Review #3:
Collisions, Rotational Motion

Momentum - Model

From *Principia:* Momentum is defined as "the quantity of motion, conjointly proportional to the mass and the velocity".

momentum is for a **system** of particles

definition ---- $$\vec{\mathbf{p}} = \sum_{i}^{system} m_i \vec{\mathbf{v}}_i$$

momentum is sum of individual momenta

Dynamics (change) of momentum - from F=ma

$$\frac{d\vec{\mathbf{p}}^{system}}{dt} = \sum_{i=1}^{forces} \vec{\mathbf{F}}_i^{external}$$

Only **external** (through system barrier) forces !

Strategies: Momentum and External Forces

1. Identify all forces acting on the masses
2. Select your system (make troublesome forces into internal forces; make $\Sigma F^{ext} = 0$)
3. $\Sigma F^{ext} = 0$ implies momentum is conserved
4. If there is a non-zero total external force:

$$\frac{d\vec{\mathbf{p}}^{system}}{dt} = \sum_{i}^{forces} \vec{\mathbf{F}}_i^{external}$$

Strategy - Impulse

Impulse most useful when time is short, simplifying the momentum change during the short time $\Delta \mathbf{t}$.

A. One large external force dominates $\Delta \mathbf{p}$:

•Ball bouncing on floor - ignore gravity

B. Finite F^{ext} for very short Δt --> $\Delta \mathbf{p}^{system} = 0$

•Colliding cars - ignore horizontal friction

•Gun and Bullet - ignore external forces

Collisions: Momentum Conserved

When the total external force on the colliding particles is much smaller than the internal forces, the collision duration is so short that the impulse on the system is approximately zero.

Then the total initial momentum of the colliding particles equals their final momentum:

$$\vec{\mathbf{p}}_f^{\,total} = \vec{\mathbf{p}}_0^{\,total}$$

Gives one equation each for x, y, and z

Elastic Collisions

- Kinetic Energy does not change.

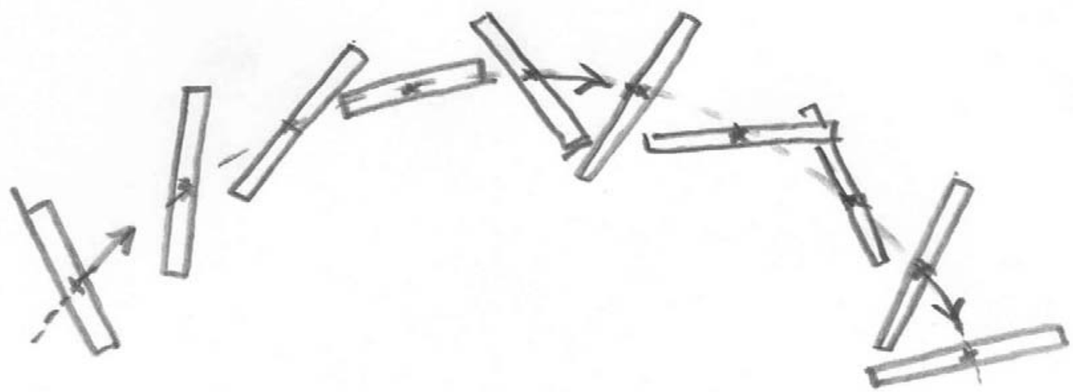
$$K_0^{total} = K_f^{total}$$

$$\frac{1}{2}m_1 v_{1,0}^{\ 2} + \frac{1}{2}m_2 v_{2,0}^{\ 2} + ... = \frac{1}{2}m_1 v_{1,f}^{\ 2} + \frac{1}{2}m_2 v_{2,f}^{\ 2} + ...$$

- Momentum conserved also

Rotation and Translation of Rigid Body

Motion of a thrown object



Translational Motion of the Center of Mass

- Total momentum $\vec{\mathbf{p}}^{total} = m^{total}\vec{\mathbf{V}}_{cm}$

- External force and acceleration of center of mass

$$\vec{\mathbf{F}}_{ext}^{total} = \frac{d\vec{\mathbf{p}}^{total}}{dt} = m^{total}\frac{d\vec{\mathbf{V}}_{cm}}{dt} = m^{total}\vec{\mathbf{A}}_{cm}$$

Rotation and Translation of Rigid Body

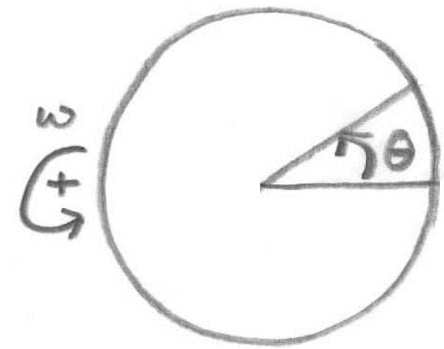
- Torque produces angular acceleration about center of mass

$$\sum_{i}^{torques} \tau_{cm,i} = I_{cm}\alpha_{cm}$$

- I_{cm} is the moment of inertial about the center of mass
- α is the angular acceleration about center of mass or any other point in a rigid body.
- This is really a vector relation; only z-component is non-zero if problem is planar

Fixed Axis Rotation: Kinematics

Angle variable θ



Angular velocity $\omega \equiv \dfrac{d\theta}{dt}$

Angular acceleration $\alpha \equiv \dfrac{d^2\theta}{dt^2}$

These are exactly analogous to the variables x, v_x, and a_x for One dimensional motion, and obey the same equations

For constant angular acceleration:

$$\theta(t) = \theta_0 + \omega_0(t - t_0) + \frac{1}{2}\alpha(t - t_0)^2$$

$$\omega(t) = \omega_0 + \alpha(t - t_0)$$

$$[\omega(t)]^2 = \omega_0^2 + 2\alpha(\theta(t) - \theta_0)$$

Fixed Axis Rotation: Tangential Velocity and Tangential Acceleration

Kinematics of individual mass elements:

$$\Delta m_i$$

- Tangential velocity

$$v_{\tan,i} = r_{\perp,i}\omega$$

- Tangential acceleration

$$a_{\tan,i} = r_{\perp,i}\alpha$$

- Radial Acceleration

$$a_{rad,i} = \frac{v_{\tan,i}^2}{r_{\perp,i}} = r_{\perp,i}\omega^2$$

PRS: Ladybug Acceleration

A ladybug sits at the outer edge of a merry-go-round that is turning and slowing down. The tangential component of the ladybug's (Cartesian) acceleration is

QuickTime™ and a
Graphics decompressor
are needed to see this picture.

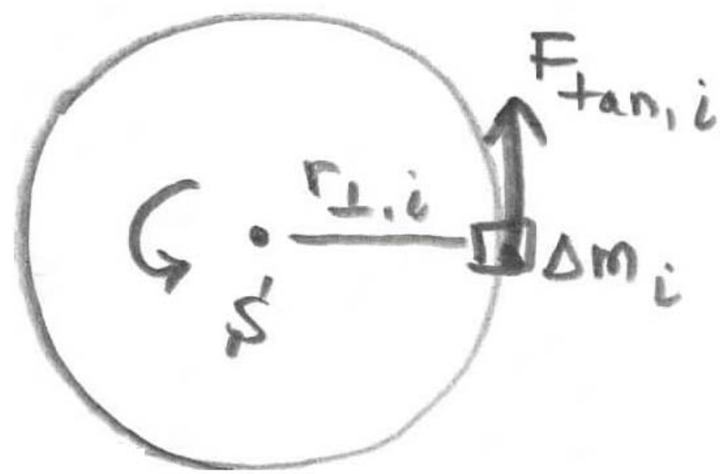
1. in the +*x* direction.
2. in the -*x* direction.
3. in the +*y* direction.
4. in the -*y* direction.
5. in the +*z* direction.
6. in the -*z* direction.
7. zero.

Torque

Torque about axis S:

$$\vec{\tau}_{S,i} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i$$

- Counterclockwise +z direction
- perpendicular to the plane



$$\tau_{S,i} = r_{S\perp,i} F_{\tan,i} = r_{S\perp,i} \Delta m_i a_y = \Delta m_i (r_{S\perp,i})^2 \alpha$$

$$\sum_i^{torques} \tau_{S,i} = I_S \alpha_{cm} = I_S \alpha_S$$

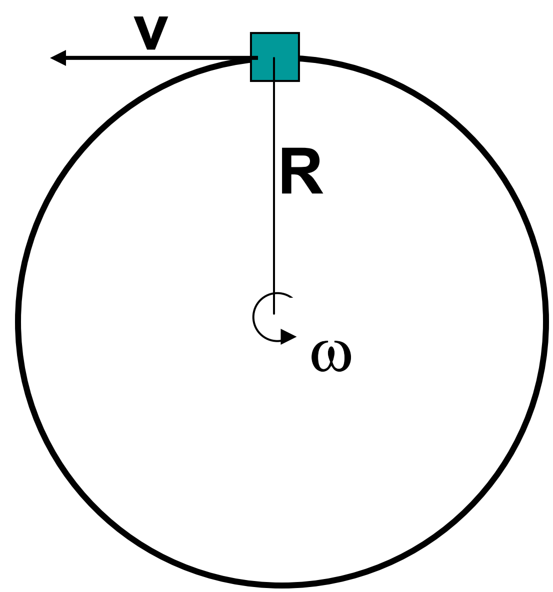
Energy of Rotating Mass

The mass is rotating:

Angular velocity ω

Radius R

Speed v

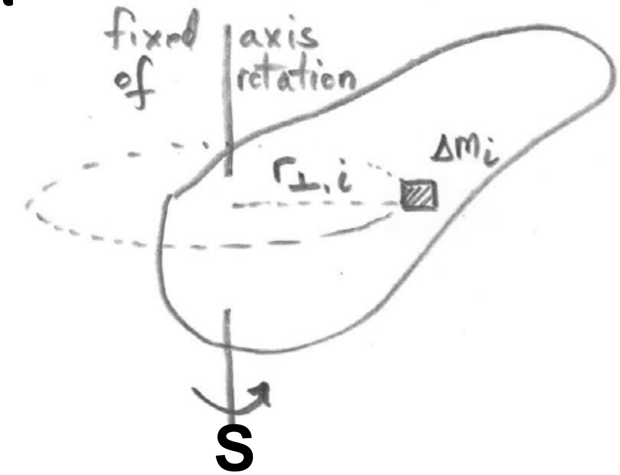


Speed $v = \omega R$

Kinetic Energy $= 1/2\, m\, v^2 = 1/2\, m\, R^2\omega^2 = 1/2\, I_s\, \omega^2$

$$K_S^{rot} = \frac{1}{2} I_S \omega_S^2$$

Moment of Inertia - Idea



Mass element Δm_i

Radius of orbit $r_{\perp,i}$

Moment of Inertia about S $I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{S\perp,i})^2$

The moment of inertia takes the place of mass in the Dynamical (F=ma) and Energy (K=mv²/2) equations for rotational motion.

$$\sum_i^{torques} \tau_{S,i} = I_S \alpha_{cm}$$

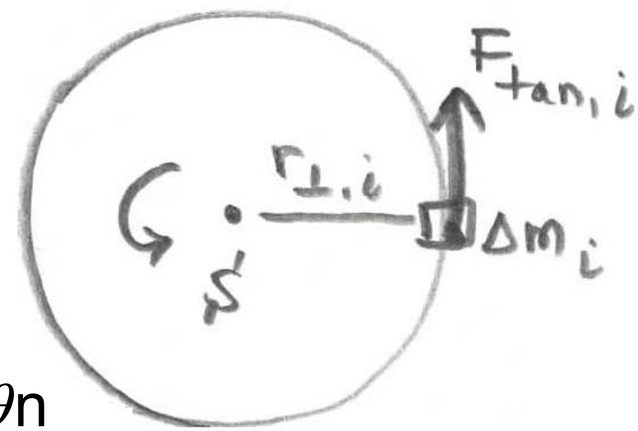
$$K_S^{rot} = \frac{1}{2} I_S \omega_S^2$$

These work about ANY NON-ACCELERATING axis

Rotational Work

Starting from

$$\Delta W_i = \vec{\mathbf{F}}_i \cdot \Delta \vec{\mathbf{r}}_i$$



- tangential force

$$\vec{\mathbf{F}}_{\tan,i} = F_{\tan,i}\,\hat{\theta}$$

- displacement vector

$$\Delta \vec{\mathbf{r}}_{S,i} = \left(r_{S,\perp}\right)_i \Delta\theta\,\hat{\theta}$$

- infinitesimal work $\Delta W_i = \vec{\mathbf{F}}_i \cdot \Delta \vec{\mathbf{r}}_i = F_{\tan,i}\left(r_{S,\perp}\right)_i \Delta\theta = \tau_i \Delta\theta$

- Rotational work:

$$W_{fi}^{rot} = \int_i^f \tau(\theta)\,d\theta = \tau_{avg}(\theta_f - \theta_i)$$

Note: if τ is constant, it equals τ_{avg}

General Work-Kinetic Energy Rel'n

- Fixed axis passing through the c of m of the body

$$W_{f0}^{rot} = \frac{1}{2} I_{cm} \omega_{cm,f}^2 - \frac{1}{2} I_{cm} \omega_{cm,0}^2 = K_{rot,f} - K_{rot,0} \equiv \Delta K_{rot}$$

- Rotation and translation combined - General Motion

$$K_f = \left(\frac{1}{2} m v_{cm,f}^2 + \frac{1}{2} I_{cm} \omega_f^2 \right) = \left(\frac{1}{2} m v_{cm,0}^2 + \frac{1}{2} I_{cm} \omega_0^2 \right) + W_{fi}^{trans} + W_{fi}^{rot} = +K_0 + W_{fi}^{total}$$

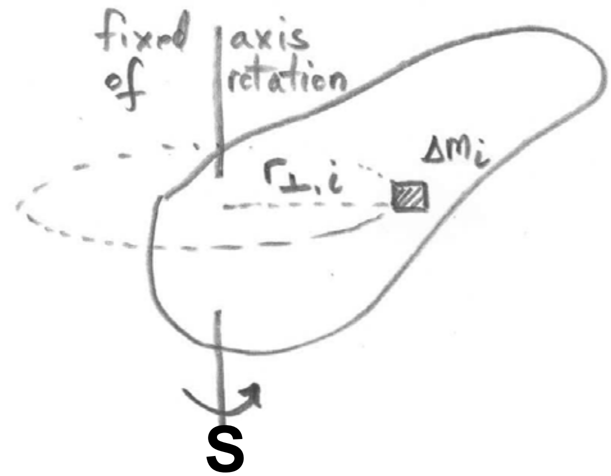
$$W_{total} = \Delta K_{trans} + \Delta K_{rot}$$

$$W_{fi}^{rot} = \int_i^f \tau(\theta) d\theta = \tau_{avg} (\theta_f - \theta_i)$$

Note: if τ is constant, it equals τ_{avg}

Strategy: Moment of Inertia

Always start from a tabulated I_{cm} plus the Parallel Axis Theorem



Use $$I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{S\perp,i})^2 = \int dm\ (r_{S\perp,dm})^2$$

When all else fails

$$r_\perp$$

Note: $$I_{cm} = \int dm\ (r_{cm\perp,dm})^2 = \alpha M R^2$$

Where α is between 0 and 1

Table of I_{cm}

Object	α
Hoop	1.0
Disk	0.5
Sphere	0.4
Rod	$ML^2/12$

Rotational Dynamics & Energy - Summary

- Dynamical Equations about axis **S**

Dynamics

$$\sum_{i}^{torques} \tau_{S,i} = I_S \alpha_{cm} = I_S \alpha_S$$

Is not true if **S** accelerates

Kinetic Energy

$$K_S^{rot} = \frac{1}{2} I_S \omega_S^2$$

Requires **S** to be stationary

Includes KE of c of m

- Most General Equations

$$\sum_{i}^{torques} \tau_{cm,i} = I_{cm} \alpha_{cm}$$

$$\Sigma_i \vec{F}_i^{ext} = M \vec{a}_{cm}$$

OK if c of m accelerates

$$K^{tot} = \frac{1}{2} I_{cm} \omega_{cm}^2 + \frac{1}{2} M \, \mathrm{v}_{cm}^2$$

KE of rotation+KE of translation

PRS - kinetic energy

A disk with mass M and radius R is spinning with angular velocity ω about an axis that passes through the rim of the disk perpendicular to its plane. Its total kinetic energy is:

1. $1/4\ M R^2 \omega^2$
2. $1/2 M R^2 \omega^2$
3. $3/4\ M R^2 \omega^2$
4. $1/4 M R \omega^2$
5. $1/2\ M R \omega^2$
6. $1/2\ M R \omega$

Dynamics: Translational & Rotational

Translational Dynamics

- Total Force $\sum_i \vec{\mathbf{F}}_i^{ext}$
- Momentum of System $\vec{\mathbf{p}}^{system}$
- Dynamics of Translation $\sum_i \vec{\mathbf{F}}_i^{ext} = \dfrac{d\vec{\mathbf{p}}^{system}}{dt}$

Rotational Dynamics of point mass about S

- Torque $\vec{\tau}_{S,i} = \vec{\mathbf{r}}_{i,S} \times \vec{\mathbf{F}}_i$
- Angular momentum $\vec{\mathbf{L}}_{S,i} = \vec{\mathbf{r}}_{i,S} \times \vec{\mathbf{p}}_i$
- Dynamics of rotation $\sum_i \vec{\tau}_{S,i}^{ext} = \dfrac{d\vec{\mathbf{L}}_S}{dt}$

Rotational Angular Momentum & Energy

- Putting it Together - Spin plus Translation

Angular Momentum

$$\vec{\mathbf{L}}_S^{system} = I_{cm}\vec{\omega} + \sum_{i=1}^{i=N} \vec{\mathbf{R}}_{cm,S} \times \vec{\mathbf{p}}$$

L of rotation+L of translation

Kinetic Energy

$$K^{tot} = \frac{1}{2} I_{cm}\omega_{cm}^2 + \frac{1}{2} M \mathrm{v}_{cm}^2$$

KE of rotation+KE of translation

Dynamics (i.e. change)

$$\sum_i \vec{\tau}_{S,i}^{ext} = \frac{d\vec{\mathbf{L}}_S^{system}}{dt}$$

$$K_f = K_0 + W_{fi}^{trans} + W_{fi}^{rot}$$

cf. $$\sum_i \vec{\mathbf{F}}_i^{ext} = \frac{d\vec{\mathbf{p}}}{dt}$$

Physical Content of $\sum_i \tau_{S,z}^{ext} = \frac{d\mathbf{L}_z^{system}}{dt}$

- In the case that $I_{S,z}$ does not change:

$$\sum_i \tau_{S,z}^{ext} = \frac{d\mathbf{L}_z^{system}}{dt} = \frac{d\left(I_{S,z}\omega_z\right)}{dt} = I_{S,z}\frac{d\omega_z}{dt} = I_{S,z}\frac{d^2\theta}{dt^2} = I_{S,z}\alpha_z$$

- But $I_{S,z}$ may change:
  - Spinning Skater pulls in arms
  - Rain falling on merry-go -round

- Conservation of L_z is richer than p_z

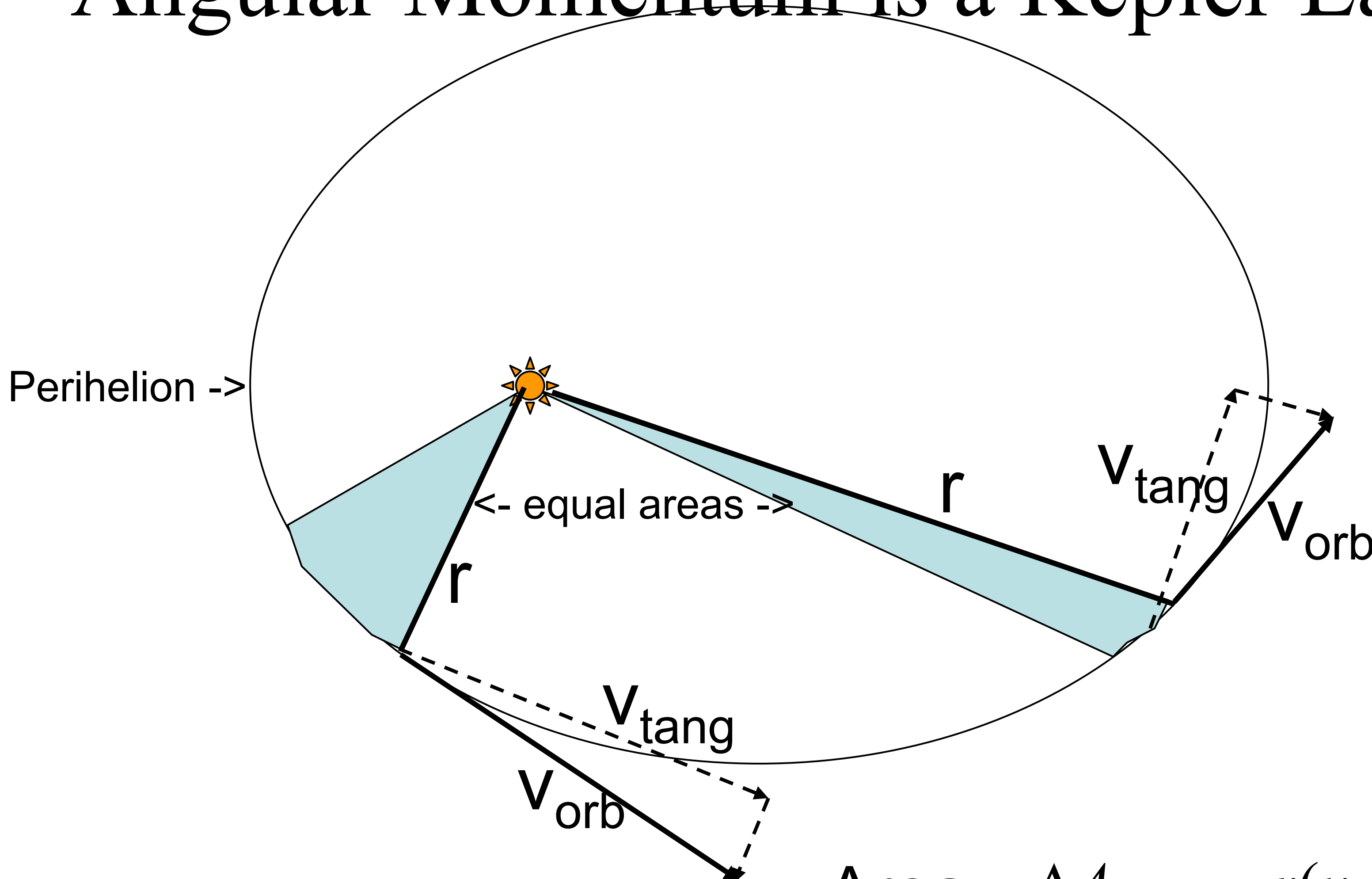
$$\sum_i \tau_{S,z}^{ext} = 0 \quad \Rightarrow \quad \mathbf{L}_{z,f}^{system} = \mathbf{L}_{z,i}^{system}$$

PRS - angular momentum

A disk with mass M and radius R is spinning with angular velocity ω about an axis that passes through the rim of the disk perpendicular to its plane. The magnitude of its angular momentum is:

1. **1/4 M R^2 ω^2**
2. **1/2M R^2 ω^2**
3. **3/4 M R^2 ω^2**
4. **1/2M R ω**
5. **3/4 M R ω**
6. **3/2 M R ω**

Angular Momentum is a Kepler Law



Area: $\Delta A_{swept} = r(v_{tang}\Delta t)/2$

Angular Momentum:

$$L_{sun} = \mu v_{tang} r$$

$$\frac{\Delta A_{swept}}{\Delta t} = \frac{1}{2} r v_{tang} = \frac{L}{2\mu}$$

Circular Orbital Mechanics
-class problem

A planet of mass m_1 is in a circular orbit of radius R around a sun of mass m_2.

a. Find the period, T
b. Find the ratio of kinetic to potential energy
c. As R increases, all of the physical properties of the orbit (e.g. velocity) decrease except one; find it.