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Assignment #6 Solutions

Problem 6.1

Flip coil. (Giancoli 29-62)

The coil starts out with its face perpendicular to the magnetic field (more formally speaking, the normal to the plane surface bounded by the coil is parallel to the magnetic field). If we choose our sign convention such that the magnetic flux throught the coil (i.e. through an open surface bounded by the coil) is initially $\Phi_{B,i} = NAB$, then after the 180° flip the flux will be $\Phi_{B,f} = -NAB$. So the total change in flux through the coil over the course of the flip is $|\Delta\Phi_B| = 2NAB$.

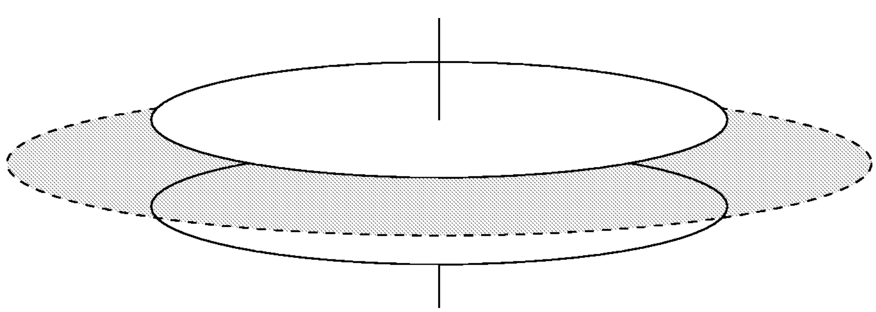
Putting Faraday's law together with Ohm's law and the definition of current as charge-flow per unit time, we have

$$-\frac{d\Phi_B}{dt} = \mathcal{E} = RI = R\frac{dQ}{dt}$$

at any instant during the flip. We can eliminate t to obtain the relation $|d\Phi_B| = R\,dQ$, which when integrated over the entire flip becomes $|\Delta\Phi_B| = RQ$. From above, $|\Delta\Phi_B| = 2NAB$, so $2NAB = RQ$, or $B = RQ/2NA$.

Problem 6.2

Displacement current. (Giancoli 32-4.)



Using Giancoli Equation (24-2) (p. 615) for the capacitance of a parallel-plate capacitor of plate area A and plate separation d (and the fact that the electric field is nearly uniform between the plates at any given time),

$$C = \epsilon_0\frac{A}{d} = \frac{Q}{V} = \frac{Q}{Ed} \Rightarrow E = \frac{Q}{\epsilon_0 A} \quad .$$

Now let's apply Ampère's law in the form of Giancoli Equation (32-1) (p. 789) to a circular Amperian loop of radius $r = 10.0\,\text{cm}$ as shown in the diagram above (note: diagram not to scale). We can expect based on the rotational symmetry of this arrangement that the magnetic field along the loop will be tangent to the loop, with magnitude dependent only upon r. The left-hand side of Ampère's law then becomes

$$\oint \mathbf{B} \cdot d\boldsymbol{l} = \oint B\,dl = B \oint dl = 2\pi r B \quad .$$

For the right-hand side of Ampère's law we consider the plane surface passing between the plates of the capacitor and bounded by our loop. There is **no** current passing through this surface, so $I_{\text{encl}} = 0$. However, there *is* an electric flux through the surface, given by $\Phi_E = EA$ (A is the area of the *capacitor* plates: there is no electric field in the region outside the capacitor). From our results above, we may calculate

$$\frac{d\Phi_E}{dt} = \frac{d}{dt}(EA) = \frac{d}{dt}\left(\frac{Q}{\epsilon_0 A}A\right) = \frac{1}{\epsilon_0}\frac{dQ}{dt} \quad .$$

Ampère's law then gives us

$$\oint \mathbf{B} \cdot d\boldsymbol{l} = \mu_0 I_{\text{encl}} + \mu_0\epsilon_0 \frac{d\Phi_E}{dt} \Longrightarrow 2\pi r B = \mu_0 \frac{dQ}{dt} \Longrightarrow B = \frac{\mu_0\, dQ/dt}{2\pi r} \quad .$$

Note that this result does *not* depend upon the precise dimensions of the capacitor. Putting in numbers for the charge-up,

$$B = \frac{(4\pi \times 10^{-7})(0.0350)}{2\pi(0.100)} = 70\,\text{nT} \quad .$$

When the capacitor is fully charged, $dQ/dT = 0$, and therefore $B = 0$.

Problem 6.3

Self-inductance of a toroid. (Giancoli 30-48.)

(a) First let's address the issue of treating the magnetic field inside the toroid as uniform. At any point inside the toroid, the distance from the center of the big circle may be expressed as $r = r_0 + \delta$, where $\delta < d$. Using the result of Giancoli Example 28-8 (p. 718) for the magnetic field inside a toroid, we have

$$B = \frac{\mu_0 NI}{2\pi r} = \frac{\mu_0 NI}{2\pi(r_0 + \delta)} = \frac{\mu_0 NI}{2\pi r_0(1 + \delta/r_0)} \approx \frac{\mu_0 NI}{2\pi r_0} \quad \text{when } r_0 \gg d > \delta.$$

Within this approximation, the magnetic flux through one winding of the toroid (and due to a current in that same toroid) is

$$\Phi_B \approx \pi(d/2)^2 B \approx \frac{\mu_0 NId^2}{8r_0} \quad .$$

By Giancoli Equation (30-4) (p. 758), the toroid's self-inductance is then

$$L = \frac{N\Phi_B}{I} \approx \frac{\mu_0 N^2 d^2}{8r_0} \quad ,$$

as we were to show. This is indeed consistent with the result for a solenoid of length $l = 2\pi r_0$ and cross-sectional area $A = \pi(d/2)^2$ (see Giancoli Example 30-3, p. 759), as it should be:

$$L = \frac{\mu_0 N^2 A}{l} = \frac{\mu_0 N^2 \pi (d/2)^2}{2\pi r_0} = \frac{\mu_0 N^2 d^2}{8r_0} \quad .$$

Our toroid is just a solenoid bent into a donut, and to the extent that we can ignore terms of order d/r_0, the bending has no effect.

(b) Plugging in the given numbers,

$$L \approx \frac{(4\pi \times 10^{-7})(550)^2(0.020)^2}{8(0.25)} = 76\,\mu\text{H} \quad .$$

Problem 6.4

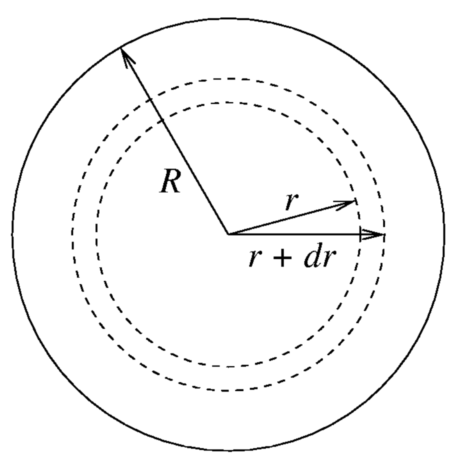
Magnetic field energy and self-inductance.

(a) From the results of problem 5.2, we have the following expression for the magnetic field magnitude inside the wire as a function of distance r from the wire's axis:

$$B = \frac{\mu_0 I r}{2\pi R^2} \quad .$$

Using Giancoli Equation (30-7) (p. 761), we can express the magnetic field energy density (that is, Joules per cubic meter) as

$$u = \frac{B^2}{2\mu_0} = \frac{1}{2\mu_0}\left(\frac{\mu_0 I r}{2\pi R^2}\right)^2 = \frac{\mu_0 I^2 r^2}{8\pi^2 R^4} \quad .$$



Consider now a cylindrical shell of length ℓ, radius r, and thickness dr within the wire. Such a shell has volume

$$d(\text{volume}) = 2\pi r \ell \, dr \quad ,$$

and in the limit $dr \to 0$ the magnetic field energy density is constant throughout the shell. The total magnetic field energy inside a length ℓ of the wire can thus be found by integrating:

$$U_{\text{in}} = \int u \, d(\text{volume}) = \int_0^R \left(\frac{\mu_0 I^2 r^2}{8\pi^2 R^4}\right)(2\pi r \ell \, dr) = \left(\frac{\mu_0 I^2 \ell}{4\pi R^4}\right)\int_0^R r^3 \, dr = \frac{\mu_0 I^2 \ell}{16\pi} \quad .$$

(b) From the relationship between stored magnetic field energy and self-inductance (Giancoli Equation (30-6), p. 761), we can identify the self-inductance due to the interior of the wire:

$$U_{\rm in} = \frac{1}{2} L_{\rm in} I^2 = \frac{\mu_0 I^2 \ell}{16\pi} \Longrightarrow L_{\rm in} = \frac{\mu_0 \ell}{8\pi} \quad .$$

The wire therefore has a self-inductance per unit length $\mu_0/8\pi$ associated with its internal magnetic field.

Problem 6.5

RL circuit.

Giancoli Equation (30-9) (p. 762) gives us an expression for the time-dependent current in an LR circuit connected to a battery of voltage V_0 at time $t = 0$:

$$I(t) = \frac{V_0}{R}\left(1 - e^{-t/\tau}\right) \ , \quad \tau = L/R \quad .$$

We will make good use of this! For our circuit, the time constant is $\tau = (0.09)/(0.05) = 1.8\,\mathrm{s}$.

(a) The final value of the current is simply V_0/R. Let $\alpha = 0.05$; the condition that the current has reached 95% of its final value can then be expressed as

$$\begin{aligned} I = \frac{V_0}{R}\left(1 - e^{-t/\tau}\right) &= (1-\alpha)\frac{V_0}{R} \\ \Rightarrow e^{-t/\tau} &= \alpha \\ \Rightarrow t &= -\tau \ln \alpha = 5.4\,\mathrm{s} \quad . \end{aligned}$$

(b) Using Giancoli Equation (30-6) (p. 761), we have for the energy stored in the magnetic field

$$U = \frac{1}{2} L I^2 = \frac{1}{2} L (1-\alpha)^2 \frac{V_0^2}{R^2} = \frac{1}{2}(0.09)(0.95)^2 \frac{(12)^2}{(0.05)^2} \simeq 2300\,\mathrm{J} \quad .$$

(c) The power delivered by the battery at any instant is

$$P(t) = V_0 I(t) = \frac{V_0^2}{R}\left(1 - e^{-t/\tau}\right) \quad .$$

To find the total energy W delivered by the battery up to the time t found in part (a) we integrate:

$$\begin{aligned} W &= \int_0^t P(t')\,dt' = \frac{V_0^2}{R}\int_0^t \left(1 - e^{-t'/\tau}\right) dt' = \frac{V_0^2}{R}\left[t' + \tau e^{-t'/\tau}\right]_0^t \\ &= \frac{V_0^2}{R}\left[t - (1-\alpha)\tau\right] = \frac{(12)^2}{(0.05)}[5.4 - (0.95)(1.8)] \simeq 10600\,\mathrm{J} \quad . \end{aligned}$$

So, $\simeq 10600 - 2300 = 8300$ J have been dissipated in the resistor.

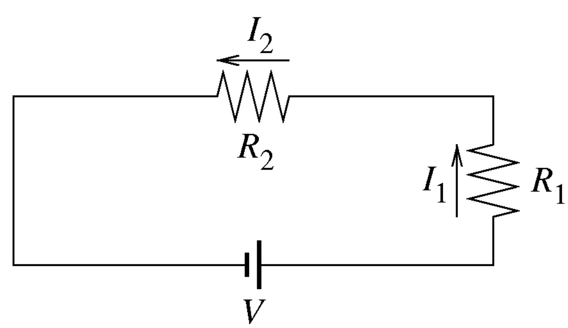
Problem 6.6

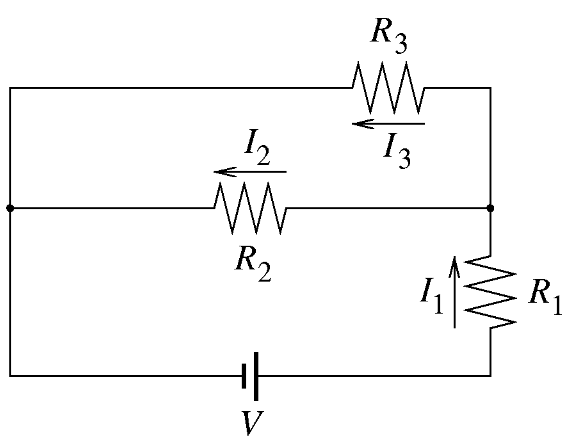
RL circuit. (Giancoli 30-30.)

The inductor in the upper branch of the circuit will resist instantaneous changes in I_3. Negligible inductance in the rest of the circuit means that I_1 and I_2 can respond essentially instantaneously to the opening and closing of the switch. (Note: these solutions use V instead of $\mathcal{E}$ to denote the battery voltage.)

(a) Before the switch is closed, all currents are zero. *Immediately* after the switch is closed, $I_3 = 0$ still. At this instant, we may regard the circuit effectively as shown at right. This situation solves easily to give

$$I_1 = I_2 = \frac{V}{R_1 + R_2} \quad .$$





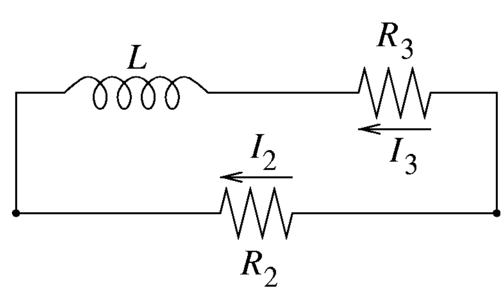
(b) After the switch has been closed for a long time, all the currents will have reached unchanging, steady-state values. Inductors have no effect when currents are not changing in time, so we may treat the circuit as shown at left. Applying Kirchhoff's rules we obtain

$$I_1 = I_2 + I_3 \ , \quad I_1R_1 + I_2R_2 = I_1R_1 + I_3R_3 = V \quad .$$

A bit of algebra then yields

$$I_1 = V\left(R_1 + \frac{R_2R_3}{R_2+R_3}\right)^{-1} , \ I_2 = V\left(R_1 + R_2 + \frac{R_1R_2}{R_3}\right)^{-1} , \ I_3 = V\left(R_1 + R_3 + \frac{R_1R_3}{R_2}\right)^{-1} \quad .$$

(c) Immediately after the switch is opened again, the branch containing the battery and R_1 is taken out of action, so $I_1 = 0$. The remaining circuit is as shown at right. I_3 must be continuous in time due to the inductor, so it has the same value as found in part (b). Kirchhoff's loop rule (a.k.a. charge conservation) then dictates $I_2 = -I_3$.

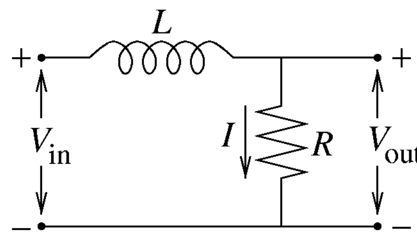


(d) A long time after the switch is reopened, the currents of part (c) will have decayed, leaving $I_1 = I_2 = I_3 = 0$.

Problem 6.7

Integrating circuit. (Giancoli 30-57.)

Let's define the current I as shown in the diagram at right. Also take the sign conventions for V_{in} and V_{out} as shown. V_{in} can be regarded as a sort of time-dependent battery voltage, and V_{out} as the readout of a voltmeter with a very large internal resistance compared to R. Essentially, V_{out} is just a measure of the current in the resistor: $V_{\text{out}} = RI$.



Expressing Faraday's law for the LRV_{in} loop, we have

$$V_{\text{in}} - L\frac{dI}{dt} - RI = 0 \quad .$$

Following Giancoli's hint, we multiply by $e^{Rt/L}$ and recognize that we can "undo the product rule":

$$\begin{aligned} e^{Rt/L}V_{\text{in}} - e^{Rt/L}L\frac{dI}{dt} - e^{Rt/L}RI &= 0 \\ \Rightarrow e^{Rt/L}V_{\text{in}} - \frac{d}{dt}\left(e^{Rt/L}LI\right) &= 0 \quad . \end{aligned}$$

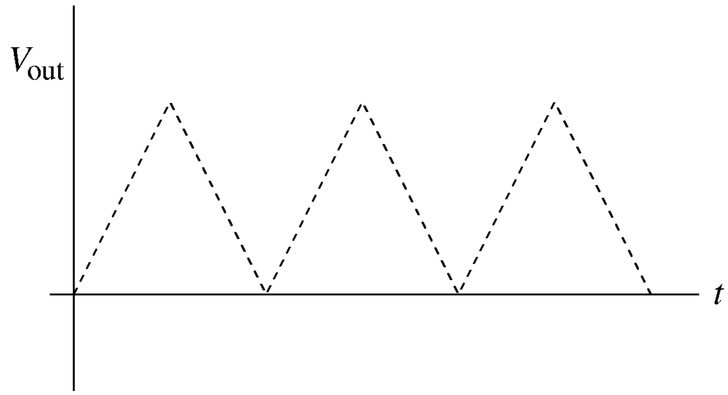
Integrating this equation from 0 to time t, we get

$$\int_0^t e^{Rt'/L}V_{\text{in}}(t')\,dt' = e^{Rt/L}LI(t) - LI_0$$

Now, if the time constant L/R is very large compared with the time over which V_{in} varies, we may take $e^{Rt/L} \approx e^0 = 1$. This leaves us with

$$\begin{aligned} \int_0^t V_{\text{in}}(t')\,dt' &= LI(t) - LI_0 \\ \Rightarrow V_{\text{out}}(t) &= V_{\text{out}}(t=0) + \frac{R}{L}\int_0^t V_{\text{in}}(t')\,dt' \quad . \end{aligned}$$

This is the sense in which the circuit integrates. Note that the assumption of L/R being much greater than the timescale of variation means that the typical magnitude of V_{out} will be much less than that of V_{in}. A sketch of the output voltage for the given square-wave input signal is shown below (taking $V_{\text{out}}(t=0) = 0$).



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