Two-Dimensional Rotational Kinematics

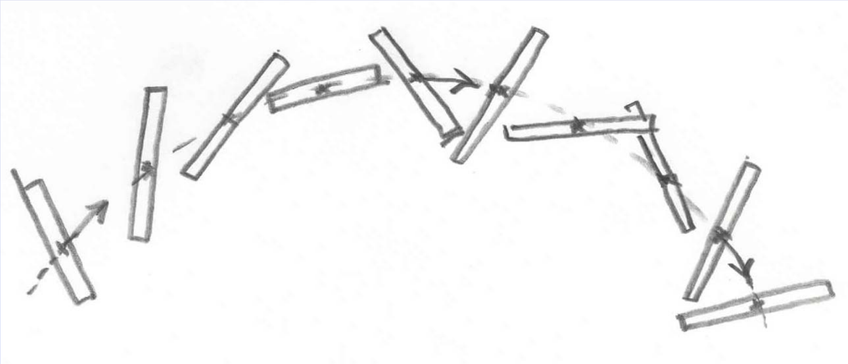
Rigid Bodies

A ***rigid body*** is an extended object in which the distance between any two points in the object is constant in time.

Springs or human bodies are non-rigid bodies.

Rotation and Translation of Rigid Body

Demonstration: Motion of a thrown baton



Translational motion: external force of gravity acts on center of mass

Rotational Motion: object rotates about center of mass

Recall: Translational Motion of the Center of Mass

Total momentum of system of particles

$$\vec{\mathbf{p}}^{\,\mathrm{sys}} = m^{\mathrm{total}}\vec{\mathbf{V}}_{\mathrm{cm}}$$

External force and acceleration of center of mass

$$\vec{\mathbf{F}}_{\mathrm{ext}}^{\mathrm{total}} = \frac{d\vec{\mathbf{p}}^{\,\mathrm{sys}}}{dt} = m^{\mathrm{total}}\frac{d\vec{\mathbf{V}}_{\mathrm{cm}}}{dt} = m^{\mathrm{total}}\vec{\mathbf{A}}_{\mathrm{cm}}$$

Main Idea: Rotation of Rigid Body

Torque produces angular acceleration about center of mass

$$\boldsymbol{\tau}_{\mathrm{cm}}^{\mathrm{total}} = I_{\mathrm{cm}}\alpha_{\mathrm{cm}}$$

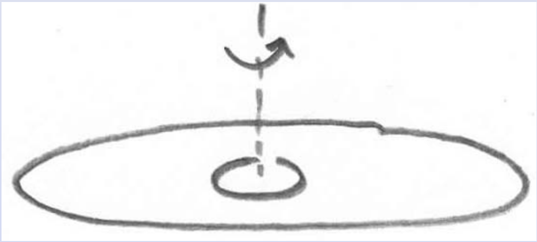
I_{cm} is the moment of inertial about the center of mass

α_{cm} is the angular acceleration about center of mass

Two-Dimensional Rotation

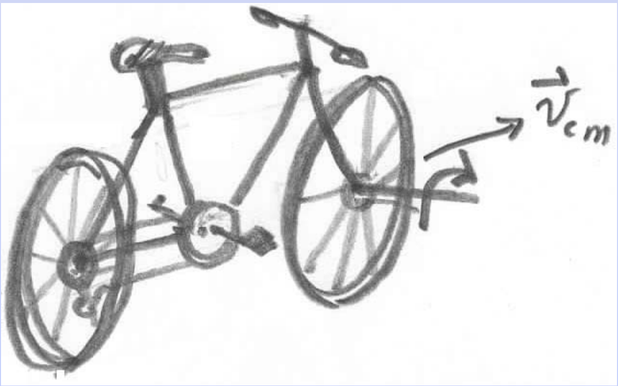
- Fixed axis rotation:

  Disc is rotating about axis passing through the center of the disc and is perpendicular to the plane of the disc.



- Plane of motion is fixed:

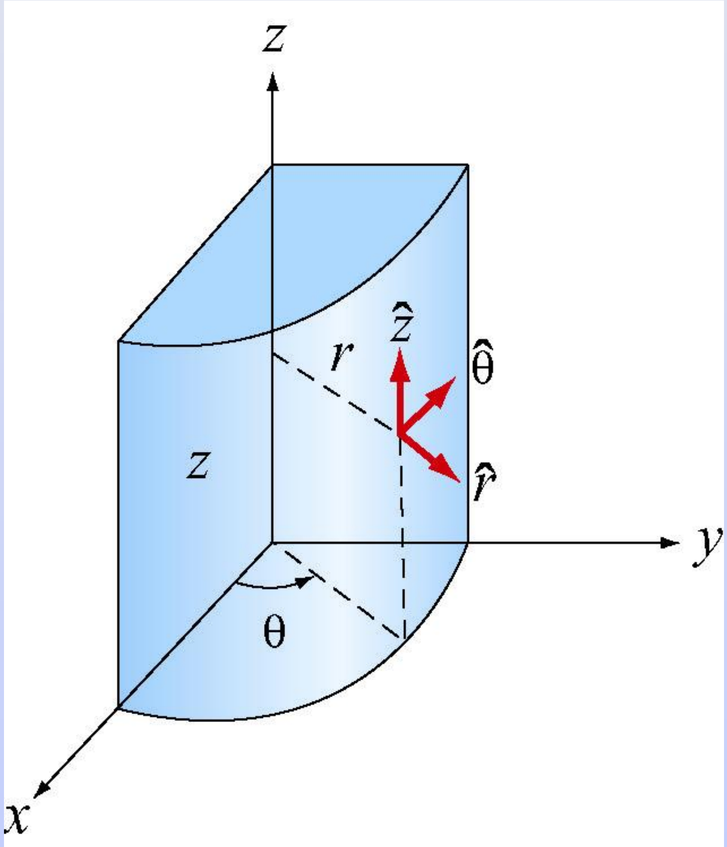
  For straight line motion, bicycle wheel rotates about fixed direction and center of mass is translating



Cylindrical Coordinate System

Coordinates (r, θ, z)

Unit vectors $(\hat{\mathbf{r}}, \hat{\theta}, \hat{\mathbf{z}})$



Circular Motion for point-like particle vector description

Use plane polar coordinates

Position
$$\vec{\mathbf{r}}(t) = R\,\hat{\mathbf{r}}(t)$$

Velocity
$$\vec{\mathbf{v}}(t) = R\frac{d\theta}{dt}\,\hat{\theta}(t) = R\,\omega\,\hat{\theta}(t)$$

Acceleration
$$\vec{\mathbf{a}} = a_r\,\hat{\mathbf{r}} + a_t\,\hat{\theta}$$

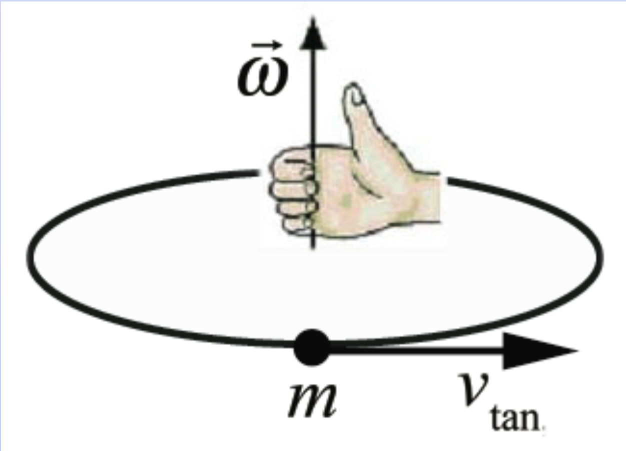
$$a_t = r\alpha, \quad a_r = -r\omega^2 = -(v^2/r)$$

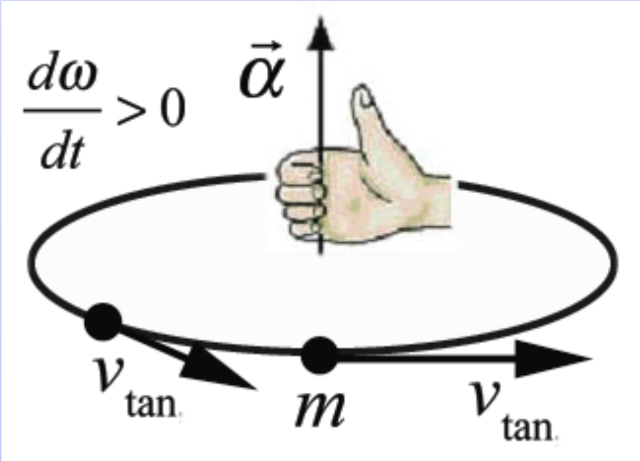
Rotational Kinematics for Fixed Axis Rotation

A point like particle undergoing circular motion at a non-constant speed has

(1)An angular velocity vector

(2) an angular acceleration vector





Fixed Axis Rotation: Angular Velocity

Angle variable θ

SI unit: $[\mathrm{rad}]$

Angular velocity $\vec{\boldsymbol{\omega}} \equiv \omega\,\hat{\mathbf{k}} \equiv \dfrac{d\theta}{dt}\hat{\mathbf{k}}$

SI unit: $\left[\mathrm{rad}\cdot\mathrm{s}^{-1}\right]$

Vector:

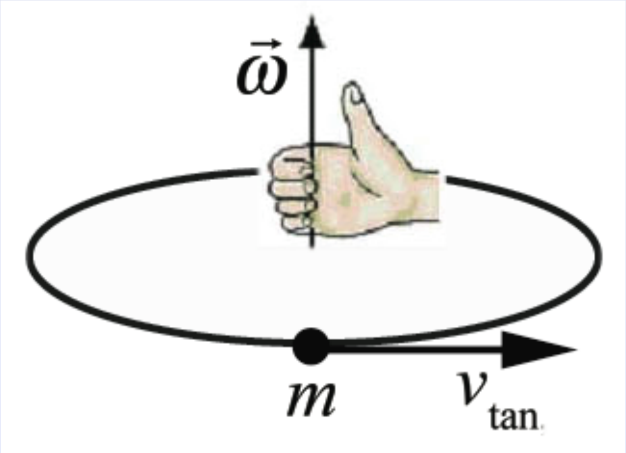
Component $\omega \equiv \dfrac{d\theta}{dt}$

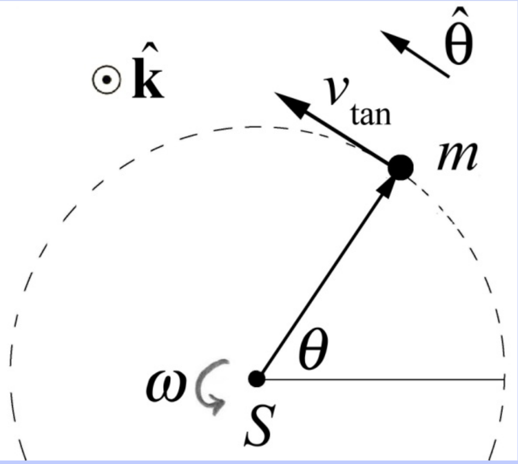
magnitude $\left|\omega\right| \equiv \left|\dfrac{d\theta}{dt}\right|$

direction

$$\omega > 0,\ \text{direction}\ +\hat{\mathbf{k}}$$

$$\omega < 0,\ \text{direction}\ -\hat{\mathbf{k}}$$

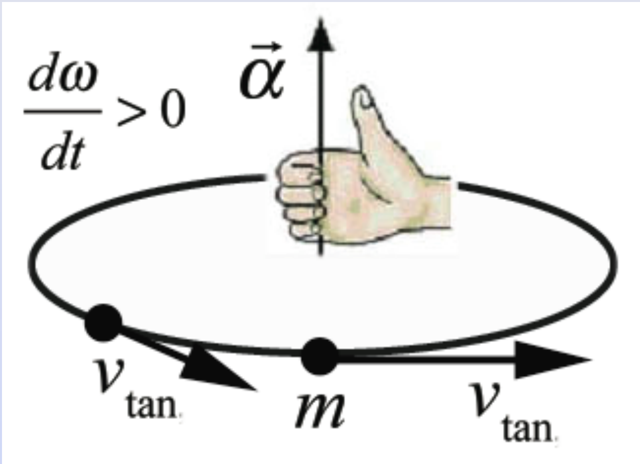




Fixed Axis Rotation: Angular Acceleration

Angular acceleration: $\vec{\alpha} \equiv \alpha\,\hat{\mathbf{k}} \equiv \dfrac{d^2\theta}{dt^2}\,\hat{\mathbf{k}}$

SI unit $\left[\mathrm{rad \cdot s^{-2}}\right]$



Vector:

Component:

$$\alpha \equiv \frac{d^2\theta}{dt^2} \equiv \frac{d\omega}{dt}$$

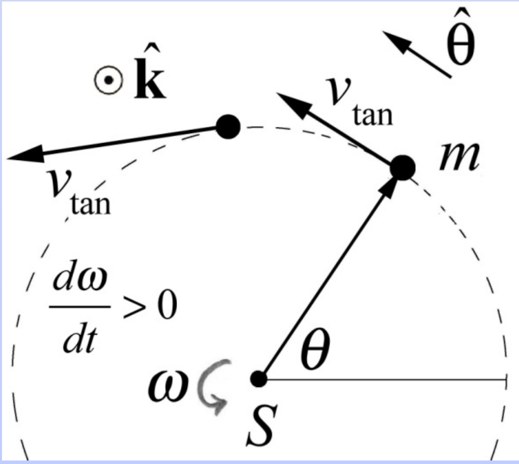
Magnitude:

$$\left|\alpha\right| \equiv \left|\frac{d\omega}{dt}\right|$$

Direction:

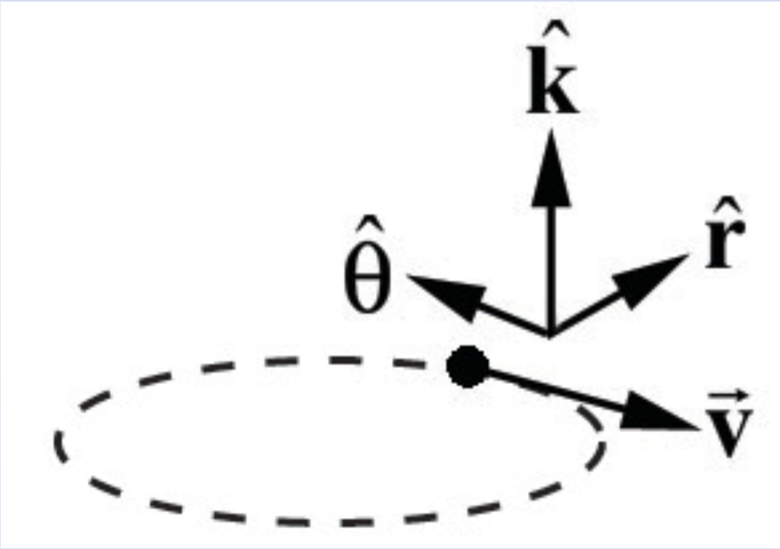
$$\frac{d\omega}{dt} > 0, \ \text{direction} \ +\hat{\mathbf{k}}$$

$$\frac{d\omega}{dt} < 0, \ \text{direction} \ -\hat{\mathbf{k}}$$



Checkpoint Problem: Angular Velocity

Consider the uniformly rotating object shown in the figure below. What is the direction of the angular velocity of the object?



Rotational Kinematics: Constant Angular Acceleration

The angular quantities $\theta, \omega, \text{and } \alpha$

are exactly analogous to the quantities $x, v_x, \text{and } a_x$

for one-dimensional motion, and obey the same type of integral relations

$$\omega(t) - \omega_0 = \int_0^t \alpha(t')dt' , \qquad \theta(t) - \theta_0 = \int_0^t \omega(t')dt' .$$

Constant angular acceleration:

$$\omega(t) = \omega_0 + \alpha\, t$$

$$\theta(t) = \theta_0 + \omega_0\, t + \frac{1}{2}\alpha\, t^2$$

$$\Longrightarrow \qquad \left(\omega(t)\right)^2 = \omega_0^2 + 2\alpha\left(\theta(t) - \theta_0\right).$$

Checkpoint Problem: Rotational Kinematics

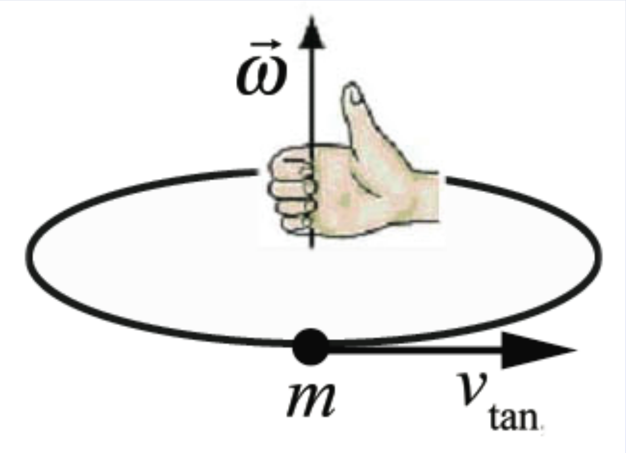
A turntable is a uniform disc of mass m and a radius R. The turntable is initially spinning clockwise when looked down on from above at a constant frequency f. The motor is turned off and the turntable slows to a stop in t seconds with constant angular deceleration.

a) What is the direction and magnitude of the initial angular velocity of the turntable?

b) What is the direction and magnitude of the angular acceleration of the turntable?

c) What is the total angle in radians that the turntable spins while slowing down?

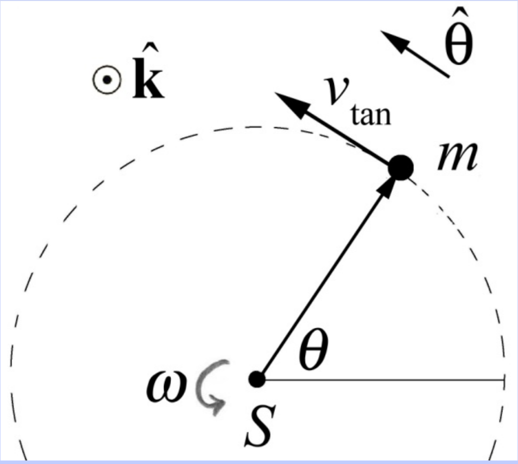
Summary: Kinematics of Circular Motion



Arc length $s = R\theta$

Tangential velocity $v_{\tan} = \dfrac{ds}{dt} = R\dfrac{d\theta}{dt} = R\omega$

Tangential acceleration $a_{\tan} = \dfrac{dv}{dt} = R\dfrac{d^2\theta}{dt^2} = R\alpha$

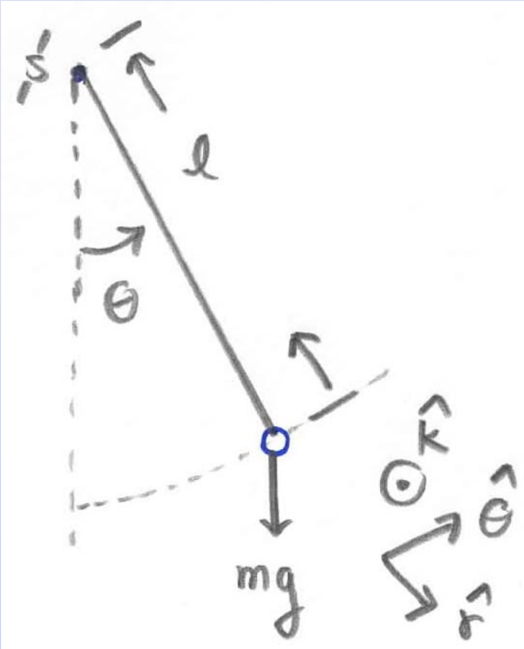


Centripetal $a_{\text{rad}} = v\omega = \dfrac{v^2}{R} = R\omega^2$

Rotational kinetic energy $K_{\text{rot}} = \dfrac{1}{2}m{v_{\tan}}^2 = \dfrac{1}{2}mR^2\omega^2$

Worked Example: Simple Pendulum

Simple Pendulum: bob of mass *m* hanging from end of massless string of length *l* pivoted at S.



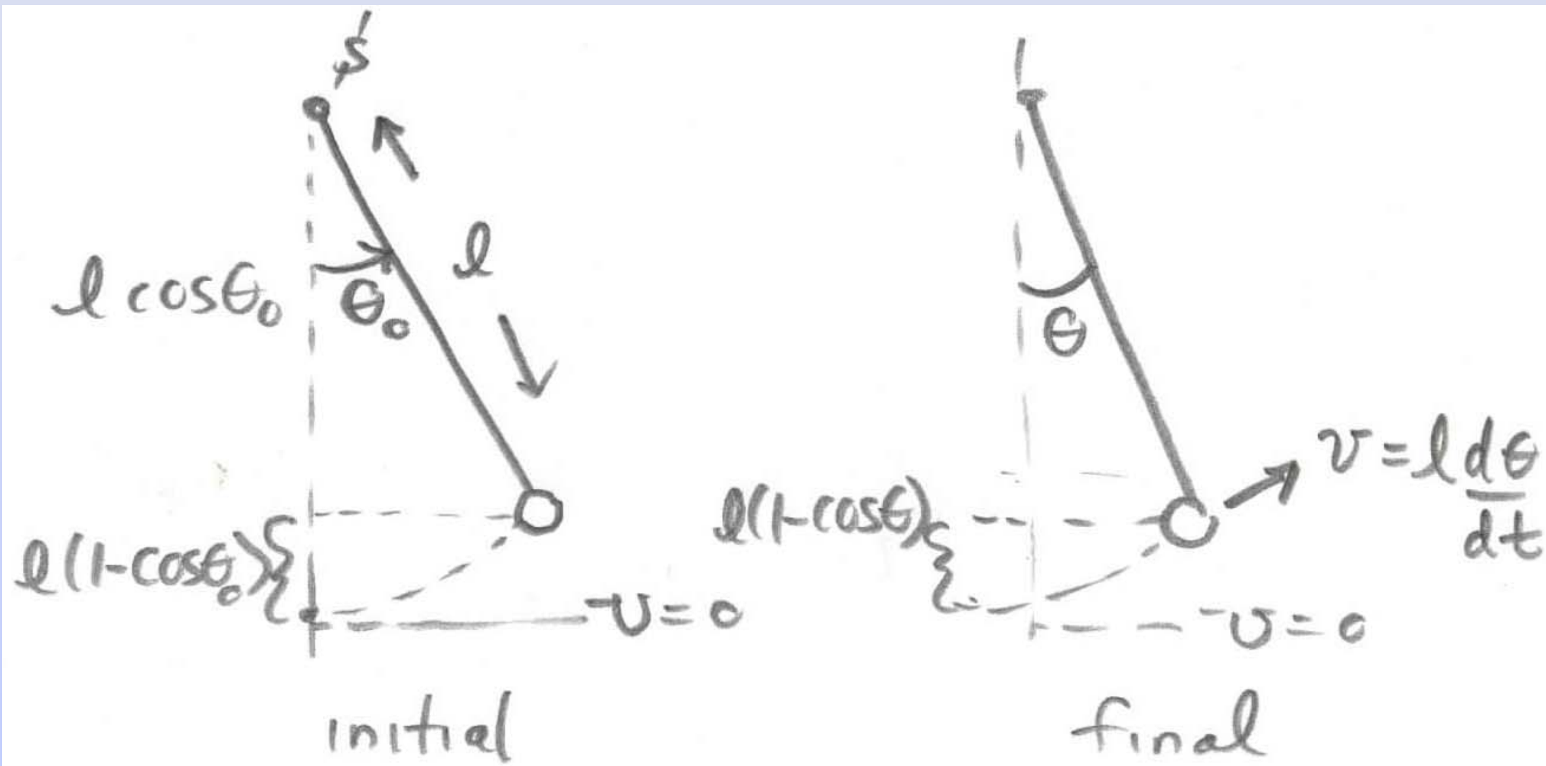
Angular velocity $$\vec{\omega} = \frac{d\theta}{dt}\hat{\mathbf{k}}$$

Angular acceleration $$\vec{\boldsymbol{\alpha}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}}$$

Kinetic energy of rotation $$K = \frac{1}{2}ml^2\omega^2$$

Worked Example: Simple Pendulum: Mechanical Energy

A simple pendulum is released from rest at an angle θ_0 . Find angular speed at angle θ



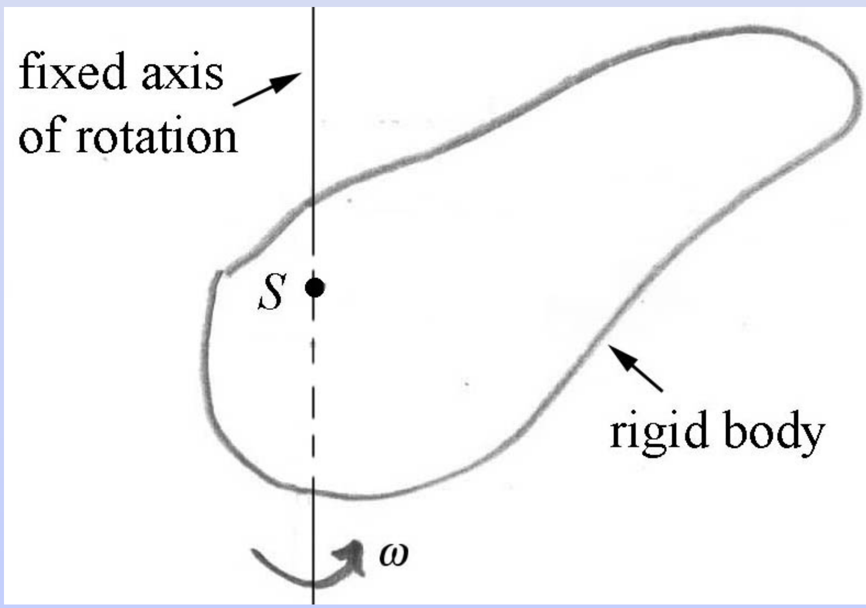
Worked Example Simple Pendulum: Mechanical Energy

- Velocity
$$v_{\tan} = l\frac{d\theta}{dt}$$
- Kinetic energy
$$K_f = \frac{1}{2}mv_{\tan}^{2} = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2$$
- Initial energy
$$E_0 = K_0 + U_0 = mgl(1-\cos\theta_0)$$
- Final energy
$$E_f = K_f + U_f = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + mgl(1-\cos\theta)$$
- Conservation of energy
$$\frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + mgl(1-\cos\theta) = mgl(1-\cos\theta_0)$$

$$\frac{d\theta}{dt} = \sqrt{\frac{2}{l}\Big(-mg(1-\cos\theta) + g(1-\cos\theta_0)\Big)}$$

Rigid Body Kinematics for Fixed Axis Rotation

Body rotates with angular velocity ω and angular acceleration α



Divide Body into Small Elements

Body rotates with angular velocity, ω

angular acceleration α

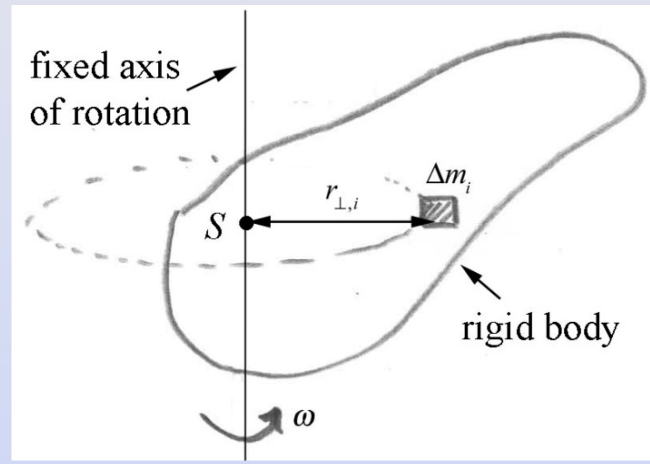
Individual elements of mass Δm_i

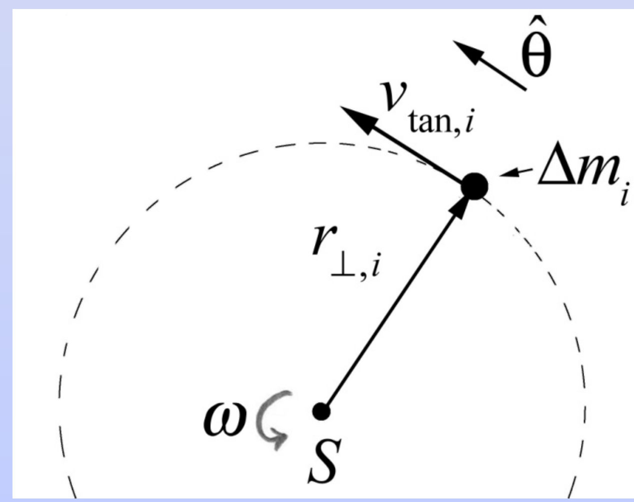
Radius of orbit $r_{\perp,i}$

Tangential velocity $v_{\tan,i} = r_{\perp,i}\omega$

Tangential acceleration $a_{\tan,i} = r_{\perp,i}\alpha$

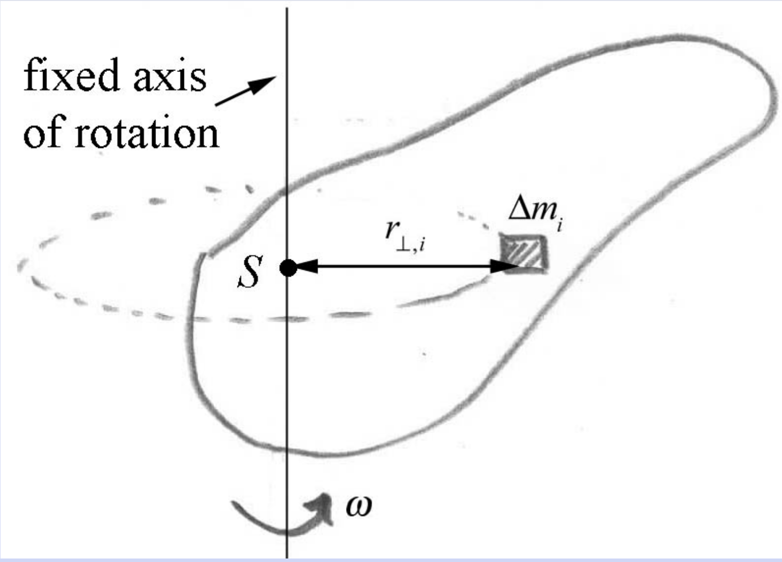
Radial Acceleration $a_{\mathrm{rad},i} = \dfrac{v_{\tan,i}^2}{r_{\perp,i}} = r_{\perp,i}\omega^2$





Rotational Kinetic Energy and Moment of Inertia

Rotational kinetic energy about axis passing through S



$$K_{\mathrm{rot},i} = \frac{1}{2}\Delta m_i v_{\tan,i}{}^2 = \frac{1}{2}\Delta m_i \left(r_{\perp,i}\right)^2 \omega^2$$

Moment of Inertia about S :

$$I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2$$

SI Unit:

$$\left[\mathrm{kg}\cdot\mathrm{m}^2\right]$$

Continuous body:

$$\Delta m_i \rightarrow dm \qquad r_{\perp,i} \rightarrow r_{\perp,dm} \qquad \sum_{i=1}^{i=N} \rightarrow \int_{\mathrm{body}}$$

Rotational Kinetic Energy:

$$I_S = \int_{\mathrm{body}} dm\,(r_{\perp,dm})^2$$

$$K_{\mathrm{rot}} = \sum_i K_{\mathrm{rot},i} = \left(\sum_i \frac{1}{2}\Delta m_i \left(r_{\perp,i}\right)^2\right)\omega^2 = \left(\frac{1}{2}\int_{\mathrm{body}} dm\,(r_{\perp,dm})^2\right)\omega^2 = \frac{1}{2} I_S \omega^2$$

Discussion: Moment of Inertia

How does moment of inertia compare to the total mass and the center of mass?

Different measures of the distribution of the mass.

Total mass: scalar

$$m^{\text{total}} = \int_{\text{body}} dm$$

Center of Mass: vector (three components)

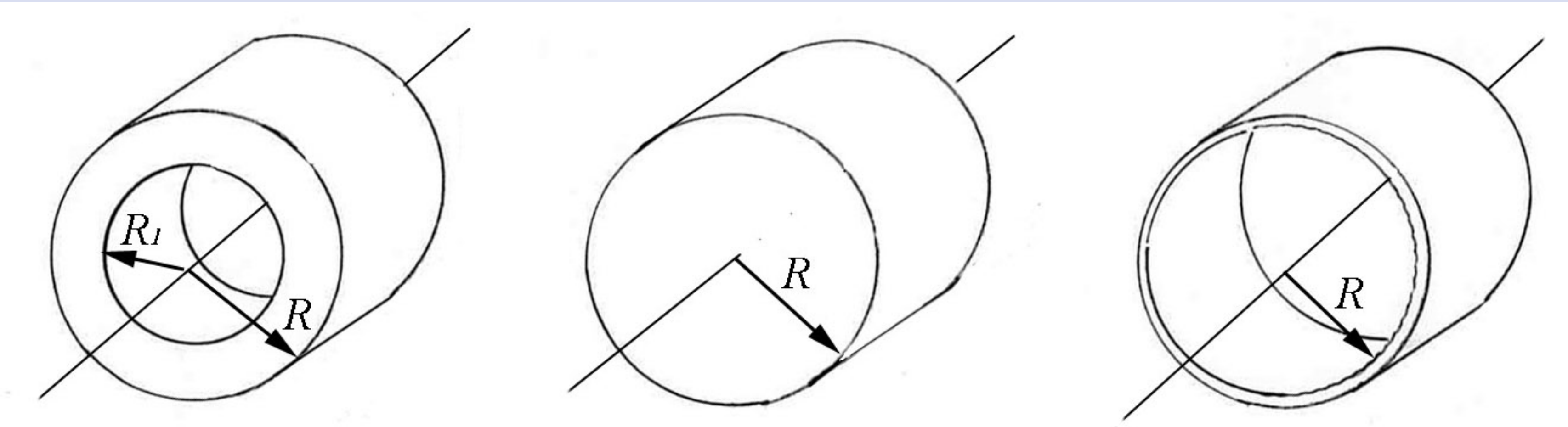
$$\vec{\mathbf{R}}_{\text{cm}} = \frac{1}{m^{\text{total}}} \int_{\text{body}} \vec{\mathbf{r}}\, dm$$

Moment of Inertia about axis passing through S: (nine possible moments)

$$I_S = \int_{\text{body}} dm\, (r_{\perp,dm})^2$$

Checkpoint Problem

All of the objects below have the same mass. Which of the objects has the largest moment of inertia about the axis shown?



1) Hollow Cylinder   2) Solid Cylinder   3) Thin-walled Hollow Cylinder

Strategy: Calculating Moment of Inertia

Step 1: Identify the axis of rotation

Step 2: Choose a coordinate system

Step 3*:* Identify the infinitesimal mass element *dm*.

Step 4: Identify the radius, $r_{\perp,dm}$, of the circular orbit of the infinitesimal mass element *dm*.

Step 5: Set up the limits for the integral over the body in terms of the physical dimensions of the rigid body.

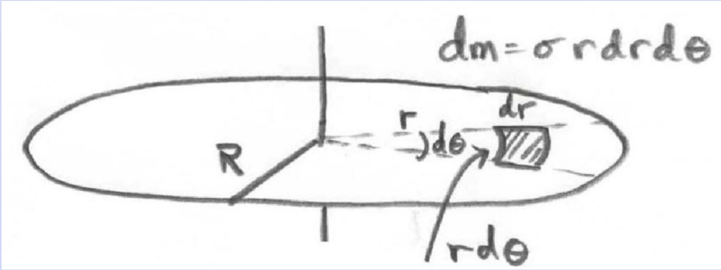
Step 6: Explicitly calculate the integrals.

Worked Example: Moment of Inertia for Uniform Disc

Consider a thin uniform disc of radius R and mass m. What is the moment of inertia about an axis that pass perpendicular through the center of the disc?

Worked Example: Moment of Inertia of a Disc

Consider a thin uniform disc of radius R and mass m. What is the moment of inertia about an axis that pass perpendicular through the center of the disc?



$$r_{\perp,dm} = r$$

$$da = r\,dr\,d\theta$$

$$\sigma = \frac{dm}{da} = \frac{m_{\text{total}}}{\text{Area}} = \frac{M}{\pi R^2}$$

$$dm = \sigma\, r\,dr\,d\theta = \frac{M}{\pi R^2} r\,dr\,d\theta$$

$$I_{\text{cm}} = \int_{\text{body}} (r_{\perp,dm})^2\,dm = \frac{M}{\pi R^2}\int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} r^3\,d\theta\,dr$$

$$I_{\text{cm}} = \frac{M}{\pi R^2}\int_{r=0}^{r=R}\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right) r^3 dr = \frac{M}{\pi R^2}\int_{r=0}^{r=R} 2\pi r^3 dr = \frac{2M}{R^2}\int_{r=0}^{r=R} r^3 dr$$

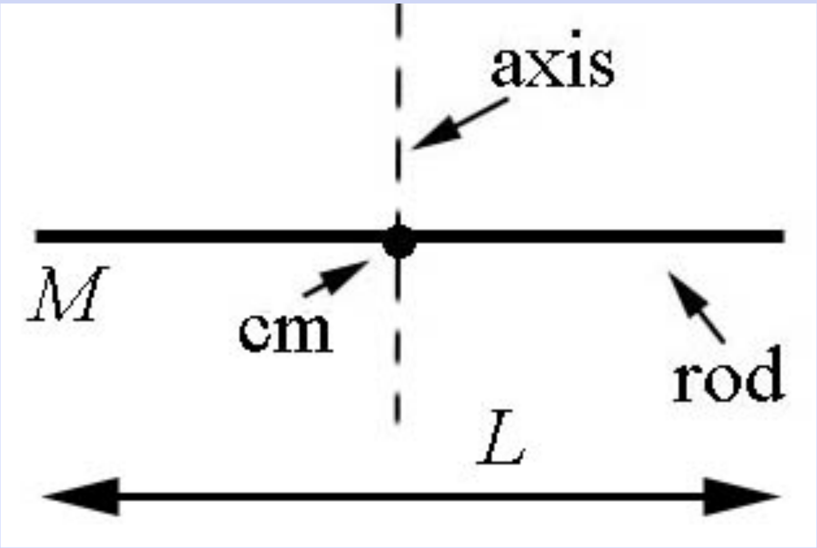
$$I_{\text{cm}} = \frac{2M}{R^2}\int_{r=0}^{r=R} r^3 dr = \frac{2M}{R^2}\left.\frac{r^4}{4}\right|_{r=0}^{r=R} = \frac{2M}{R^2}\frac{R^4}{4} = \frac{1}{2}MR^2$$

Checkpoint Problem: Moment of Inertia of a Rod

Consider a thin uniform rod of length L and mass M.

a) Calculate the moment of inertia about an axis that passes perpendicular through the center of mass of the rod.

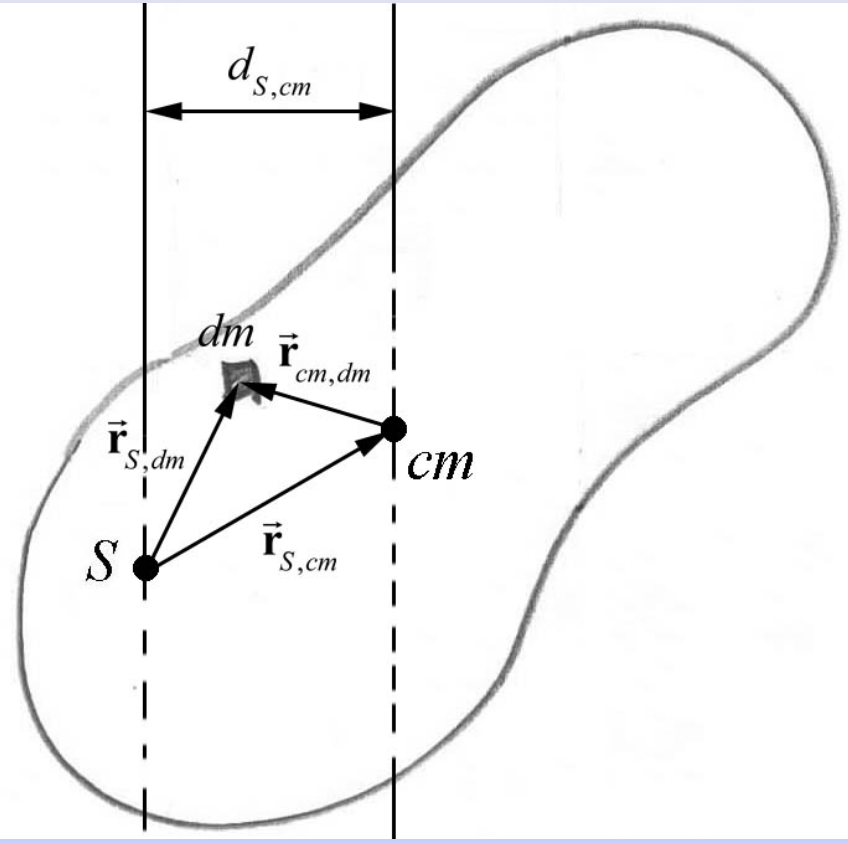
b) Calculate I about an axis that passes perpendicular through the end of the rod.



Parallel Axis Theorem

- Rigid body of mass m.
- Moment of inertia I_{cm} about axis through center of mass of the body.
- Moment of inertia I_S about parallel axis through point S in body.
- $d_{S,\mathrm{cm}}$ perpendicular distance between two parallel axes.

$$I_S = I_{\mathrm{cm}} + m d_{S,\mathrm{cm}}^2$$



Summary: Moment of Inertia

Moment of Inertia about S:

$$I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,S,i})^2 = \int_{\text{body}} r_{\perp,S}{}^2 \, dm$$

Examples: Let S be the center of mass

- rod of length l and mass m $\quad I_{\text{cm}} = \dfrac{1}{12} m l^2$
- disc of radius R and mass m $\quad I_{\text{cm}} = \dfrac{1}{2} m R^2$

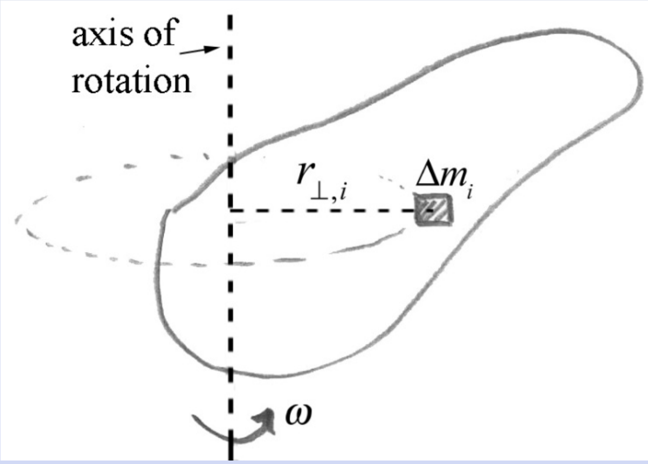
Parallel Axis theorem:

$$I_S = I_{\text{cm}} + m d_{S,\text{cm}}^2$$

Checkpoint Problem: Kinetic Energy

A disk with mass M and radius R is spinning with angular speed ω about an axis that passes through the rim of the disk perpendicular to its plane. The moment of inertia about the cm is $(1/2)M R^2$. What is the kinetic energy of the disk?

Summary: Fixed Axis Rotation Kinematics



Angle variable θ

Angular velocity $\omega \equiv d\theta / dt$

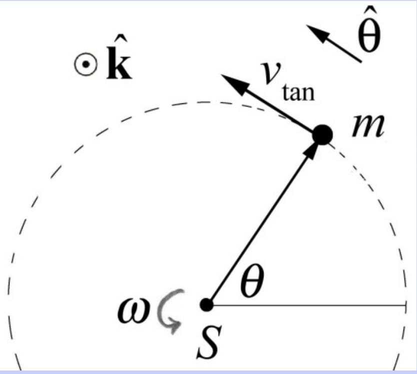
Angular acceleration $\alpha \equiv d^2\theta / dt^2$

Mass element Δm_i

Radius of orbit $r_{\perp,i}$

Moment of inertia

$$I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2 \rightarrow \int_{body} dm (r_\perp)^2$$



Parallel Axis Theorem

$$I_S = Md^2 + I_{cm}$$

Checkpoint Problem: Moment of Inertia Wheel

Using energy techniques, Calculate the speed of block 2 as a function of distance that it moves down the inclined plane using energy techniques. Let I_P denote the moment of inertia of the pulley about its center of mass. Assume there are no energy losses due to friction and that the rope does slip around the pulley.