Translation and Rotation Dynamics

Review: Angular Momentum and Torque for a System of Particles

Change in total angular momentum about a point S equals the total torque about the point S

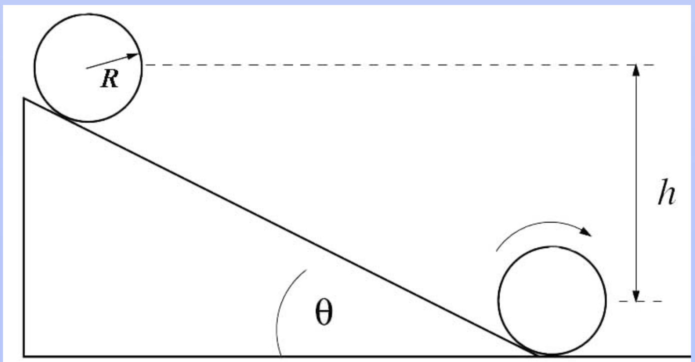
$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \sum_{i=1}^{i=N} \vec{\mathbf{L}}_{S,i} = \sum_{i=1}^{i=N} \left(\frac{d\vec{\mathbf{r}}_{S,i}}{dt} \times \vec{\mathbf{p}}_i + \vec{\mathbf{r}}_{S,i} \times \frac{d\vec{\mathbf{p}}_i}{dt} \right)$$

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \sum_{i=1}^{i=N} \left(\vec{\mathbf{r}}_{S,i} \times \frac{d\vec{\mathbf{p}}_i}{dt} \right) = \sum_{i=1}^{i=N} \left(\vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i \right) = \sum_{i=1}^{i=N} \vec{\boldsymbol{\tau}}_{S,i} = \vec{\boldsymbol{\tau}}_S^{\ \text{total}}$$

$$\frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} = \vec{\boldsymbol{\tau}}_S^{\ \text{total}}$$

Checkpoint Problem: Cylinder on Inclined Plane Torque Method

A hollow cylinder of outer radius R and mass m with moment of inertia I_{cm} about the center of mass starts from rest and moves down an incline tilted at an angle θ from the horizontal. The center of mass of the cylinder has dropped a vertical distance h when it reaches the bottom of the incline. Let g denote the gravitational constant. The coefficient of static friction between the cylinder and the surface is μ_s. The cylinder rolls without slipping down the incline. Using torque about a fixed point lying along the line of contact between the cylinder and the surface, calculate the acceleration of the center of mass of the cylinder when it reaches the bottom of the incline.



Torque for Rotation and Translation

The total torque about *S* is given by

$$\vec{\boldsymbol{\tau}}_S = \vec{\boldsymbol{\tau}}_{S,\mathrm{cm}} + \vec{\boldsymbol{\tau}}_{\mathrm{cm}}$$

where the first term torque about *S* due to the total external force acting at the center-of-mass and the second term is torque about the center-of-mass is only due to the forces as seen in the laboratory frame.

$$\vec{\boldsymbol{\tau}}_{S,\mathrm{cm}} = \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \vec{\mathbf{F}}^{\mathrm{ext}} \qquad \vec{\boldsymbol{\tau}}_{\mathrm{cm}} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\mathrm{cm,i}} \times \vec{\mathbf{F}}_i$$

The total external torque produces an angular acceleration about the center-of-mass

$$\vec{\boldsymbol{\tau}}_{\mathrm{cm}}^{\mathrm{ext}} = I_{\mathrm{cm}} \vec{\alpha}_{\mathrm{cm}} = \frac{d\vec{\mathbf{L}}_{\mathrm{cm}}}{dt}$$

Torque and Angular Momentum for Rotation and Translation

The torque about a point *S* is the time derivative of the angular momentum about *S*,

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{L}}_S}{dt}$$

$$\vec{\boldsymbol{\tau}}_S = \frac{d\vec{\mathbf{R}}_{S,\mathrm{cm}}}{dt} \times \vec{\mathbf{p}}^{\,\mathrm{sys}} + \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \frac{d\vec{\mathbf{p}}^{\,\mathrm{sys}}}{dt} + \sum_{i=1}^{i=N} \frac{d\vec{\mathbf{r}}_{\mathrm{cm,i}}}{dt} \times m_i \vec{\mathbf{v}}_{\mathrm{cm,i}} + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\mathrm{cm,i}} \times \left(\frac{d}{dt} m_i \vec{\mathbf{v}}_{\mathrm{cm,i}} \right)$$

Once again the first and third terms vanish because

$$\frac{d\vec{\mathbf{R}}_{S,\mathrm{cm}}}{dt} \times \vec{\mathbf{p}}^{\,\mathrm{sys}} = \vec{\mathbf{V}}_{\mathrm{cm}} \times m^{\mathrm{total}} \vec{\mathbf{V}}_{\mathrm{cm}} = \vec{\mathbf{0}} \qquad \frac{d\vec{\mathbf{r}}_{\mathrm{cm,i}}}{dt} \times m_i \vec{\mathbf{v}}_{\mathrm{cm,i}} = \vec{\mathbf{v}}_{\mathrm{cm,i}} \times m_i \vec{\mathbf{v}}_{\mathrm{cm,i}} = \vec{\mathbf{0}}$$

So the torque about *S* becomes,

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \frac{d\vec{\mathbf{p}}^{\,\mathrm{sys}}}{dt} + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\mathrm{cm,i}} \times \left(\frac{d}{dt} m_i \vec{\mathbf{v}}_{\mathrm{cm,i}} \right)$$

Torque and Angular Momentum for Rotation and Translation

The torque about a point *S* is

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \frac{d\vec{\mathbf{p}}^{\,\mathrm{sys}}}{dt} + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\mathrm{cm,i}} \times \left(\frac{d}{dt} m_i \vec{\mathbf{v}}_{\mathrm{cm,i}} \right)$$

Recall that the external force is the time change of the momentum of the center-of-mass,

$$\vec{\mathbf{F}}^{\mathrm{ext}} = \frac{d\vec{\mathbf{p}}^{\,\mathrm{sys}}}{dt}$$

So the first term is the torque about *S* due to the total external force acting at the center-of-mass

$$\vec{\boldsymbol{\tau}}_{S,\mathrm{cm}} = \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \vec{\mathbf{F}}^{\mathrm{ext}}$$

Torque for Rotation and Translation

$$\vec{\boldsymbol{\tau}}_S = \vec{\boldsymbol{\tau}}_{S,\mathrm{cm}} + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\mathrm{cm,i}} \times \left(\frac{d}{dt} m_i \vec{\mathbf{v}}_{\mathrm{cm,i}} \right)$$

The time derivative that appears in the second term in the above expression, the time derivative of the momentum of a mass element in the center-of mass-frame, is equal to the force acting on that element which include both inertial and fictitious forces,

$$\frac{d}{dt} m_i \vec{\mathbf{v}}_{\mathrm{cm,i}} = \vec{\mathbf{F}}_i'$$

The torque about the center-of-mass is then

$$\vec{\boldsymbol{\tau}}_{\mathrm{cm}} = \vec{\mathbf{r}}_{\mathrm{cm,i}} \times \left(\frac{d}{dt} m_i \vec{\mathbf{v}}_{\mathrm{cm,i}} \right) = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\mathrm{cm,i}} \times \vec{\mathbf{F}}_i'$$

Torque for Rotation and Translation

When we sum the torques over all the elements in the body, the fictitious forces act at the center-of-mass, so the torque from these fictitious forces is zero, so, the torque about the center-of-mass is only due to the forces as seen in the laboratory frame.

$$\begin{aligned}\vec{\boldsymbol{\tau}}_{\text{cm}} &= \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\text{cm,i}} \times \vec{\mathbf{F}}_i' = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\text{cm,i}} \times \left(\vec{\mathbf{F}}_i - m_i \vec{\mathbf{A}}\right) \\ &= \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\text{cm,i}} \times \vec{\mathbf{F}}_i - \left(\sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_{\text{cm,i}}\right) \times \vec{\mathbf{A}} \\ &= \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\text{cm,i}} \times \vec{\mathbf{F}}_i\end{aligned}$$

Rules to Live By: Angular Momentum and Torque

1) About any fixed point S

$$\vec{\mathbf{L}}_S = \vec{\mathbf{L}}_{about\ cm} + \vec{\mathbf{L}}_{of\ cm} = \vec{\mathbf{L}}_{about\ cm} + \vec{r}_{s,cm} \times m_{total} \vec{v}_{cm}$$

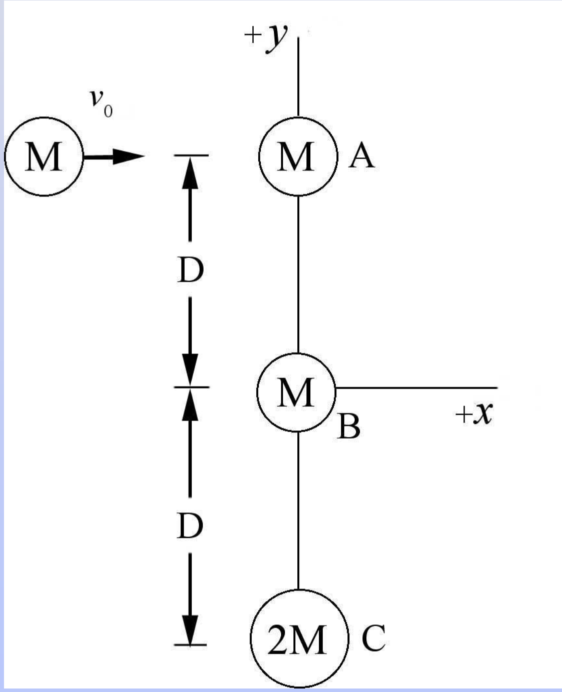
$$\vec{\boldsymbol{\tau}}_S = \sum_i \vec{\boldsymbol{\tau}}_{S,i}^{ext} = \frac{d\vec{\mathbf{L}}_S}{dt}$$

2) Independent of the CM motion, even if $\vec{\mathbf{L}}_{about\ cm}$ and $\vec{\omega}$ are not parallel

$$\vec{\tau}_{about\ cm} = \frac{d\vec{\mathbf{L}}_{about\ cm}}{dt}$$

Checkpoint Problem: Collision

Three point-like objects located at the points A, B and C of respective masses M_A=M , M_B=M and, M_C=2M are shown in the figure. The three objects are initially at rest, oriented along the y-axis, and connected by rods of negligible mass each of length D, forming a rigid body. A fourth object of mass M moving with speed v_0 in the +x-direction collides and sticks to the object at rest at point A. Neglect gravity. The z-axis points out of the page.



a) Describe qualitatively in words how the system moves after the collision: direction, translation and rotation.
b) What is the direction and magnitude of the linear velocity of the center of mass after the collision?
c) What is the magnitude of the angular velocity of the system after the collision?
d) What is the direction and magnitude of the velocity of the object located at the point C immediately after the collision?

Rotational and Translational Comparison

Quantity	Rotation	Translation
Momentum		$p = mV_{\text{cm}}$
Ang Momentum	$L_{\text{cm}} = I_{\text{cm}}\omega$	
Force		$\vec{\mathbf{F}}^{\text{ext}} = d\vec{\mathbf{p}}^{\text{sys}} / dt = m^{\text{total}}\vec{\mathbf{A}}_{cm}$
Torque	$\vec{\boldsymbol{\tau}}_{\text{cm}} = d\vec{\mathbf{L}}_{\text{cm}} / dt$	
Kinetic Energy	$K_{\text{rot}} = (1/2)I_{\text{cm}}\omega^2$ $K_{\text{rot}} = L_{\text{cm}}^2 / 2I_{\text{cm}}$	$K_{\text{trans}} = (1/2)mV_{\text{cm}}^2$ $K_{\text{trans}} = p^2 / 2m$
Work	$W = \int_{\theta_0}^{\theta_f} \tau_S \, d\theta$	$W = \int_0^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$
Power	$P_{rot} = \tau_S \, \omega$	$P = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$

Checkpoint Problem: Cylinder on Inclined Plane translation and rotation methods

A hollow cylinder of outer radius R and mass m with moment of inertia I_{cm} about the center of mass starts from rest and moves down an incline tilted at an angle θ from the horizontal. The center of mass of the cylinder has dropped a vertical distance h when it reaches the bottom of the incline. Let g denote the gravitational constant. The coefficient of static friction between the cylinder and the surface is μ_s. The cylinder rolls without slipping down the incline. Use the method of translation of center of mass and rotation about center of mass to calculate the velocity of the center of mass of the cylinder when it reaches the bottom of the incline.

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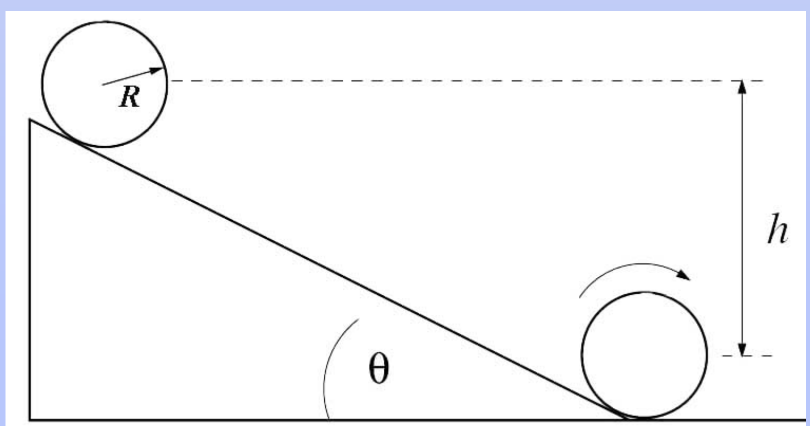
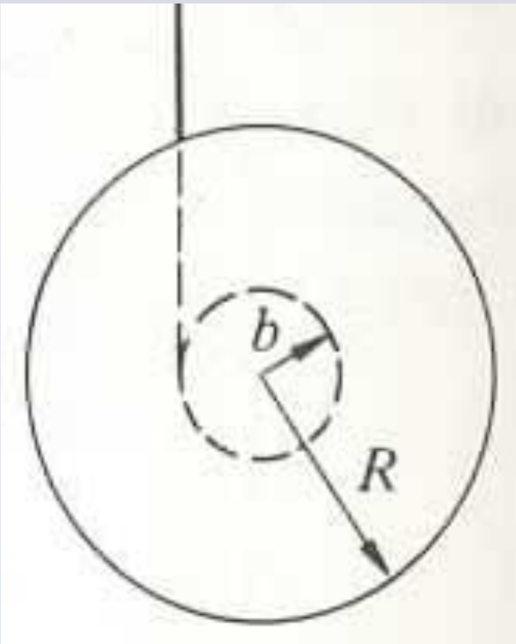


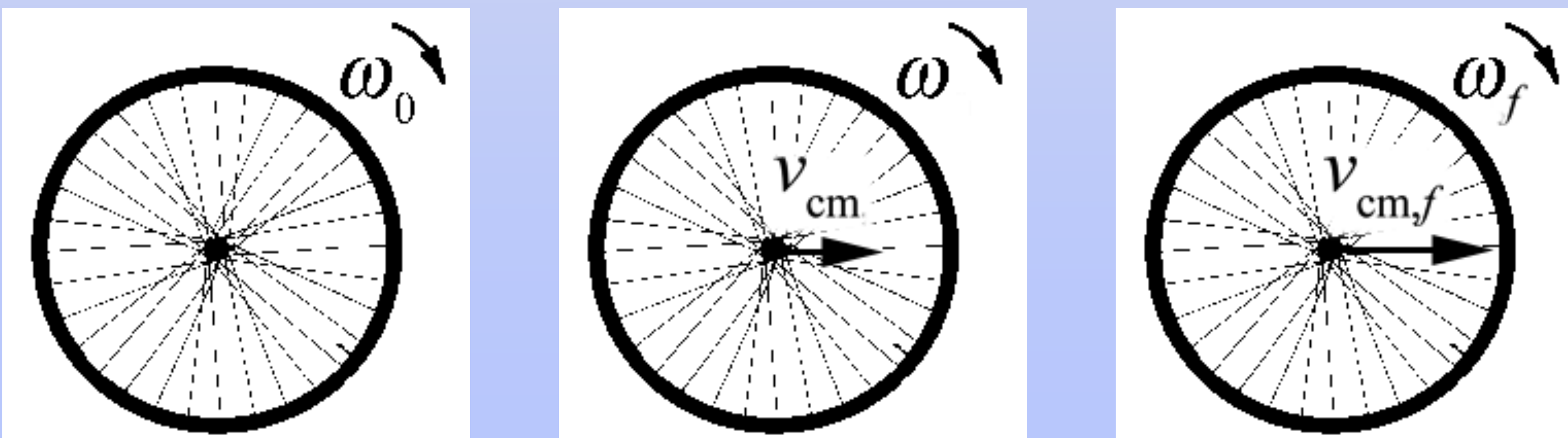
Checkpoint Problem: Descending Yo-Yo

A Yo-Yo of mass m has an axle of radius b and a spool of radius R. It's moment of inertia about the center of mass can be taken to be $I = (1/2)mR^2$ and the thickness of the string can be neglected. The Yo-Yo is released from rest. What is the tension in the string as the yo-yo descends?



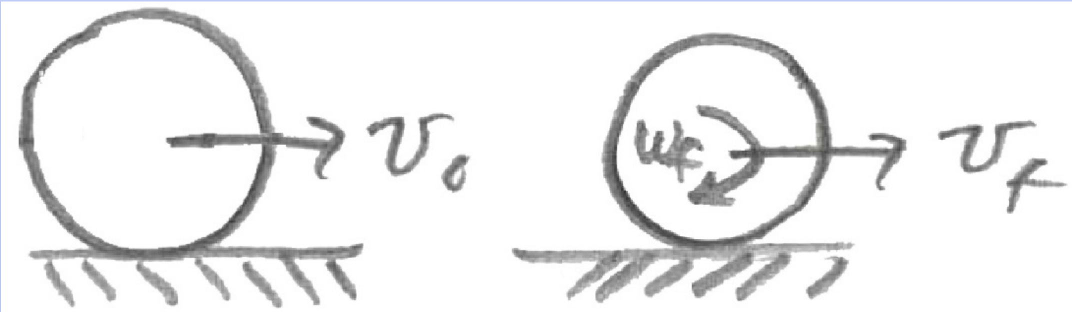
Checkpoint Problem: Bicycle Wheel

Consider a bicycle wheel of radius *R* and mass *m* with moment of inertia I_{cm} about an axis passing perpendicular to the plane of the wheel and through the center-of-mass. The bicycle wheel is initially spinning with angular velocity ω_0 about the center-of-mass. The wheel is lowered to the ground without bouncing. As soon as the wheel touches the level ground, the wheel starts to accelerate forward until it begins to roll without slipping with a final angular velocity ω_f and center-of-mass velocity $v_{cm,f}$. By cleverly choosing a point about which to calculate the angular momentum, use conservation of angular momentum to find is the velocity of the center-of-mass when the wheel rolls without slipping.



Checkpoint Problem: Bowling Ball

A bowling ball of mass m and radius R is initially thrown down an alley with an initial speed v_0, and it slides without rolling but due to friction it begins to roll. The moment of inertia of the ball about its center of mass is $I = (2/5)mR^2$. Use the torque method to find the speed v_f of the bowling ball when it just start to roll without slipping.



Checkpoint Problem: Angular impulse

Two point-like objects are located at the points A, and B, of respective masses M_A=2M and M_B=M, as shown in the figure below. The two objects are initially oriented along the y-axis and connected by a rod of negligible mass of length D, forming a rigid body. Initially the rigid body is at rest. A force of magnitude F acting along the *x* direction is applied to the object at A at t = 0 for a short time interval Δt. Neglect gravity. What is the magnitude of the angular velocity of the system after the force is applied?