Module 22: Simple Harmonic Oscillation and Torque

22.1 Introduction

We have already used Newton's Second Law or Conservation of Energy to analyze systems like the bloc-spring system that oscillate. We shall now use torque and the rotational equation of motion to study oscillating systems like pendulums or torsional springs.

22.2 Simple Pendulum

A pendulum consists of an object hanging from the end of a string or rigid rod pivoted about the point S. The object is pulled to one side and allowed to oscillate. If the object has negligible size and the string or rod is massless, then the pendulum is called a *simple pendulum*. The force diagram for the simple pendulum is shown in Figure 22.1.

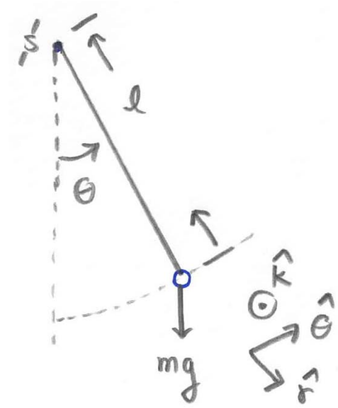


Figure 22.1 A simple pendulum.

The string or rod exerts no torque about the pivot point S. The weight of the object has radial $\hat{\mathbf{r}}$ - and $\hat{\boldsymbol{\theta}}$ - components given by

$$m\vec{\mathbf{g}} = mg(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \tag{22.2.1}$$

and the torque about the pivot point S is given by

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,m} \times m\vec{\mathbf{g}} = l\,\hat{\mathbf{r}} \times m\,g(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = -l\,m\,g\sin\theta\,\hat{\mathbf{k}} \tag{22.2.2}$$

and so the component of the torque in the z -direction (into the page in Figure 22.1 for θ positive, out of the page for θ negative) is

$$(\tau_S)_z = -mgl\sin\theta\,. \tag{22.2.3}$$

The moment of inertia of a point mass about the pivot point S is

$$I_S = ml^2 . \tag{22.2.4}$$

From the rotational dynamical equation is

$$\begin{aligned}(\tau_S)_z &= I_S \alpha \equiv I_S \frac{d^2\theta}{dt^2} \\ -mgl\sin\theta &= ml^2 \frac{d^2\theta}{dt^2}.\end{aligned} \tag{22.2.5}$$

Thus we have the equation of motion for the simple pendulum,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta . \tag{22.2.6}$$

When the angle of oscillation is small, then we can use the small angle approximation

$$\sin\theta \cong \theta ; \tag{22.2.7}$$

the rotational dynamical equation for the pendulum becomes

$$\frac{d^2\theta}{dt^2} \cong -\frac{g}{l}\theta . \tag{22.2.8}$$

This equation is similar to the object-spring simple harmonic oscillator differential equation from **(add reference),**

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x , \tag{22.2.9}$$

which describes the oscillation of a mass about the equilibrium point of a spring. Recall that in **(add reference)**, the angular frequency of oscillation was given by

$$\omega_{\text{spring}} = \sqrt{\frac{k}{m}} . \tag{22.2.10}$$

By comparison, the frequency of oscillation for the pendulum is approximately

$$\omega_{\text{pendulum}} \cong \sqrt{\frac{g}{l}} , \tag{22.2.11}$$

with period

$$T = \frac{2\pi}{\omega_{\text{pendulum}}} \cong 2\pi\sqrt{\frac{l}{g}} \, . \qquad (22.2.12)$$

A procedure for determining the period for larger angles is given in Appendix 22.A.

22.3 Physical Pendulum

A *physical pendulum* consists of a rigid body that undergoes fixed axis rotation about a fixed point S (Figure 22.2). The gravitational force acts at the center of mass of the physical pendulum. Suppose the center of mass is a distance l_{cm} from the pivot point S.

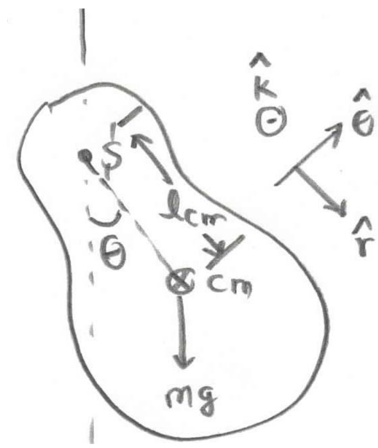


Figure 22.2 Physical pendulum.

The analysis is nearly identical to the simple pendulum. The torque about the pivot point is given by

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,\text{cm}} \times m\vec{\mathbf{g}} = l_{\text{cm}}\hat{\mathbf{r}} \times mg(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = -l_{\text{cm}} mg\sin\theta\,\hat{\mathbf{k}} \, . \qquad (22.3.1)$$

Following the same steps that led from Equation (22.2.2) to Equation (22.2.6), the rotational dynamical equation for the physical pendulum is

$$\begin{aligned} (\tau_S)_z &= I_S\alpha = I_S\frac{d^2\theta}{dt^2} \\ -mgl_{\text{cm}}\sin\theta &= I_S\frac{d^2\theta}{dt^2}. \end{aligned} \qquad (22.3.2)$$

Thus we have the equation of motion for the physical pendulum,

$$\frac{d^2\theta}{dt^2} = -\frac{mgl_{\mathrm{cm}}}{I_S}\sin\theta\ . \tag{22.3.3}$$

As with the simple pendulum, for small angles $\sin\theta \approx \theta$ and Equation (22.3.3) reduces to the simple harmonic oscillator equation with angular frequency

$$\omega_{\mathrm{pendulum}} \approx \sqrt{\frac{mg\,l_{\mathrm{cm}}}{I_S}} \tag{22.3.4}$$

and period

$$T_{\mathrm{physical}} = \frac{2\pi}{\omega_{\mathrm{pendulum}}} \cong 2\pi\sqrt{\frac{I_S}{m\,g\,l_{\mathrm{cm}}}}\ . \tag{22.3.5}$$

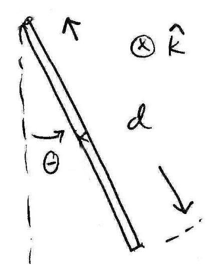
It is sometimes convenient to express the moment of inertia about the pivot point in terms of l_{cm} and I_{cm} using the parallel axis theorem **(add link)**, with $d_{S,\mathrm{cm}} \equiv l_{\mathrm{cm}}$, $I_S = I_{\mathrm{cm}} + ml_{\mathrm{cm}}^2$, with the result

$$T_{\mathrm{physical}} \cong 2\pi\sqrt{\frac{l_{\mathrm{cm}}}{g} + \frac{I_{\mathrm{cm}}}{m\,g\,l_{\mathrm{cm}}}}\ . \tag{22.3.6}$$

Thus, if the object is “small” in the sense that $I_{\mathrm{cm}} \ll ml_{\mathrm{cm}}^2$, the expressions for the physical pendulum reduce to those for the simple pendulum. Note that this is *not* the case shown in Figure 22.11.

22.3.1 Example: Oscillating rod

A physical pendulum consists of a uniform rod of length d and mass m pivoted at one end. The pendulum is initially displaced to one side by a small angle θ_0 and released from rest. You can then approximate $\sin\theta \cong \theta$ (with θ measured in radians). Find the period of the pendulum.



We shall find the period of the pendulum using two different methods.

1. Applying the torque equation about the pivot point.

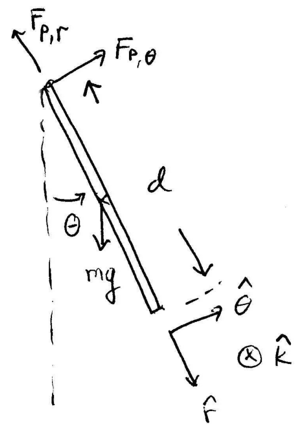
2. Applying the energy equation.

Applying the torque equation about the pivot point.

With our choice of rotational coordinate system the angular acceleration is given by

$$\vec{\boldsymbol{\alpha}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}} . \qquad (22.3.7)$$

The force diagram on the pendulum is shown below. In particular, there is an unknown pivot force, the gravitational force acting at the center of mass of the rod.



The torque about the pivot point is given by

$$\vec{\boldsymbol{\tau}}_P = \vec{\mathbf{r}}_{P,cm} \times m\vec{\mathbf{g}} \;. \qquad (22.3.8)$$

The rod is uniform, therefore the center of mass is a distance $d/2$ from the pivot point. The gravitational force acts at the center of mass, so the torque about the pivot point is given by

$$\vec{\boldsymbol{\tau}}_P = (d/2)\hat{\mathbf{r}} \times mg(-\sin\theta\,\hat{\boldsymbol{\theta}} + \cos\hat{\mathbf{r}}) = -(d/2)mg\sin\theta\,\hat{\mathbf{k}} . \qquad (22.3.9)$$

The rotational dynamical equation (torque equation) is

$$\vec{\boldsymbol{\tau}}_P = I_P\vec{\boldsymbol{\alpha}} . \qquad (22.3.10)$$

Therefore

$$-(d/2)mg\sin\theta\,\hat{\mathbf{k}} = I_P\frac{d^2\theta}{dt^2}\hat{\mathbf{k}} . \qquad (22.3.11)$$

When the angle of oscillation is small, then we can use the small angle approximation

$$\sin\theta \cong \theta\,. \tag{22.3.12}$$

Then the torque equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{(d/2)mg}{I_P}\theta = 0 \tag{22.3.13}$$

which is the simple harmonic oscillator equation. The angular frequency of oscillation for the pendulum is approximately

$$\omega_0 \cong \sqrt{\frac{(d/2)mg}{I_P}}\,. \tag{22.3.14}$$

The moment of inertia of a rod about the end point P is $I_P = (1/3)md^2$ therefore the angular frequency is

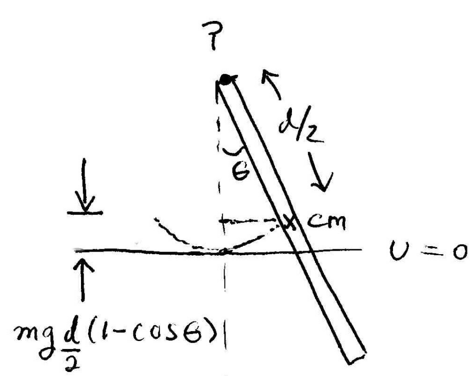
$$\omega_0 \cong \sqrt{\frac{(d/2)mg}{(1/3)md^2}} = \sqrt{\frac{(3/2)g}{d}} \tag{22.3.15}$$

with period

$$T = \frac{2\pi}{\omega_0} \cong 2\pi\sqrt{\frac{2}{3}\frac{d}{g}}\,. \tag{22.3.16}$$

Applying the energy equation.

Take the zero point of gravitational potential energy to be the point where the center of mass of the pendulum is at its lowest point, that is, $\theta = 0$.



When the pendulum is at an angle θ the potential energy is

$$U = m\,g\frac{d}{2}\left(1-\cos\theta\right). \qquad \textbf{(22.3.17)}$$

The kinetic energy of rotation about the pivot point is

$$K_{rot} = \frac{1}{2}I_p\omega^2. \qquad (22.3.18)$$

The mechanical energy is then

$$E = U + K_{rot} = m\,g\frac{d}{2}\left(1-\cos\theta\right)+\frac{1}{2}I_p\omega^2, \qquad (22.3.19)$$

with $I_P = (1/3)md^2$. There are no non-conservative forces acting, so the mechanical energy is constant, and therefore its time derivative is zero

$$0 = \frac{dE}{dt} = m\,g\frac{d}{2}\text{sinè}\frac{d\theta}{dt} + I_p\omega\frac{d\omega}{dt}. \qquad (22.3.20)$$

Recall that $\omega = d\theta/dt$ and $d\omega/dt = d^2\theta/dt^2$ so Eq. (22.3.20) becomes

$$0 = m\,g\frac{d}{2}\sin\theta\omega + I_p\omega\frac{d^2\theta}{dt^2}. \qquad (22.3.21)$$

Therefore two solutions, $\omega = 0$, in which the same remains at the bottom of the swing and

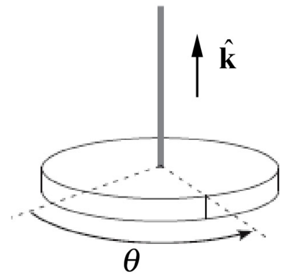
$$0 = m\,g\frac{d}{2}\sin\theta + I_p\frac{d^2\theta}{dt^2} \qquad (22.3.22)$$

Using the small angle approximation, we have the simple harmonic oscillator equation (Eq. (22.3.13))

$$\frac{d^2\theta}{dt^2} + \frac{m\,g(d/2)}{I_p}\theta = 0. \qquad \textbf{(22.3.23)}$$

22.3.2 Example: Torsional Oscillator Solution

A disk with moment of inertia about the center of mass I_{cm} rotates in a horizontal plane. It is suspended by a thin, massless rod. If the disk is rotated away from its equilibrium position by an angle θ, the rod exerts a restoring torque given by $\tau_{cm} = -\gamma\theta$. At $t = 0$, the disk is released from rest at an angular displacement of θ_0. Find the subsequent time dependence of the angular displacement $\theta(t)$.



Solution: Choose a coordinate system such that $\hat{\mathbf{k}}$ is pointing upwards, then the angular acceleration is given by

$$\vec{\boldsymbol{\alpha}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}}. \qquad \textbf{(22.3.24)}$$

The torque about the center of mass is given in the statement of the problem as a restoring torque

$$\vec{\boldsymbol{\tau}}_{cm} = -\gamma\theta\,\hat{\mathbf{k}}. \qquad (22.3.25)$$

Therefore the $\hat{k}$-component of the torque equation $\vec{\boldsymbol{\tau}}_{cm} = I_{cm}\vec{\boldsymbol{\alpha}}$ is

$$-\gamma\theta = I_{cm}\frac{d^2\theta}{dt^2}. \qquad (22.3.26)$$

This is a simple harmonic oscillator equation with solution

$$\theta(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \qquad (22.3.27)$$

where the angular frequency of oscillation is given by

$$\omega_0 = \sqrt{\gamma / I_{cm}}. \qquad (22.3.28)$$

The z -component of the angular velocity is given by

$$\frac{d\theta}{dt}(t) = -\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t). \qquad (22.3.29)$$

The initial conditions at $t = 0$, are that $\theta(t = 0) = A = \theta_0$, and $(d\theta / dt)(t = 0) = \omega_0 B = 0$. Therefore

$$\theta(t) = \theta_0 \cos(\sqrt{\gamma / I_{cm}}\; t)\,. \tag{22.3.30}$$

Appendix 22.A: Higher-Order Corrections to the Period for Larger Amplitudes of a Simple Pendulum

In Section 22.2 we found that using the small angle approximation the period for a simple pendulum is (22.2.12)

$$T = \frac{2\pi}{\omega_{\text{pendulum}}} \cong 2\pi\sqrt{\frac{l}{g}}\,.$$

How good is this approximation? If the pendulum is pulled out to an initial angle θ_0 that is not small (such that our first approximation $\sin\theta \cong \theta$ no longer holds) then our expression for the period is no longer valid. We then would like to calculate the first-order (or higher-order) correction to the period of the pendulum.

Let's first consider the mechanical energy, a conserved quantity in this system. Choose an initial state when the pendulum is released from rest at an angle θ_0; this need not be at time $t = 0$, and in fact later in this derivation we'll see that it's inconvenient to choose this position to be at $t = 0$. Choose for the final state the position and velocity of the bob at an arbitrary time t. Choose the zero point for the potential energy to be at the bottom of the bob's swing (Figure 22.A.1).

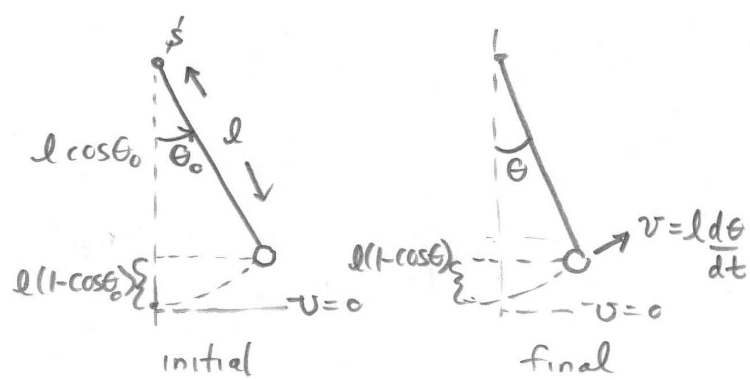


Figure 22.A.1 Energy states for a simple pendulum.

The initial mechanical energy is

$$E_0 = K_0 + U_0 = m\,g\,l(1 - \cos\theta_0)\,. \tag{22.A.1}$$

The tangential velocity of the bob at an arbitrary time t is given by

$$v_{\tan} = l\frac{d\theta}{dt}, \tag{22.A.2}$$

and the kinetic energy at the final state is

$$K_f = \frac{1}{2}mv_{\tan}^2 = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2. \tag{22.A.3}$$

The final mechanical energy is then

$$E_f = K_f + U_f = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + m\,g\,l(1-\cos\theta). \tag{22.A.4}$$

Since the tension in the string is always perpendicular to the displacement of the bob, the tension does no work and mechanical energy is conserved, $E_f = E_0$. Thus

$$\begin{gathered}\frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + m\,g\,l(1-\cos\theta) = m\,g\,l(1-\cos\theta_0)\\ \left(l\frac{d\theta}{dt}\right)^2 = 2\frac{g}{l}(\cos\theta - \cos\theta_0).\end{gathered} \tag{22.A.5}$$

We can solve Equation (22.A.5) for the angular velocity as a function of θ,

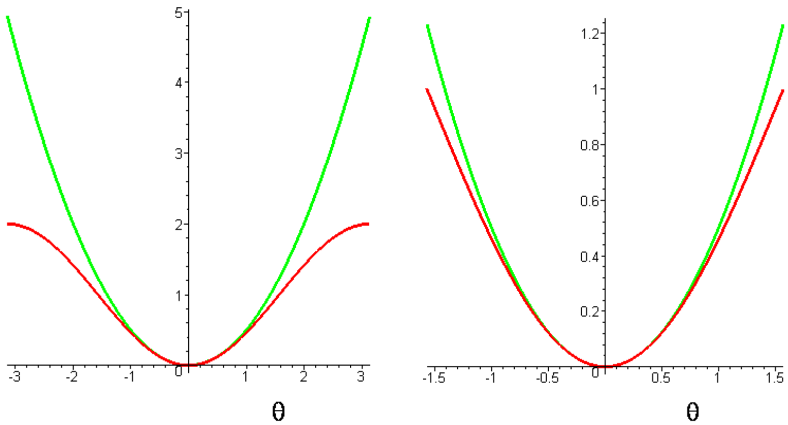
$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_0}. \tag{22.A.6}$$

Note that we have taken the positive square root, implying that $d\theta/dt \geq 0$. This clearly cannot always be the case, and we should change the sign of the square root every time the pendulum's direction of motion changes. For our purposes, this is not an issue. If we wished to find an explicit form for either $\theta(t)$ or $t(\theta)$, we would have to consider the signs in Equation (22.A.6) more carefully.

Before proceeding, it's worth considering the difference between Equation (22.A.6) and the equation for the simple pendulum in the simple harmonic oscillator limit,

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{\frac{\theta_0^2}{2} - \frac{\theta^2}{2}}. \tag{22.A.7}$$

In both Equations (22.A.6) and (22.A.7) the last term in the square root is proportional to the difference between the initial potential energy and the final potential energy. The final potential energy for the two cases is plotted in Figures 22.A.2 below for $-\pi < \theta < \pi$ on the left and $-\pi/2 < \theta < \pi/2$ on the right (the vertical scale is in units of mgl).



Figures 22.A.2 Potential energies as a function of displacement angle.

It would seem to be to our advantage to express the potential energy for an arbitrary displacement of the pendulum as the difference between two squares. This is done by recalling the trigonometric identity

$$1 - \cos\theta = 2\sin^2(\theta/2) \tag{22.A.8}$$

with the result that Equation (22.A.6) may be re-expressed as

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{2(\sin^2(\theta_0/2) - \sin^2(\theta/2))}\,. \tag{22.A.9}$$

(Note that using Equation (22.A.8) is "undoing" one step of (22.A.5).)

Equation (22.A.9) is separable,

$$\frac{d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}} = 2\sqrt{\frac{g}{l}}\,dt \tag{22.A.10}$$

What comes next may seem like it's been pulled out of a hat. The motivation is analogous to for the simple harmonic oscillator. That is, rewrite Equation (22.A.10) as

$$\frac{d\theta}{\sin(\theta_0/2)\sqrt{1 - \dfrac{\sin^2(\theta/2)}{\sin^2(\theta_0/2)}}} = 2\sqrt{\frac{g}{l}}\,dt\,. \tag{22.A.11}$$

The ratio $\sin(\theta/2)/\sin(\theta_0/2)$ in the square root in the denominator will oscillate (but *not* with simple harmonic motion) between -1 and 1, and so we will make the identification

$$\sin\phi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)}. \qquad (22.A.12)$$

If ϕ were proportional to the time t, the motion would be simple harmonic motion. We don't expect this to be the case, but we might make some simplifications in doing the needed integration, and allow comparison to simple harmonic motion.

Let $k = \sin(\theta_0/2)$, so that

$$\begin{aligned} \sin\frac{\theta}{2} &= k\sin\phi \\ \cos\frac{\theta}{2} &= \left(1-\sin^2\frac{\theta}{2}\right)^{1/2} = (1-k^2\sin^2\phi)^{1/2}. \end{aligned} \qquad (22.A.13)$$

Please be sure to note that here $k = \sin(\theta_0/2)$ is a dimensionless parameter of the system, not a spring constant.

The integral in (22.A.11) can then be rewritten as

$$\int\frac{d\theta}{k\sqrt{1-\sin^2\phi}} = 2\int\sqrt{\frac{g}{l}}\,dt. \qquad (22.A.14)$$

From differentiating the first expression in Equation (22.A.13), we have that

$$\begin{aligned} \frac{1}{2}\cos\frac{\theta}{2}\,d\theta &= k\cos\phi\,d\phi \\ d\theta &= 2k\frac{\cos\phi}{\cos(\theta/2)}d\phi = 2k\frac{\sqrt{1-\sin^2\phi}}{\sqrt{1-\sin^2(\theta/2)}}d\phi \\ &= 2k\frac{\sqrt{1-\sin^2\phi}}{\sqrt{1-k^2\sin^2\phi}}d\phi. \end{aligned} \qquad (22.A.15)$$

Substituting the last equation in (22.A.15) into the left-hand side of the integral in (22.A.14) yields

$$\int \frac{2k}{k\sqrt{1-\sin^2\phi}} \frac{\sqrt{1-\sin^2\phi}}{\sqrt{1-k^2\sin^2\phi}} d\phi = 2\int \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} . \tag{22.A.16}$$

Thus the integral in Equation (22.A.14) becomes

$$\int \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \int \sqrt{\frac{g}{l}}\, dt . \tag{22.A.17}$$

This integral is one of a class of integrals known as *elliptic integrals*. We will encounter a similar integral when we solve the *Kepler Problem*, where the orbits of an object under the influence of an inverse square gravitational force are determined.

We find a power series solution to this integral by expanding the function

$$(1-k^2\sin^2\phi)^{-1/2} = 1 + \frac{1}{2}k^2\sin^2\phi + \frac{3}{8}k^4\sin^4\phi + \cdots . \tag{22.A.18}$$

The integral in Equation (22.A.17) then becomes

$$\int \left(1 + \frac{1}{2}k^2\sin^2\phi + \frac{3}{8}k^4\sin^4\phi + \cdots\right) d\phi = \int \sqrt{\frac{g}{l}}\, dt . \tag{22.A.19}$$

Now let's integrate over one period. Set $t = 0$ when $\theta = 0$, the lowest point of the swing, so that $\sin\phi = 0$ and $\phi = 0$. One period T has elapsed the second time the bob returns to the lowest point, or when $\phi = 2\pi$. Putting in the limits of the ϕ-integral, we can integrate term by term, noting that

$$\begin{aligned} \int_0^{2\pi} \frac{1}{2}k^2\sin^2\phi\, d\phi &= \int_0^{2\pi} \frac{1}{2}k^2 \frac{1}{2}(1-\cos(2\phi))\, d\phi \\ &= \frac{1}{2}k^2 \frac{1}{2}\left(\phi - \frac{\sin(2\phi)}{2}\right)\Bigg|_0^{2\pi} \\ &= \frac{1}{2}\pi k^2 = \frac{1}{2}\pi \sin^2\frac{\theta_0}{2}. \end{aligned} \tag{22.A.20}$$

Thus, from Equation (22.A.19) we have that

$$\int_0^{2\pi}\left(1+\frac{1}{2}k^2\sin^2\phi+\frac{3}{8}k^4\sin^4\phi+\cdots\right)d\phi=\int_0^T\sqrt{\frac{g}{l}}\,dt \\ 2\pi+\frac{1}{2}\pi\sin^2\frac{\theta_0}{2}+\cdots=\sqrt{\frac{g}{l}}\,T \quad , \tag{22.A.21}$$

We can now solve for the period,

$$T=2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\frac{\theta_0}{2}+\cdots\right). \tag{22.A.22}$$

If the initial angle is small compared to 1 (measured in radians), $\theta_0 << 1$, then $\sin^2(\theta_0/2)\cong\theta_0{}^2/4$ and the period is approximately

$$T\cong 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{16}\theta_0{}^2\right)=T_0\left(1+\frac{1}{16}\theta_0{}^2\right), \tag{22.A.23}$$

where

$$T_0=2\pi\sqrt{\frac{l}{g}} \tag{22.A.24}$$

is the period of the simple pendulum with the standard small angle approximation.

The first order correction to the period of the pendulum is then

$$\Delta T_1=\frac{1}{16}\theta_0{}^2\,T_0\,. \tag{22.A.25}$$

Figure 22.A.3 below shows the three functions given in Equation (22.A.24) (the horizontal, or red plot if seen in color), Equation (22.A.23) (the middle, parabolic or green plot) and the numerically-integrated function obtained by integrating the expression in Equation (22.A.17) (the upper, or blue plot) between $\phi=0$ and $\phi=2\pi$ as a function of $k=\sin(\theta_0/2)$. The plots demonstrate that Equation (22.A.24) is a valid approximation for small values of θ_0, and that Equation (22.A.23) is a very good approximation for all but the largest amplitudes of oscillation. The vertical axis is in units of $\sqrt{l/g}$. Note the displacement of the horizontal axis.

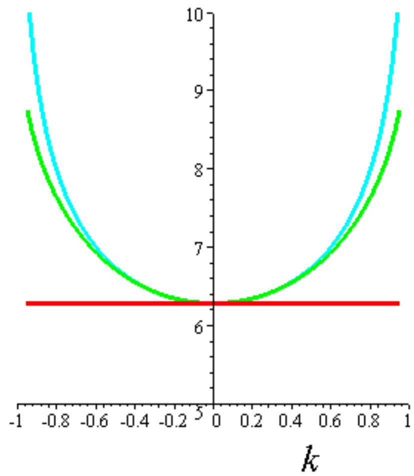


Figure 22.A.3 Pendulum Period Approximations as Functions of Amplitude.