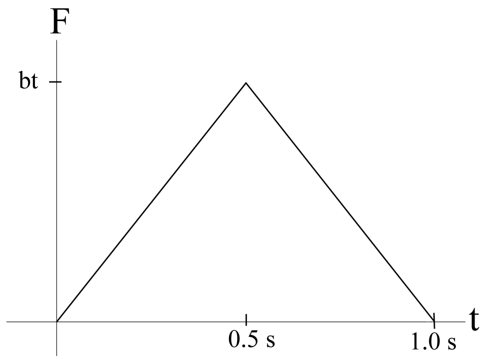

Figure 16.3 Graph of force vs. time

Answer: We can find the impulse by calculating the area under the force vs. time curve. Since the force vs. time graph consists of two triangles, the area under the curve is easy to calculate and is given by

$$\begin{aligned}\vec{\mathbf{I}} &= \left[\frac{1}{2}(b\Delta t)(\Delta t/2) + \frac{1}{2}(b\Delta t)(\Delta t/2)\right]\hat{\mathbf{i}} \\ &= \frac{1}{2}b(\Delta t)^2\,\hat{\mathbf{i}} = \frac{1}{2}(2.0\times 10^1\ \mathrm{N}\cdot\mathrm{s}^{-1})(1.0\,\mathrm{s})^2\,\hat{\mathbf{i}} = (1.0\times 10^1\ \mathrm{N}\cdot\mathrm{s})\hat{\mathbf{i}}.\end{aligned} \tag{16.2.14}$$

16.3 External and Internal Forces and the Change in Momentum of a System

So far we have restricted ourselves to considering how the momentum of an object changes under the action of a force. For example, if we analyze in detail the forces acting on the cart rolling down the inclined plane (Figure 16.1), we determine that there are three forces acting on the cart: the force $\vec{\mathbf{F}}_{\text{cart, spring}}$ the spring applies to the cart; the gravitational interaction $\vec{\mathbf{F}}_{\text{cart, earth}}$ between the cart and the earth; and the contact force $\vec{\mathbf{F}}_{\text{cart, plane}}$ between the inclined plane and the cart. If we define the cart as our *system*, then everything else acts as the *surroundings*. We illustrate this division of system and surroundings in Figure 16.4.

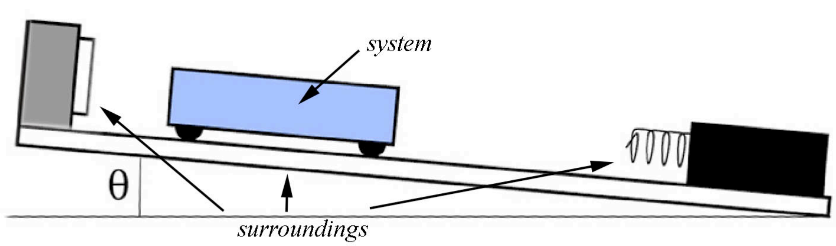

Figure 16.4 A diagram of a cart as a system and its surroundings

The forces acting on the cart are *external* forces. We refer to the vector sum of these external forces that are applied to the system (the cart) as the total external force,

$$\vec{\mathbf{F}}_{\text{ext}}^{\text{total}} = \vec{\mathbf{F}}_{\text{cart,spring}} + \vec{\mathbf{F}}_{\text{cart,earth}} + \vec{\mathbf{F}}_{\text{cart,plane}}. \quad (16.3.1)$$

Then Newton's Second Law applied to the cart, in terms of impulse, is

$$\Delta\vec{\mathbf{p}}_{\text{system}} = \int_{t_0}^{t_f} \vec{\mathbf{F}}_{\text{ext}}^{\text{total}}\, dt \equiv \vec{\mathbf{I}}_{\text{system}}. \quad (16.3.2)$$

Let's extend our system to two interacting objects, for example the cart and the spring. The forces between the spring and cart are now *internal* forces, $\vec{\mathbf{F}}_{\text{int}}$. Both objects, the cart and the spring, experience these internal forces, which by Newton's Third Law are equal in magnitude and applied in opposite directions. So when we sum up the internal forces for the whole system, they cancel. Thus the total internal force is always zero,

$$\vec{\mathbf{F}}_{\text{int}}^{\text{total}} = \vec{\mathbf{0}}. \quad (16.3.3)$$

External forces are still acting on our system; the gravitational force, the contact force between the inclined plane and the cart, and also a new external force, the force between the spring and the force sensor. The total force acting on the system is the sum of the internal and the external forces. However, as we have shown, the internal forces cancel, so we have that

$$\vec{\mathbf{F}}^{\text{total}} = \vec{\mathbf{F}}_{\text{ext}}^{\text{total}} + \vec{\mathbf{F}}_{\text{int}}^{\text{total}} = \vec{\mathbf{F}}_{\text{ext}}^{\text{total}}. \quad (16.3.4)$$