8.022 (E&M) – Lecture 5

Topics:

- More on conductors… and many demos!
- Capacitors

Last time…

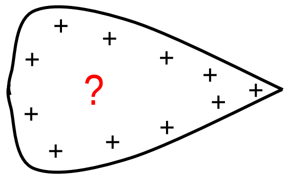
- Curl: $\text{curl } \vec{F} = \nabla \times \vec{F}$
  - Stoke's theorem: $\oint_C \vec{F} \bullet d\vec{s} = \int_A \text{curl } \vec{F} \bullet d\vec{A} \quad \Rightarrow \quad \nabla \times \vec{E} = 0$
- Laplacian: $\nabla^2 \phi \equiv \nabla \bullet \nabla \phi$

$$\nabla^2 \phi = -4\pi\rho \text{ (Poisson)} \xrightarrow{\text{in vacuum}} \nabla^2 \phi = 0 \text{ (Laplace)}$$

- Conductors
  - Materials with free electrons (e.g. metals)
  - Properties:
    - Inside a conductor E=0
    - $E_{surface} = 4\pi\sigma$
    - Field lines perpendicular to the surface → surface is equipotential
- Uniqueness Theorem
  - Given $\rho(xyz)$ and boundary conditions, the solution $\phi(xyz)$ is unique

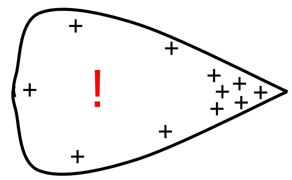
Charge distribution on a conductor

- Let's deposit a charge Q on a tear drop-shaped conductor
- How will the charge distribute on the surface? Uniformly?



- Experimental answer: NO! (Demo D28)
  - $\sigma_{tip} >> \sigma_{flat}$
- Important consequence
  - Although ϕ=const, $E=4\pi\sigma$

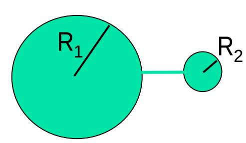
    → $E_{tip} >> E_{flat}$
- Why?



Charge distribution on a conductor (2)

- Qualitative explanation
  - Consider 2 spherical conductors connected by conductive wire
  - Radii: R_1 and R_2 with $R_1 >> R_2$
  - Deposit a charge Q on one of them

    → charge redistributes itself until ϕ=constant



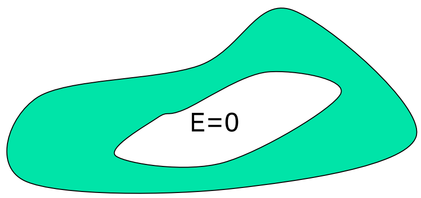
$$\begin{cases} \phi_1 = \dfrac{Q_1}{R_1} = \dfrac{Q_2}{R_2} = \phi_2 \\ E_1 = \dfrac{Q_1}{R_1^{\,2}} = \dfrac{\phi_1}{R_1} \\ E_2 = \dfrac{Q_2}{R_2^{\,2}} = \dfrac{\phi_2}{R_2} \end{cases} \Rightarrow \boxed{\frac{E_1}{E_2} = \frac{R_2}{R_1}} \Rightarrow \boxed{\frac{\sigma_1}{\sigma_2} = \frac{R_2}{R_1}}$$

- Conclusion:

  Electric field is stronger where curvature (1/R) is larger
- More experimental evidence: D29 (Lightning with Van der Graaf)

Shielding

We proved that in a hollow region inside a conductor E=0

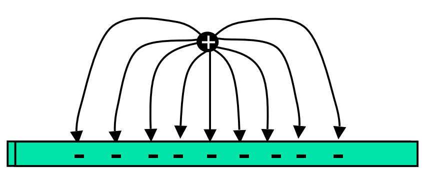


- This is the principle of shielding
- Do we need a solid conductor or would a mesh do?
  - Demo D32 (Faraday's cage in Van der Graaf)
  - Is shielding perfect?

Application of Uniqueness Theorem:

Method of images

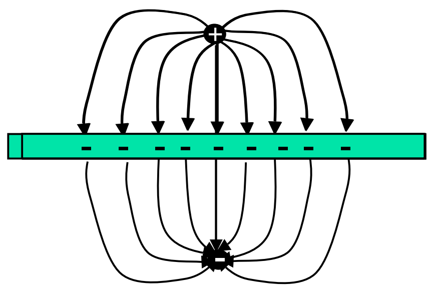
- What is the electric potential created by a point charge +Q at a distance y from an infinite conductive plane?
- Consider field lines:
  - Radial around the charge
  - Perpendicular to the surface conductor



- The point charge +Q induces – charges on the conductor

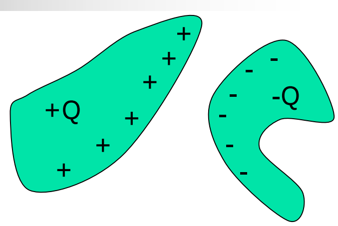
Method of images

- Apply the uniqueness theorem
  - It does not matter how you find the potential ϕ as long as the boundary conditions are satisfied. The solution is unique.
  - In our case: on the conductor surface: $\phi=0$ and always perpendicular
- Can we find an easier configuration of charges that will create the same field lines above the conductor surface?
  - YES!
  - For this system of point charges we can calculate $\phi(x,y,z)$ anywhere
  - This is THE solution (uniqueness)
  - NB: we do not care what happens below the surface of the conductor: that is nor the region under study



Capacitance

- Consider 2 conductors at a certain distance
- Deposit charge +Q on one and –Q on the other
  - They are conductors
    → each surface is equipotential



- What is the $\Delta\phi$ between the 2?
  - Let's try to calculate:

$$V \equiv \phi_2 - \phi_1 = -\int_1^2 \vec{E} \bullet d\vec{s} = Q \times (\text{constant depending on geometry})$$

  - Caveat: C is proportional to Q only if there is enough Q, uniformly spread...
- Naming the proportionality constant 1/C: $\Rightarrow$ $\boxed{Q = CV}$
- Definitions:
  - C = capacitance of the system
  - Capacitor: system of 2 oppositely charged conductors

Units of capacitance

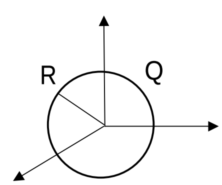
- Definition of capacitance:

$$Q = CV \quad \Rightarrow \quad C = \frac{Q}{V}$$

- Units:
  - SI: Farad (F) = Coulomb/Volt
  - cgs: cm = esu/(esu/cm)
  - Conversion: 1 cm = 1.11 x 10^{-12} F ~ 1 pF
- Remember:
  - 1 Coulomb is a BIG charge: 1 F is a BIG capacitance
  - Usual C ~ pF-μF

Simple capacitors:
Isolated Sphere

- Conductive sphere of radius R in (0,0,0) with a charge Q
  - Review questions:
    - Where is the charge located?
      - Hollow sphere? Solid sphere? Why?
    - What is the E everywhere in space?



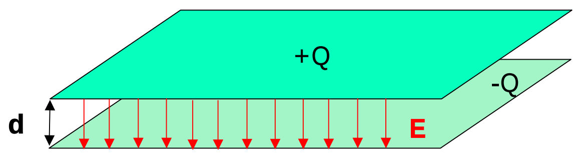
- Is this a capacitor?
  - Yes! The second conductor is a virtual one: infinity
  - Calculate the capacitance:

$$\begin{cases} V = \phi_R - \phi_\infty = Q/R \\ Q = Q \end{cases} \quad \Rightarrow \quad C_{sphere} = R$$

- Capacitors are everywhere!

The prototypical capacitor:
Parallel plates

- Physical configuration:
  - 2 parallel plates, each of area A, at a distance d
    - NB: if $d^2 << A$ → ~ infinite parallel planes
  - Deposit +Q on top plate and –Q on bottom plate



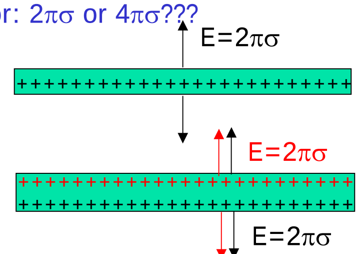
- Capacitance:

$$\begin{cases} V = \int_{top}^{bottom} \vec{E} \bullet d\vec{s} = \int_{top}^{bottom} (4\pi\sigma)\hat{n} \bullet d\vec{n} = 4\pi \left(\frac{Q}{A}\right) d \\ Q = Q \end{cases} \Rightarrow \boxed{C = \frac{Q}{V} = \frac{A}{4\pi d}}$$

Parallel plates capacitor: discussion

- Parallel plates capacitor: $\boxed{C = \frac{Q}{V} = \frac{A}{4\pi d}}$
- Observations:
  - C depends only on the geometry of the arrangement
    - As it should, not on Q deposited or V between the plates!
  - Electric field on surface of conductor: 2πσ or 4πσ???
    - Infinite plane of charges: 2πσ
      - With σ=Q/A
    - Conductor surface: 4πσ
      - With σ=Q/2A

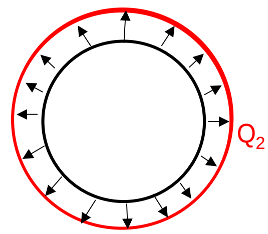
  → No contradiction if σ correctly defined!



  - What is the E outside the capacitor? Zero!

More review questions:

E in Nested Spherical Shells



- Configuration:
  - 2 concentric spherical shells
  - Charge: +Q (–Q) on inner (outer) sphere
- Calculate **E** in the following regions:
  - $r<R_1$, $R_1<r<R_2$, $r>R_2$

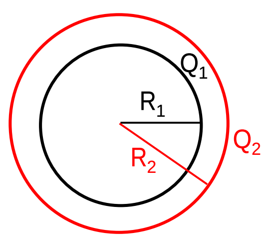
Gauss's law is the key.

- Φ_E on spherical surface with $r<R_1$. $Q_{enc}=0 \Rightarrow E=0$
- Φ_E on spherical surface with $r>R_2$. $Q_{enc}=+Q-Q=0 \Rightarrow E=0$
- Φ_E on spherical surface with $R_1<r<R_2$. $Q_{enc}=+Q \Rightarrow E \neq 0$

$$\Phi_E = \int \vec{E}\bullet d\vec{s} = E(4\pi r) = 4\pi Q \quad \Rightarrow \vec{E}=\frac{Q}{r^2}\hat{r}$$

More capacitors:

Nested Spherical Shells



- Same configuration:
  - 2 concentric spherical shells
  - Charge: +Q (–Q) on inner (outer) sphere
- Capacitance:
  - Key: finding the potential difference V

$$V=\phi_1-\phi_2=-\int_{R2}^{R1}\vec{E}\bullet d\vec{s}=-\int_{R2}^{R1}\frac{Q}{r^2}dr=\frac{Q}{R_1}-\frac{Q}{R_2} \Rightarrow \boxed{C=\frac{Q}{V}=\frac{R_1R_2}{R_2-R_1}}$$

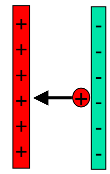
- If $R_2-R_1=d<<R_2 \rightarrow 0$

$$C=\frac{R_1R_2}{R_2-R_1} \sim \frac{R_1^2}{d}=\frac{4\pi R_1^2}{4\pi d}=\frac{A_{sphere}}{4\pi d}$$ same as plane capacitor!

Energy stored in a capacitor

- Consider a capacitor with charge +/-q
- How much work is needed to bring a positive charge dq from the negative plate to the positive plate?
  - NB: we are charging the capacitor!

$$dW = V(q)dq = \frac{q}{C}dq$$



- How much work is needed to charge the capacitor from scratch?

$$W = \int_0^Q dW = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C}$$

- Energy stored in the capacitor: $U = \frac{Q^2}{2C} = \frac{1}{2}CV^2$
- Is this result consistent with what we found earlier?
  - Example: parallel plate capacitor

$$U = \frac{1}{8\pi}\int E^2 dV = \frac{1}{8\pi}E^2 Ad = \frac{1}{8\pi}(4\pi\sigma)^2 Ad\frac{A}{A} = \frac{Q^2}{2}(4\pi\frac{d}{A}) = \frac{1}{2}\frac{Q^2}{C}$$

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Cylindrical Capacitor

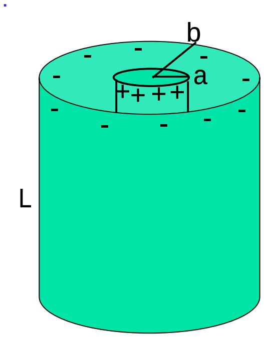
- Concentric cylindrical shells with charge +/-Q. Calculate:
  - Electric Field in between plates

r<a and r>b: E=0 (Gauss)

a<r<b: Gauss's law on cylinder of radius r: $\vec{E}(r) = \frac{2Q}{L}\frac{\hat{r}}{r}$

  - V between plates: $V = \int_a^b \vec{E}\cdot d\vec{r} = \int_a^b \frac{2Q}{L}\frac{dr}{r} = \frac{2Q}{L}\ln\frac{b}{a}$
  - Capacitance C: $C = \frac{Q}{V} = \frac{L}{2\ln\frac{b}{a}}$
  - Calculate energy stored in capacitor:

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{L}{2\ln\frac{b}{a}}\left(\frac{2Q}{L}\ln\frac{b}{a}\right)^2 = \frac{Q^2}{L}\ln\frac{b}{a}$$



Next time...

- More on capacitors
- Charges in motion: currents
- Some help to get ready for quiz #1?
  - Review of Electrostatics?