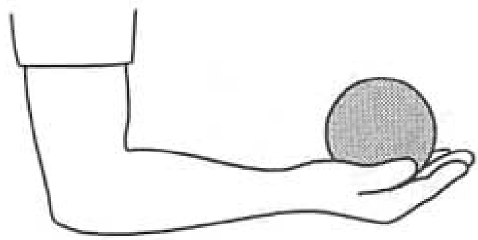
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

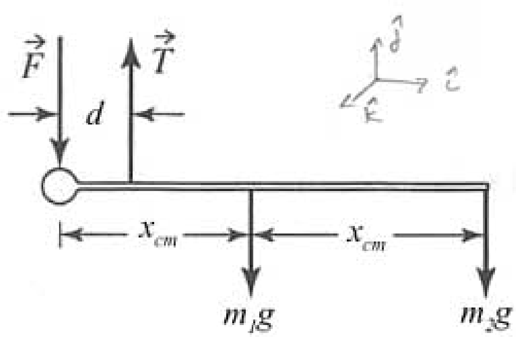
Physics 8.01T **Fall Term 2004**

In-Class Problems 17-19: Static Equilibrium Solutions

In-Class-Problem 17 Static Equilibrium: Forearm

The human forearm (including the hand) can be regarded as a lever pivoted at the joint of the elbow with a downward force $\vec{\mathbf{F}}$ acting on the forearm at the elbow. The forearm is pulled upward by the tendon of the biceps $\vec{\mathbf{T}}$ which connects to the forearm a distance d from the joint. The tendon $\vec{\mathbf{T}}$ makes an angle of 90^0 with the horizontal. The forearm and hand have a mass of m_1 **.** Assume the center of mass of the forearm and hand is a distance x_{cm} from the joint. Suppose an object of mass m_2 rests in the hand a distance $2x_{cm}$ away from the joint of the elbow. See the accompanying figure.





a) What are the equations for static equilibrium of the forces?

b) About point will you choose to analyze the torques? What are the equations for static equilibrium of the torques?

c) What upward force $\vec{\mathbf{T}}$ must the tendon exert to keep the forearm horizontal?

d) What is the downward force $\vec{\mathbf{F}}$ on the forearm at the elbow joint?

Force Equations:

$\hat{j}$: $T - F - m_1 g - m_2 g = 0$ (1)

Torque: Choose pivot point to calculate torque.

$$\vec{0} = \vec{\tau}_P = \vec{r}_{P,T} \times \vec{T} + \vec{r}_{P,m_1 g} \times m_1 \vec{g} + \vec{r}_{P,m_2 g} \times m_2 \vec{g}$$

$$= (d\,\hat{i} \times T\,\hat{j}) + x_{cm}\,\hat{i} \times (-m_1 g\,\hat{j}) + 2x_{cm}\,\hat{i} \times (-m_2 g\,\hat{j})$$

$$\vec{0} = dT\,\hat{k} - m_1 g x_{cm}\,\hat{k} - 2 m_2 g x_{cm}\,\hat{k}$$

$\hat{k}$: $dT - m_1 g x_{cm} - 2m_2 g x_{cm} = 0$

$$\Rightarrow T = \frac{x_{cm} g (m_1 + 2m_2)}{d} \quad (2)$$

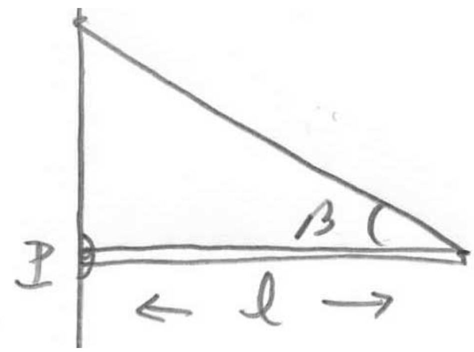
The force equation becomes

$$F = T - m_1 g - m_2 g$$

$$F = \frac{x_{cm} g (m_1 + 2m_2)}{d} - g(m_1 + m_2)$$

In-Class-Problem 18 Static Equilibrium: Suspended Beam

A rod of length 2.0 m and mass 4.0 kg is hinged to a wall at one end and suspended from the wall by a cable which is attached to the other end of the rod at an angle of $\beta = 30^{o}$ to the rod (see sketch). Assume the cable is massless. There is a pivot force acting on the beam. The magnitude and direction of this force is unknown. One of the most difficult parts of these types of problems is to introduce an angle for the pivot force and then solve for that angle if possible. In this problem you will solve for the magnitude of the tension in the cable and the direction and magnitude of the pivot force.



a) Draw a free body force diagram for the beam. Clearly indicate your choice of angle for the pivot force.

b) What are the equations for static equilibrium of the forces?

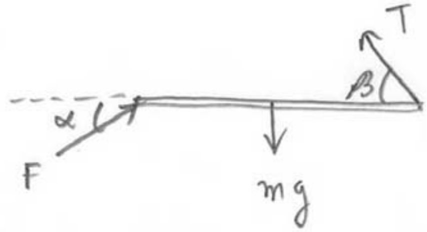
c) About point will you choose to analyze the torques? What are the equations for static equilibrium of the torques?

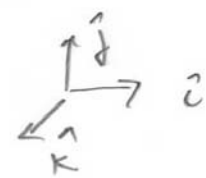
d) What is the tension in the cable?

e) What angle does the pivot force make with the beam?

f) What is the magnitude of the pivot force?

Assume an unknown force $\vec{F}$ acts at pivot point at an angle α





Force Decomposition:

$$\hat{i}: \ -T\cos\beta + F\cos\alpha = 0 \qquad (1)$$

$$\hat{j}: \ T\sin\beta + F\sin\alpha - mg = 0 \qquad (2)$$

Choose pivot to calculate torque

$$\vec{\tau}_{P,mg} = \vec{r}_{P,mg} \times m\vec{g} = \frac{l}{2}\,\hat{i} \times mg(-\hat{j}) = \frac{l}{2}mg(-\hat{k})$$

$$\vec{\tau}_{P,T} = \vec{r}_{P,T} \times \vec{T} = l\,\hat{i} \times (-T\cos\beta\,\hat{i} + T\sin\beta\,\hat{j})$$

$$= lT\sin\beta\,\hat{k}$$

$$\vec{\tau}_P^{\,total} = \vec{\tau}_{P,mg} + \vec{\tau}_{P,T} = -\frac{l}{2}mg\,\hat{k} + lT\sin\beta\,\hat{k} = 0$$

$$\hat{k}: \ -\frac{l}{2}mg + lT\sin\beta = 0$$

$$\Rightarrow \quad T\sin\beta = \frac{mg}{2} \qquad (3)$$

$$T = \frac{mg}{2\sin\beta} = \frac{(4.0\,kg)(9.8\,m/s^2)}{2\sin(30^\circ)} = 39.2\,N$$

The force equations are

$$T\cos\beta = F\cos\alpha \qquad (1a)$$

$$T\sin\beta + F\sin\alpha = mg \qquad (2a)$$

Substitute eq (3) into eq (2a) yields

$$\frac{mg}{2} + F\sin\alpha = mg$$

or

$$F\sin\alpha = \frac{mg}{2} \qquad (4)$$

recall

$$F\cos\alpha = T\cos\beta \qquad (1a)$$

if we divide eq (4) by eq (1a) we can solve for the angle α

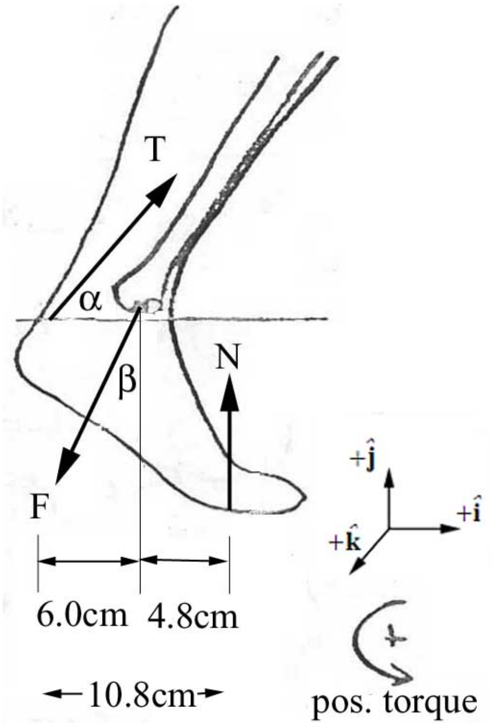
$$\tan\alpha = \frac{\frac{mg}{2}}{T\cos\beta} = \frac{\frac{mg}{2}}{\frac{mg}{2}\frac{\cos\beta}{\sin\beta}} = \tan\beta$$

$\Rightarrow \alpha = \beta = 30^\circ$, then

eq (1a) implies

$$F = T = 39.2\,N$$

In-Class-Problem 19 Static Equilibrium: The Ankle



A person of mass $m = 75\,kg$ is crouching with his/her weight evenly distributed on both tiptoes. The forces on the skeletal part of the foot are shown in the diagram. In this position, the Achilles tendon in under considerable tension, $\vec{\mathbf{T}}$, and makes an angle of $\alpha = 37^0$ with the horizontal. The tendon acts on the ankle a horizontal distance $d = 10.8\,cm$ from the point where the foot contacts the floor. The tibia acts on the foot with magnitude, $F \equiv |\vec{\mathbf{F}}|$ and makes an angle β with the vertical. This force acts on the ankle a horizontal distance $s = 4.8\,cm$ from the point where the foot contacts the floor. You may ignore weight of the foot.

a) Find the magnitude of the tension in the Achilles tendon, $T \equiv |\vec{\mathbf{T}}|$.

b) Find the magnitude, $F \equiv |\vec{\mathbf{F}}|$, and the angle , β, of the tibia force on the ankle.

Solution:

We shall apply the two conditions of static equilibrium,

(1) The sum of the forces acting on the rigid body is zero,

$$\vec{\mathbf{F}}_{total} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \ldots = \vec{\mathbf{0}}.$$

(2) The vector sum of the torques about any point S in a rigid body is zero,

$$\vec{\boldsymbol{\tau}}_S^{total} = \vec{\boldsymbol{\tau}}_{S,T} + \vec{\boldsymbol{\tau}}_{S,N} = \vec{\mathbf{0}}.$$

The first condition that the sum of the forces is zero becomes

$$\hat{\mathbf{i}} : -F\sin\beta + T\cos\alpha = 0$$

$$\hat{\mathbf{j}}: N - F\cos\beta + T\sin\alpha = 0\,.$$

Since the weight is evenly distributed on the two feet, the normal force on one foot is equal to half the weight, or

$$N = (1/2)mg\,,$$

the second equation becomes

$$\hat{\mathbf{j}}: (1/2)mg - F\cos\beta + T\sin\alpha = 0$$

Choose the point of action of the tendon on the ankle as the point S to compute the torque about. Note that the force, $\vec{\mathbf{F}}$, that the tibia exerts on the ankle will make no contribution to the torque about this point S. Choose counterclockwise as the positive direction for the torque, this is the positive $\hat{\mathbf{k}}$ direction. The torque force diagram on the ankle is shown below.

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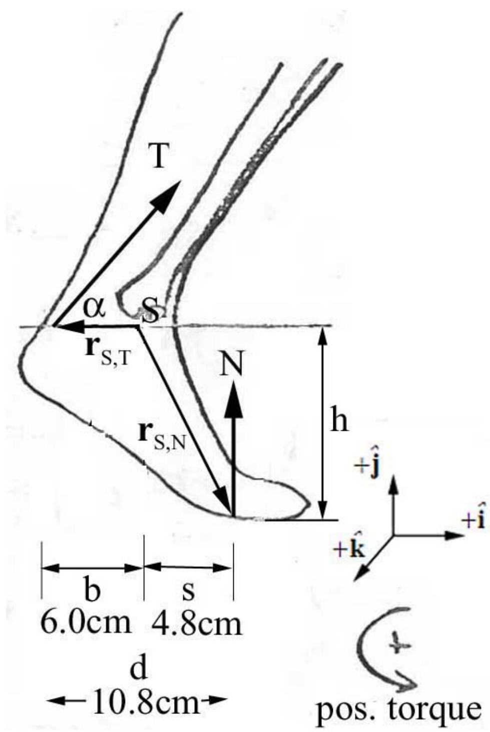

Then the torque due to the force of the tendon $\vec{\mathbf{T}}$ on the ankle is

$$\vec{\boldsymbol{\tau}}_{S,T} = \vec{\mathbf{r}}_{S,T} \times \vec{\mathbf{T}} = -b\hat{\mathbf{i}} \times \left(T\cos\alpha\hat{\mathbf{i}} + T\sin\alpha\hat{\mathbf{j}}\right) = -bT\sin\alpha\hat{\mathbf{k}} .$$

The torque to the normal force of the ground is

$$\vec{\boldsymbol{\tau}}_{S,N} = \vec{\mathbf{r}}_{S,N} \times N\hat{\mathbf{j}} = \left(s\hat{\mathbf{i}} - h\hat{\mathbf{j}}\right) \times N\hat{\mathbf{j}} = sN\hat{\mathbf{k}} = (1/2)smg\hat{\mathbf{k}} .$$

So the condition that the total torque about the point S vanishes becomes

$$\vec{\boldsymbol{\tau}}_S^{total} = \vec{\boldsymbol{\tau}}_{S,T} + \vec{\boldsymbol{\tau}}_{S,N} = \vec{\mathbf{0}}$$

becomes

$$-bT\sin\alpha\hat{\mathbf{k}} + (1/2)smg\hat{\mathbf{k}} = \vec{\mathbf{0}} .$$

We can use this torque condition to find the tension in the tendon,

$$T = \frac{(1/2)smg}{b\sin\alpha} = \frac{(1/2)\left(4.8\times10^{-2}\,m\right)\left(75\,kg\right)\left(9.8\,m\cdot s^{-2}\right)}{\left(6.0\times10^{-2}\,m\right)\sin\left(37^0\right)} = 4.9\times10^{2}\,N .$$

Recalling our other two force equations

$$\hat{\mathbf{i}} : -F\sin\beta + T\cos\alpha = 0$$

$$\hat{\mathbf{j}} : (1/2)mg - F\cos\beta + T\sin\alpha = 0$$

We can now solve for the direction β of the force of the tibia $\vec{\mathbf{F}}$ on the ankle as follows. Rewrite the two force equations as

$$F\cos\beta = (1/2)mg + T\sin\alpha$$

$$F\sin\beta = T\cos\alpha$$

Dividing these equations yields

$$\frac{F\,cos\,\beta}{F\,sin\,\beta} = cot\,an\,\beta = \frac{(1/2)mg + T\,sin\,\alpha}{T\,cos\,\alpha} .$$

So

$$\beta = cot\,an^{-1}\left(\frac{(1/2)mg + T\,sin\,\alpha}{T\,cos\,\alpha}\right)$$

$$\alpha = cot\,an^{-1}\left(\frac{(1/2)(75\,kg)(9.8\,m\cdot s^{-2}) + (4.9\times10^{2}\,N)\,sin(37^{0})}{(4.9\times10^{2}\,N)\,cos(37^{0})}\right) = 31^{0}$$

We can now use the horizontal force equation to calculate the magnitude $\left|\vec{\mathbf{F}}\right|$ of the force of the tibia $\vec{\mathbf{F}}$ on the ankle,

$$F = \frac{(4.9\times10^{2}\,N)\cos(37^{0})}{\sin(31^{0})} = 7.7\times10^{2}\,N$$