Module 5: One Dimensional Kinematics Non-Constant Acceleration

5.1 Integration and Kinematics

Change of Velocity as the Integral of Non-constant Acceleration

When the acceleration is a non-constant function, we would like to know how the x-component of the velocity changes for a time interval $\Delta t = [0, t]$. Since the acceleration is non-constant we cannot simply multiply the acceleration by the time interval. We shall calculate the change in the x-component of the velocity for a small time interval $\Delta t_i \equiv [t_i, t_{i+1}]$ and sum over these results. We then take the limit as the time intervals become very small and the summation becomes an integral of the x-component of the acceleration.

For a time interval $\Delta t = [0, t]$, we divide the interval up into N small intervals $\Delta t_i \equiv [t_i, t_{i+1}]$, where the index $i = 1, 2, \ldots, N$, and $t_1 \equiv 0, t_N \equiv t$. Over the interval Δt_i, we can approximate the acceleration as a constant, $\overline{a_x(t_i)}$. Then the change in the x-component of the velocity is the area under the acceleration *vs*. time curve,

$$\Delta v_{x,i} \equiv v_x(t_{i+1}) - v_x(t_i) = \overline{a_x(t_i)}\,\Delta t_i + E_i \tag{5.1.1}$$

where E_i is the error term (see Figure 5.11a). Then the sum of the changes in the x-component of the velocity is

$$\sum_{i=1}^{i=N} \Delta v_{x,i} = (v_x(t_2) - v_x(t_1 = 0)) + (v_x(t_3) - v_x(t_2)) + \cdots + (v_x(t_N = t) - v_x(t_{N-1})). \tag{5.1.2}$$

In this summation pairs of terms of the form $\left(v_x(t_2) - v_x(t_2)\right) = 0$ sum to zero, and the overall sum becomes

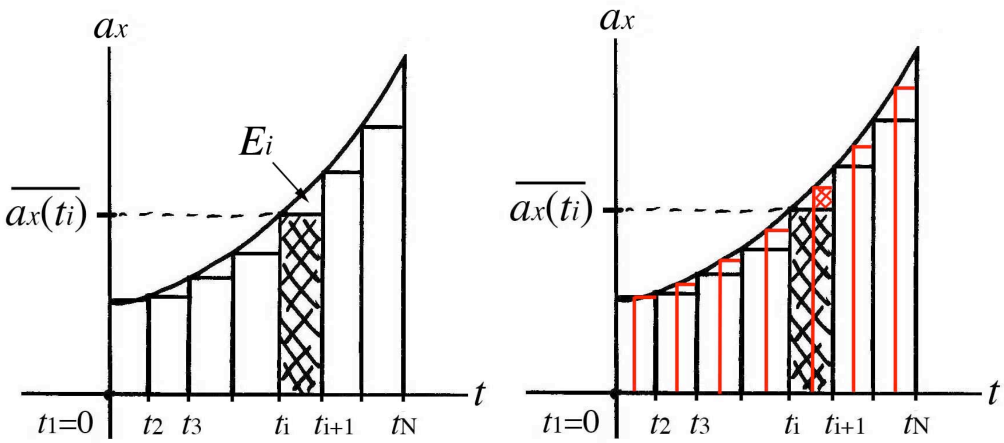
$$v_x(t) - v_x(0) = \sum_{i=1}^{i=N} \Delta v_{x,i}\,. \tag{5.1.3}$$

Substituting Equation (5.1.1) into Equation (5.1.3),

$$v_x(t) - v_x(0) = \sum_{i=1}^{i=N} \Delta v_{x,i} = \sum_{i=1}^{i=N} \overline{a_x(t_i)}\;\Delta t_i + \sum_{i=1}^{i=N} E_i\,. \tag{5.1.4}$$

We now approximate the area under the graph in Figure 5.11a by summing up all the rectangular area terms,

$$\text{Area}_N(a_x,t) = \sum_{i=1}^{i=N} \overline{a_x(t_i)}\,\Delta t_i\,. \qquad (5.1.5)$$



Figures 5.11a and 5.11b Approximating the area under the graph of the x-component of the acceleration *vs*. time

Suppose we make a finer subdivision of the time interval $\Delta t = [0,t]$ by increasing N, as shown in Figure 5.11b. The error in the approximation of the area decreases. We now take the limit as N approaches infinity and the size of each interval Δt_i approaches zero. For each value of N, the summation in Equation (5.1.5) gives a value for $\text{Area}_N(a_x,t)$, and we generate a sequence of values

$$\{\text{Area}_1(a_x,t), \text{Area}_2(a_x,t), \ldots, \text{Area}_N(a_x,t)\}\,. \qquad (5.1.6)$$

The limit of this sequence is the area, $\text{Area}(a_x,t)$, under the graph of the x-component of the acceleration *vs*. time. When taking the limit, the error term vanishes in Equation (5.1.4),

$$\lim_{N\to\infty} \sum_{i=1}^{i=N} E_i = 0\,. \qquad (5.1.7)$$

Therefore in the limit as N approaches infinity, Equation (5.1.4) becomes

$$v_x(t) - v_x(0) = \lim_{N\to\infty}\sum_{i=1}^{i=N}\overline{a_x(t_i)}\;\Delta t_i + \lim_{N\to\infty}\sum_{i=1}^{i=N} E_i = \lim_{N\to\infty}\sum_{i=1}^{i=N}\overline{a_x(t_i)}\;\Delta t_i = \text{Area}(a_x, t)\,, \quad (5.1.8)$$

and thus the change in the x-component of the velocity is equal to the area under the graph of x-component of the acceleration *vs*. time.

Definition: **Integral of acceleration**

The *integral of the x-component of the acceleration* for the interval $[0, t]$ is defined to be the limit of the sequence of areas, $\text{Area}_N(a_x, t)$, and is denoted by

$$\int_{t'=0}^{t'=t} a_x(t')\,dt' \equiv \lim_{\Delta t_i \to 0}\sum_{i=1}^{i=N} a_x(t_i)\;\Delta t_i = \text{Area}(a_x, t)\,. \quad (5.1.9)$$

Equation (5.1.8) shows that the change in the x–component of the velocity is the integral of the x-component of the acceleration with respect to time.

$$v_x(t) - v_x(0) = \int_{t'=0}^{t'=t} a_x(t')\,dt'\,. \quad (5.1.10)$$

Using integration techniques, we can in principle find the expressions for the velocity as a function of time for any acceleration.

Integral of Velocity

We can repeat the same argument for approximating the area $\text{Area}(v_x, t)$ under the graph of the x-component of the velocity *vs*. time by subdividing the time interval into N intervals and approximating the area by

$$\text{Area}_N(a_x, t) = \sum_{i=1}^{i=N}\overline{v_x(t_i)}\;\Delta t_i\,. \quad (5.1.11)$$

The displacement for a time interval $\Delta t = [0, t]$ is limit of the sequence of sums $\text{Area}_N(a_x, t)$,

$$\Delta x = x(t) - x(0) = \lim_{N\to\infty}\sum_{i=1}^{i=N}\overline{v_x(t_i)}\;\Delta t_i\,. \quad (5.1.12)$$

This approximation is shown in Figure 5.12.

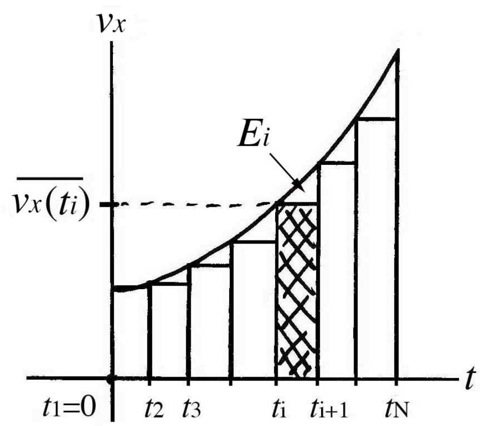


Figure 5.12 Approximating the area under the graph of the x-component of the velocity *vs*. time.

Definition: **Integral of Velocity**

The integral of the x-component of the velocity for the interval $[0,t]$ is the limit of the sequence of areas, $\text{Area}_N(a_x, t)$, and is denoted by

$$\int_{t'=0}^{t'=t} v_x(t')\,dt' \equiv \lim_{\Delta t_i \to 0} \sum_{i=1}^{i=N} v_x(t_i)\,\Delta t_i = \text{Area}(v_x, t). \tag{5.1.13}$$

The displacement is then the integral of the x-component of the velocity with respect to time,

$$\Delta x = x(t) - x(0) = \int_{t'=0}^{t'=t} v_x(t')\,dt'. \tag{5.1.14}$$

Using integration techniques, we can in principle find the expressions for the position as a function of time for any acceleration.

Example:

Let's consider a case in which the acceleration, $a_x(t)$, is not constant in time,

$$a_x(t) = b_0 + b_1 t + b_2 t^2 \tag{5.1.15}$$

The graph of the x-component of the acceleration *vs*. time is shown in Figure 5.13

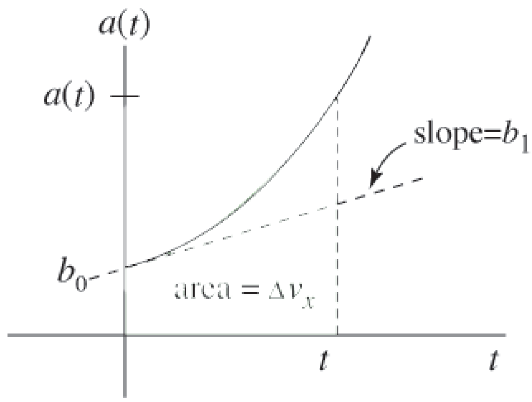


Figure 5.13 A non-constant acceleration *vs*. time graph.

Let's find the change in the x-component of the velocity as a function of time. Denote the initial velocity at $t = 0$ by $v_{x,0} \equiv v_x(t = 0)$. Then,

$$v_x(t) - v_{x,0} = \int_{t'=0}^{t'=t} a_x(t')\,dt' = \int_{t'=0}^{t'=t} (b_o + b_1 t' + b_2 t'^2)\,dt' = b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}. \quad (5.1.16)$$

The x-component of the velocity as a function in time is then

$$v_x(t) = v_{x,0} + b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}. \quad (5.1.17)$$

Denote the initial position by $x_0 \equiv x(t = 0)$. The displacement as a function of time is the integral

$$x(t) - x_0 = \int_{t'=0}^{t'=t} v_x(t')\,dt'. \quad (5.1.18)$$

Use Equation (5.1.17) for the x-component of the velocity in Equation (5.1.18) to find

$$x(t) - x_0 = \int_{t'=0}^{t'=t} \left(v_{x,0} + b_0 t' + \frac{b_1 t'^2}{2} + \frac{b_2 t'^3}{3} \right) dt' = v_{x,0}\, t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}. \quad (5.1.19)$$

Finally the position is then

$$x(t) = x_0 + v_{x,0}\,t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}. \qquad (5.1.20)$$

Example 2: A car is driving through a green light at $t = 0$ located at $x = 0$ with an initial speed $v_{c,0} = 12\ \mathrm{m \cdot s^{-1}}$. The acceleration of the car as a function of time is given by

$$a_c = \begin{cases} 0; & 0 < t < t_1 = 1\ \mathrm{s} \\ -(6\ \mathrm{m \cdot s^{-3}})(t - t_1); & 1\ \mathrm{s} < t < t_2 \end{cases}.$$

a) Find the speed and position of the car as a function of time.
b) Graph the speed and position of the car as a function of time.
c) A bicycle rider is riding at a constant speed of $v_{b,0}$ and at $t = 0$ is $17\ \mathrm{m}$ behind the car. The bicyclist reaches the car when the car just comes to rest. Find the speed of the bicycle.

Solution: a) We need to integrate the acceleration for both intervals. The first interval is easy, the speed is constant. For the second integral we need to be careful about the endpoints of the integral and the fact that the integral is the change in speed so we must subtract $v_c(t_1) = v_{c0}$

$$v_c(t) = \begin{cases} v_{c0}; & 0 < t < t_1 = 1\ \mathrm{s} \\ v_c(t_1) + \int_{t_1}^{t} -(6\ \mathrm{m \cdot s^{-3}})(t - t_1); & 1\ \mathrm{s} < t < t_2 \end{cases}.$$

After integrating we get

$$v_c(t) = \begin{cases} v_{c0}; & 0 < t < t_1 = 1\ \mathrm{s} \\ v_{c0} - (3\ \mathrm{m \cdot s^{-3}})(t - t_1)^2 \Big|_{t_1}^{t}; & 1\ \mathrm{s} < t < t_2 \end{cases}.$$

Now substitute the endpoint so the integral to finally yield

$$v_c(t) = \begin{cases} v_{c0} = 12\ \mathrm{m \cdot s^{-1}}; & 0 < t < t_1 = 1\ \mathrm{s} \\ 12\ \mathrm{m \cdot s^{-1}} - (3\ \mathrm{m \cdot s^{-3}})(t - t_1)^2; & 1\ \mathrm{s} < t < t_2 \end{cases}.$$

For this one dimensional motion the change in position is the integral of the speed so

$$x_c(t) = \begin{cases} x_c(0) + \int_0^{t_1} (12\ \text{m}\cdot\text{s}^{-1})dt; & 0 < t < t_1 = 1\ \text{s} \\ x_c(t_1) + \int_{t_1}^{t} \left(12\ \text{m}\cdot\text{s}^{-1} - (3\text{m}\cdot\text{s}^{-3})(t-t_1)^2\right)dt; & 1\ \text{s} < t < t_2 \end{cases}.$$

Upon integration we have

$$x_c(t) = \begin{cases} x_c(0) + (12\ \text{m}\cdot\text{s}^{-1})t; & 0 < t < t_1 = 1\ \text{s} \\ x_c(t_1) + \left((12\ \text{m}\cdot\text{s}^{-1})(t-t_1) - (1\ \text{m}\cdot\text{s}^{-3})(t-t_1)^3\right)\Big|_{t_1}^{t}; & 1\ \text{s} < t < t_2 \end{cases}.$$

We choose our coordinate system such that $x_c(0) = 0$, therefore $x_c(t_1) = (12\ \text{m}\cdot\text{s}^{-1})(1\ \text{s}) = 12\ \text{m}$. So after substituting in the endpoints of the integration interval we have that

$$x_c(t) = \begin{cases} (12\ \text{m}\cdot\text{s}^{-1})t; & 0 < t < t_1 = 1\ \text{s} \\ 12\ \text{m} + (12\ \text{m}\cdot\text{s}^{-1})(t-t_1) - (1\ \text{m}\cdot\text{s}^{-3})(t-t_1)^3; & 1\ \text{s} < t < t_2 \end{cases}.$$

b) Graph the speed and position of the car as a function of time.

Solution: The graphs of the speed and position are shown below.

c) A bicycle rider is riding at a constant speed of $v_{b,0}$ and at $t = 0$ is $17\ \text{m}$ behind the car. The bicyclist reaches the car when the car just comes to rest. Find the speed of the bicycle.

Solution: we are looking for the instant that t_2 the car has come to rest. So we use our expression for the speed for the interval $1\ \text{s} < t < t_2$,

$$0 = v_c(t_2) = 12\ \text{m}\cdot\text{s}^{-1} - (3\ \text{m}\cdot\text{s}^{-3})(t_2 - t_1)^2.$$

We can solve this for t_2:

$$(t_2 - t_1)^2 = 4\ \text{s}^2.$$

We have two solutions: $(t_2 - t_1) = 2\ \text{s}$ or $(t_2 - t_1) = -2\ \text{s}$. The second solution $t_2 = t_1 - 2\ \text{s} = 1\ \text{s} - 2\ \text{s} = -1\ \text{s}$ does not apply to our time interval and so

$$t_2 = t_1 + 2\text{ s} = 1\text{ s} + 2\text{ s} = 3\text{ s}.$$

During the position of the car at t_2 is then given by

$$\begin{aligned} x_c(t_2) &= 12\text{ m}+(12\text{ m}\cdot\text{s}^{-1})(t_2 - t_1) - (1\text{ m}\cdot\text{s}^{-3})(t_2 - t_1)^3 \\ &= 12\text{ m}+(12\text{ m}\cdot\text{s}^{-1})(2\text{ s}) - (1\text{ m}\cdot\text{s}^{-3})(2\text{ s})^3 = 28\text{ m} \end{aligned}.$$

Since the bicycle is traveling at a constant speed with an initial position $x_{b0} = -17\text{ m}$, the position of the bicycle is given by

$$x_b(t) = -17\text{ m} + v_b t .$$

The bicycle and car intersect at instant $t_2 = 3\text{ s}$:

$$x_b(t_2) = x_c(t_2).$$

Therefore

$$-17\text{ m} + v_b(3\text{ s}) = 28\text{ m} .$$

So the speed of the bicycle is

$$v_b = \frac{(28\text{ m} + 17\text{ m})}{(3\text{ s})} = 15\text{ m}\cdot\text{s}^{-1}.$$