

Module 2: Math Review

A. Vector Analysis

A.1 Vectors

A.1.1 Introduction

Certain physical quantities such as mass or the absolute temperature at some point only have magnitude. These quantities can be represented by numbers alone, with the appropriate units, and they are called scalars. There are, however, other physical quantities which have both magnitude and direction; the magnitude can stretch or shrink, and the direction can reverse. These quantities can be added in such a way that takes into account both direction and magnitude. Force is an example of a quantity that acts in a certain direction with some magnitude that we measure in newtons. When two forces act on an object, the sum of the forces depends on both the direction and magnitude of the two forces. Position, displacement, velocity, acceleration, force, momentum and torque are all physical quantities that can be represented mathematically by vectors. We shall begin by defining precisely what we mean by a vector.

A.1.2 Properties of a Vector

A vector is a quantity that has both direction and magnitude. Let a vector be denoted by the symbol \vec{A} . The magnitude of \vec{A} is $|\vec{A}| \equiv A$. We can represent vectors as geometric objects using arrows. The length of the arrow corresponds to the magnitude of the vector. The arrow points in the direction of the vector (Figure A.1.1).

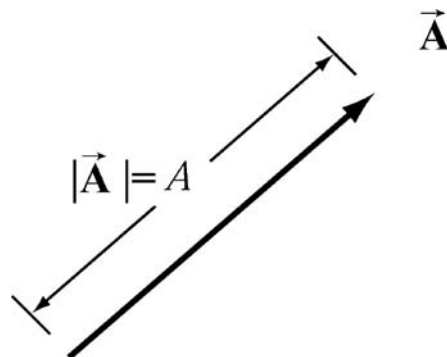


Figure A.1.1 Vectors as arrows.

There are two defining operations for vectors:

(1) **Vector Addition:** Vectors can be added.

Let \vec{A} and \vec{B} be two vectors. We define a new vector, $\vec{C} = \vec{A} + \vec{B}$, the “vector addition” of \vec{A} and \vec{B} , by a geometric construction. Draw the arrow that represents \vec{A} . Place the tail of the arrow that represents \vec{B} at the tip of the arrow for \vec{A} as shown in Figure A.1.2(a). The arrow that starts at the tail of \vec{A} and goes to the tip of \vec{B} is defined to be the “vector addition” $\vec{C} = \vec{A} + \vec{B}$. There is an equivalent construction for the law of vector addition. The vectors \vec{A} and \vec{B} can be drawn with their tails at the same point. The two vectors form the sides of a parallelogram. The diagonal of the parallelogram corresponds to the vector $\vec{C} = \vec{A} + \vec{B}$, as shown in Figure A.1.2(b).

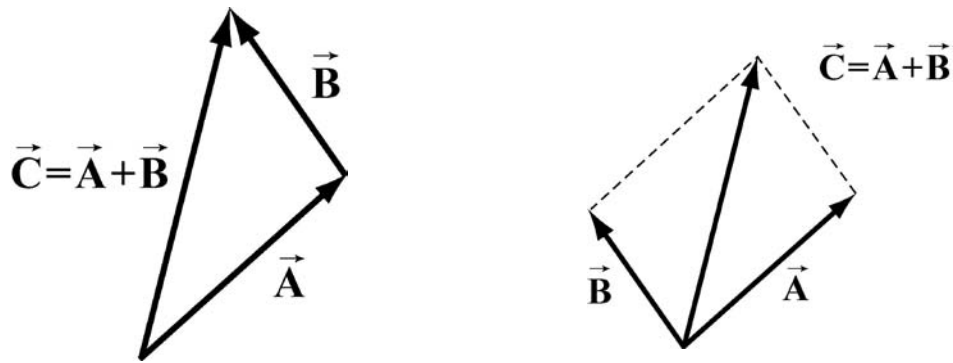


Figure A.1.2 Geometric sum of vectors.

Vector addition satisfies the following four properties:

(i) **Commutivity:** The order of adding vectors does not matter.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A} \quad (\text{A.1.1})$$

Our geometric definition for vector addition satisfies the commutivity property (i) since in the parallelogram representation for the addition of vectors, it doesn't matter which side you start with as seen in Figure A.1.3.

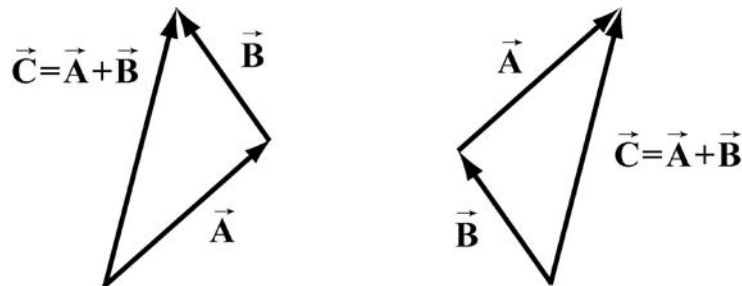


Figure A.1.3 Commutative property of vector addition

(ii) **Associativity:** When adding three vectors, it doesn't matter which two you start with

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}) \quad (\text{A.1.2})$$

In Figure A.1.4(a), we add $(\vec{A} + \vec{B}) + \vec{C}$, while in Figure A.1.4(b) we add $\vec{A} + (\vec{B} + \vec{C})$. We arrive at the same vector sum in either case.

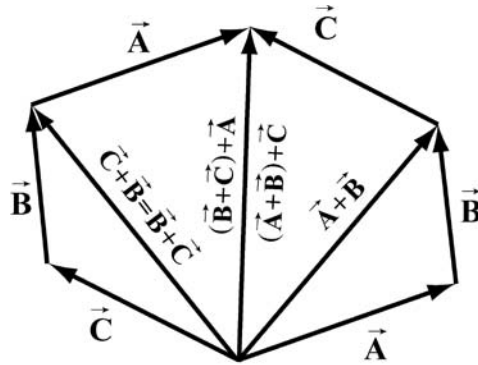


Figure A.1.4 Associative law.

(iii) Identity Element for Vector Addition: There is a unique vector, $\vec{0}$, that acts as an identity element for vector addition.

This means that for all vectors \vec{A} ,

$$\vec{A} + \vec{0} = \vec{0} + \vec{A} = \vec{A} \quad (\text{A.1.3})$$

(iv) Inverse element for Vector Addition: For every vector \vec{A} , there is a unique inverse vector

$$(-1)\vec{A} \equiv -\vec{A} \quad (\text{A.1.4})$$

such that

$$\vec{A} + (-\vec{A}) = \vec{0}$$

This means that the vector $-\vec{A}$ has the same magnitude as \vec{A} , $|\vec{A}| = |-\vec{A}| = A$, but they point in opposite directions (Figure A.1.5).

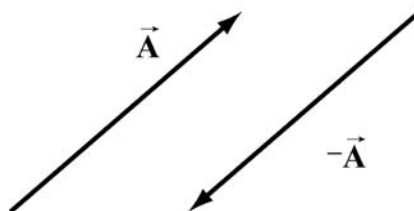


Figure A.1.5 additive inverse.

(2) Scalar Multiplication of Vectors: Vectors can be multiplied by real numbers.

Let \vec{A} be a vector. Let c be a real positive number. Then the multiplication of \vec{A} by c is a new vector which we denote by the symbol $c\vec{A}$. The magnitude of $c\vec{A}$ is c times the magnitude of \vec{A} (Figure A.1.6a),

$$cA = Ac \quad (\text{A.1.5})$$

Since $c > 0$, the direction of $c\vec{A}$ is the same as the direction of \vec{A} . However, the direction of $-c\vec{A}$ is opposite of \vec{A} (Figure A.1.6b).

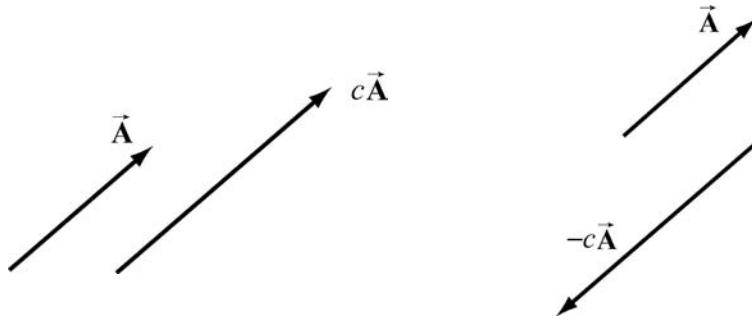


Figure A.1.6 Multiplication of vector \vec{A} by (a) $c > 0$, and (b) $-c < 0$.

Scalar multiplication of vectors satisfies the following properties:

(i) Associative Law for Scalar Multiplication: The order of multiplying numbers is doesn't matter.

Let b and c be real numbers. Then

$$b(c\vec{A}) = (bc)\vec{A} = (cb\vec{A}) = c(b\vec{A}) \quad (\text{A.1.6})$$

(ii) Distributive Law for Vector Addition: Vector addition satisfies a distributive law for multiplication by a number.

Let c be a real number. Then

$$c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B} \quad (\text{A.1.7})$$

Figure A.1.7 illustrates this property.

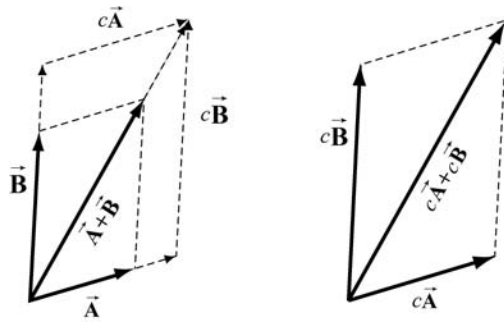


Figure A.1.7 Distributive Law for vector addition.

(iii) Distributive Law for Scalar Addition: The multiplication operation also satisfies a distributive law for the addition of numbers.

Let b and c be real numbers. Then

$$(b+c)\vec{A} = b\vec{A} + c\vec{A} \quad (\text{A.1.8})$$

Our geometric definition of vector addition satisfies this condition as seen in Figure A.1.8.

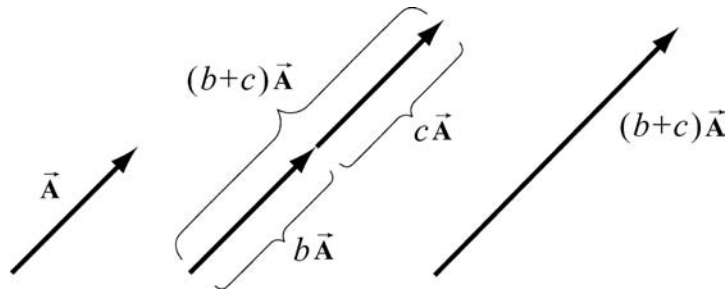


Figure A.1.8 Distributive law for scalar multiplication

(iv) Identity Element for Scalar Multiplication: The number 1 acts as an identity element for multiplication,

$$1\vec{A} = \vec{A} \quad (\text{A.1.9})$$

A.1.3 Application of Vectors

When we apply vectors to physical quantities it's nice to keep in the back of our minds all these formal properties. However from the physicist's point of view, we are interested in representing physical quantities such as displacement, velocity, acceleration, force, impulse, momentum, torque, and angular momentum as vectors. We can't add force to velocity or subtract momentum from torque. We must always understand the physical context for the vector quantity. Thus, instead of approaching vectors as formal

mathematical objects we shall instead consider the following essential properties that enable us to represent physical quantities as vectors.

- (1) Vectors can exist at any point P in space.
- (2) Vectors have direction and magnitude.
- (3) Vector Equality: Any two vectors that have the same direction and magnitude are equal no matter where in space they are located.
- (4) Vector Decomposition: Choose a coordinate system with an origin and axes. We can decompose a vector into component vectors along each coordinate axis. In Figure A.1.9 we choose Cartesian coordinates for the x - y plane (we ignore the z -direction for simplicity but we can extend our results when we need to). A vector $\vec{\mathbf{A}}$ at P can be decomposed into the vector sum,

$$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y \quad (\text{A.1.10})$$

where $\vec{\mathbf{A}}_x$ is the x -component vector pointing in the positive or negative x -direction, and $\vec{\mathbf{A}}_y$ is the y -component vector pointing in the positive or negative y -direction (Figure A.1.9).

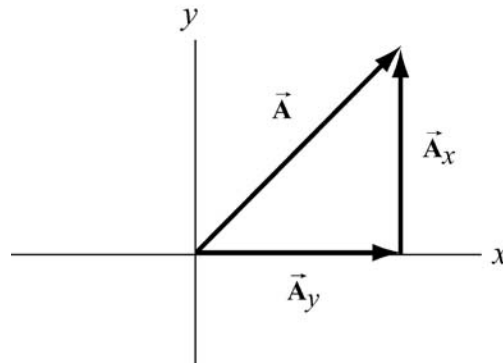


Figure A.1.9 Vector decomposition

- (5) Unit vectors: The idea of multiplication by real numbers allows us to define a set of unit vectors at each point in space. We associate to each point P in space, a set of three unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$. A unit vector means that the magnitude is one: $|\hat{\mathbf{i}}|=1$, $|\hat{\mathbf{j}}|=1$, and $|\hat{\mathbf{k}}|=1$. We assign the direction of $\hat{\mathbf{i}}$ to point in the direction of the increasing x -coordinate at the point P . We call $\hat{\mathbf{i}}$ the unit vector at P pointing in the $+x$ -direction. Unit vectors $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ can be defined in a similar manner (Figure A.1.10).

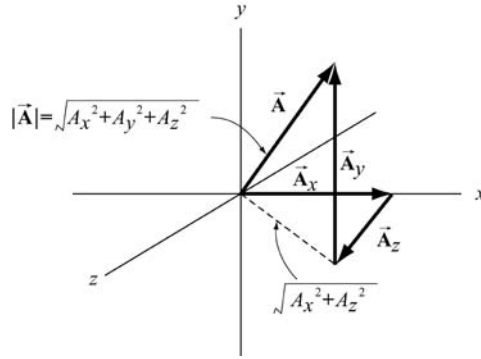


Figure A.1.10 Choice of unit vectors in Cartesian coordinates.

(6) Vector Components: Once we have defined unit vectors, we can then define the x -component and y -component of a vector. Recall our vector decomposition, $\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y$. We can write the x -component vector, $\vec{\mathbf{A}}_x$, as

$$\vec{\mathbf{A}}_x = A_x \hat{\mathbf{i}} \quad (\text{A.1.11})$$

In this expression the term A_x , (without the arrow above) is called the x -component of the vector $\vec{\mathbf{A}}$. The x -component A_x can be positive, zero, or negative. It is not the magnitude of $\vec{\mathbf{A}}_x$ which is given by $(A_x^2)^{1/2}$. Note the difference between the x -component, A_x , and the x -component vector, $\vec{\mathbf{A}}_x$.

In a similar fashion we define the y -component, A_y , and the z -component, A_z , of the vector $\vec{\mathbf{A}}$

$$\vec{\mathbf{A}}_y = A_y \hat{\mathbf{j}}, \quad \vec{\mathbf{A}}_z = A_z \hat{\mathbf{k}} \quad (\text{A.1.12})$$

A vector $\vec{\mathbf{A}}$ can be represented by its three components $\vec{\mathbf{A}} = (A_x, A_y, A_z)$. We can also write the vector as

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \quad (\text{A.1.13})$$

(7) Magnitude: In Figure A.1.10, we also show the vector components $\vec{\mathbf{A}} = (A_x, A_y, A_z)$. Using the Pythagorean theorem, the magnitude of the $\vec{\mathbf{A}}$ is,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (\text{A.1.14})$$

(8) Direction: Let's consider a vector $\vec{\mathbf{A}} = (A_x, A_y, 0)$. Since the z -component is zero, the vector $\vec{\mathbf{A}}$ lies in the x - y plane. Let θ denote the angle that the vector $\vec{\mathbf{A}}$ makes in the

counterclockwise direction with the positive x -axis (Figure A.1.12). Then the x -component and y -components are

$$A_x = A \cos \theta, \quad A_y = A \sin \theta \quad (\text{A.1.15})$$

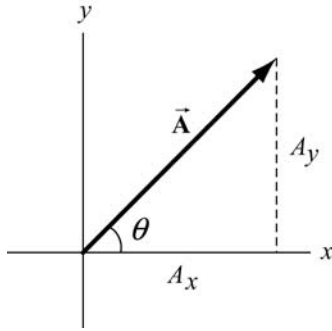


Figure A.1.12 Components of a vector in the x - y plane.

We can now write a vector in the x - y plane as

$$\vec{\mathbf{A}} = A \cos \theta \hat{\mathbf{i}} + A \sin \theta \hat{\mathbf{j}} \quad (\text{A.1.16})$$

Once the components of a vector are known, the tangent of the angle θ can be determined by

$$\frac{A_y}{A_x} = \frac{A \sin \theta}{A \cos \theta} = \tan \theta \quad (\text{A.1.17})$$

which yields

$$\theta = \tan^{-1} \left(\frac{A_y}{A_x} \right) \quad (\text{A.1.18})$$

Clearly, the direction of the vector depends on the sign of A_x and A_y . For example, if both $A_x > 0$ and $A_y > 0$, then $0 < \theta < \pi/2$, and the vector lies in the first quadrant. If, however, $A_x > 0$ and $A_y < 0$, then $-\pi/2 < \theta < 0$, and the vector lies in the fourth quadrant.

(9) Vector Addition: Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors in the x - y plane. Let θ_A and θ_B denote the angles that the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ make (in the counterclockwise direction) with the positive x -axis. Then

$$\vec{\mathbf{A}} = A \cos \theta_A \hat{\mathbf{i}} + A \sin \theta_A \hat{\mathbf{j}} \quad (\text{A.1.19})$$

$$\vec{\mathbf{B}} = B \cos \theta_B \hat{\mathbf{i}} + B \sin \theta_B \hat{\mathbf{j}} \quad (\text{A.1.20})$$

In Figure A.1.13, the vector addition $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ is shown. Let θ_C denote the angle that the vector $\vec{\mathbf{C}}$ makes with the positive x -axis.

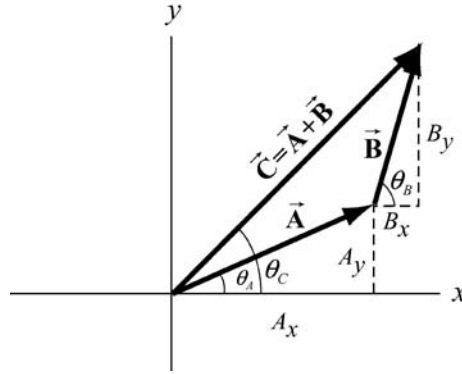


Figure A.1.13 Vector addition with components

Then the components of $\vec{\mathbf{C}}$ are

$$C_x = A_x + B_x, \quad C_y = A_y + B_y \quad (\text{A.1.21})$$

In terms of magnitudes and angles, we have

$$\begin{aligned} C_x &= C \cos \theta_C = A \cos \theta_A + B \cos \theta_B \\ C_y &= C \sin \theta_C = A \sin \theta_A + B \sin \theta_B \end{aligned} \quad (\text{A.1.22})$$

We can write the vector $\vec{\mathbf{C}}$ as

$$\vec{\mathbf{C}} = (A_x + B_x) \hat{\mathbf{i}} + (A_y + B_y) \hat{\mathbf{j}} = C (\cos \theta_C \hat{\mathbf{i}} + \sin \theta_C \hat{\mathbf{j}}) \quad (\text{A.1.23})$$

A.2 Dot Product [\(link\)](#)

A.2.1 Introduction

We shall now introduce a new vector operation, called the “dot product” or “scalar product” that takes any two vectors and generates a scalar quantity (a number). We shall see that the physical concept of work can be mathematically described by the dot product between the force and the displacement vectors.

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Since any two non-collinear vectors form a plane, we define the angle θ to be the angle between the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as shown in Figure A.2.1. Note that θ can vary from 0 to π .

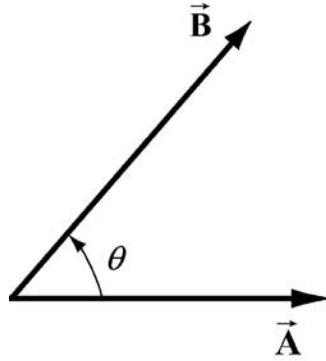


Figure A.2.1 Dot product geometry.

A.2.2 Definition

The dot product $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the cosine of the angle θ between the two vectors:

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB \cos \theta \quad (\text{A.2.1})$$

Where $A = |\vec{\mathbf{A}}|$ and $B = |\vec{\mathbf{B}}|$ represent the magnitude of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ respectively. The dot product can be positive, zero, or negative, depending on the value of $\cos \theta$. The dot product is always a scalar quantity.

We can give a geometric interpretation to the dot product by writing the definition as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (A \cos \theta) B \quad (\text{A.2.2})$$

In this formulation, the term $A \cos \theta$ is the projection of the vector $\vec{\mathbf{A}}$ in the direction of the vector $\vec{\mathbf{B}}$. This projection is shown in Figure A.2.2a. So the dot product is the product of the projection of the length of $\vec{\mathbf{A}}$ in the direction of $\vec{\mathbf{B}}$ with the length of $\vec{\mathbf{B}}$. Note that we could also write the dot product as

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A(B \cos \theta) \quad (\text{A.2.3})$$

Now the term $B \cos \theta$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction of the vector $\vec{\mathbf{A}}$ as shown in Figure A.2.2b. From this perspective, the dot product is the product of the projection of the length of $\vec{\mathbf{B}}$ in the direction of $\vec{\mathbf{A}}$ with the length of $\vec{\mathbf{A}}$.

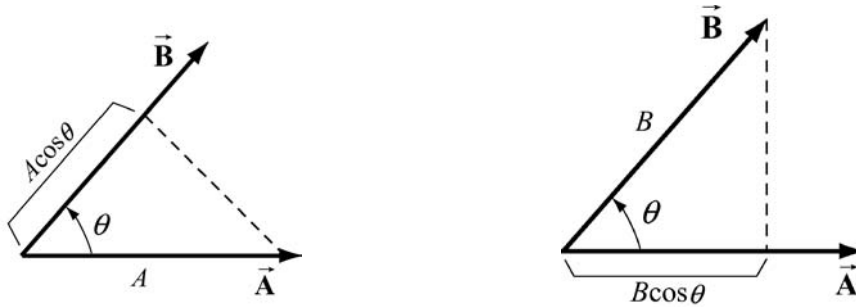


Figure A.2.2a and A.2.2b Projection of vectors and the dot product.

From our definition of the dot product we see that the dot product of two vectors that are perpendicular to each other is zero since the angle between the vectors is $\pi/2$ and $\cos(\pi/2) = 0$.

A.2.3 Properties of Dot Product

The first property involves the dot product between a vector $c\vec{A}$ where c is a scalar and a vector \vec{B} ,

$$(1a) \quad c\vec{A} \cdot \vec{B} = c(\vec{A} \cdot \vec{B}) \quad (A.2.4)$$

The second involves the dot product between the sum of two vectors \vec{A} and \vec{B} with a vector \vec{C} ,

$$(2a) \quad (\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C} \quad (A.2.5)$$

Since the dot product is a commutative operation

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \quad (A.2.6)$$

the similar definitions hold

$$(1b) \quad \vec{A} \cdot c\vec{B} = c(\vec{A} \cdot \vec{B}) \quad (A.2.7)$$

$$(2b) \quad \vec{C} \cdot (\vec{A} + \vec{B}) = \vec{C} \cdot \vec{A} + \vec{C} \cdot \vec{B} \quad (A.2.8)$$

A.2.4 Vector Decomposition and the Dot Product

With these properties in mind we can now develop an algebraic expression for the dot product in terms of components. Let's choose a Cartesian coordinate system with the

vector $\vec{\mathbf{B}}$ pointing along the positive x -axis with positive x -component B_x , i.e., $\vec{\mathbf{B}} = B_x \hat{\mathbf{i}}$. The vector $\vec{\mathbf{A}}$ can be written as

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \quad (\text{A.2.9})$$

We first calculate that the dot product of the unit vector $\hat{\mathbf{i}}$ with itself is unity:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = |\hat{\mathbf{i}}| |\hat{\mathbf{i}}| \cos(0) = 1 \quad (\text{A.2.10})$$

since the unit vector has magnitude $|\hat{\mathbf{i}}|=1$ and $\cos(0)=1$. We note that the same rule applies for the unit vectors in the y and z directions:

$$\hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \quad (\text{A.2.11})$$

The dot product of the unit vector $\hat{\mathbf{i}}$ with the unit vector $\hat{\mathbf{j}}$ is zero because the two unit vectors are perpendicular to each other:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = |\hat{\mathbf{i}}| |\hat{\mathbf{j}}| \cos(\pi/2) = 0 \quad (\text{A.2.12})$$

Similarly, the dot product of the unit vector $\hat{\mathbf{i}}$ with the unit vector $\hat{\mathbf{k}}$, and the unit vector $\hat{\mathbf{j}}$ with the unit vector $\hat{\mathbf{k}}$ are also zero:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0 \quad (\text{A.2.13})$$

The dot product of the two vectors now becomes

$$\begin{aligned} \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} &= (A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}) \cdot B_x \hat{\mathbf{i}} \\ &= A_x \hat{\mathbf{i}} \cdot B_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} \cdot B_x \hat{\mathbf{i}} + A_z \hat{\mathbf{k}} \cdot B_x \hat{\mathbf{i}} \quad \text{property (2a)} \\ &= A_x B_x (\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}) + A_y B_x (\hat{\mathbf{j}} \cdot \hat{\mathbf{i}}) + A_z B_x (\hat{\mathbf{k}} \cdot \hat{\mathbf{i}}) \quad \text{property (1a) and (1b)} \\ &= A_x B_x \end{aligned} \quad (\text{A.2.14})$$

This third step is the crucial one because it shows that it is only the unit vectors that undergo the dot product operation.

Since we assumed that the vector $\vec{\mathbf{B}}$ points along the positive x -axis with positive x -component B_x , our answer can be zero, positive, or negative depending on the x -component of the vector $\vec{\mathbf{A}}$. In Figure A.2.3, we show the three different cases.

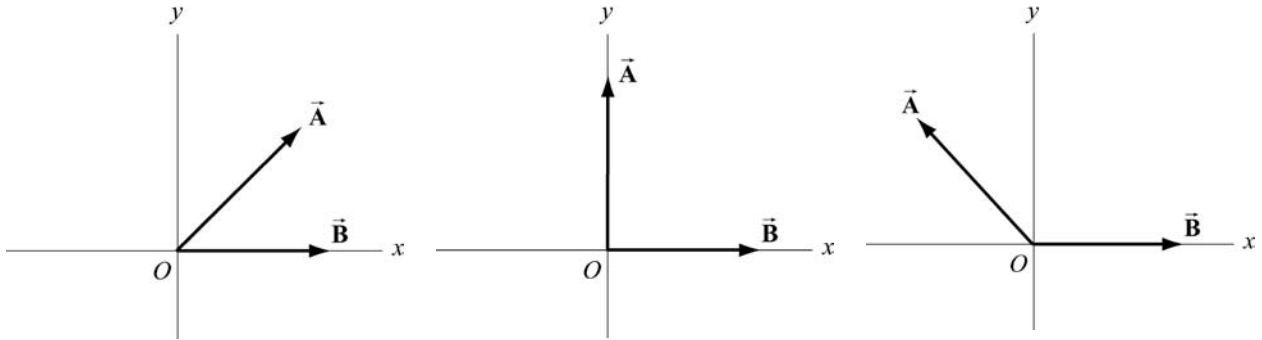


Figure A.2.3 Dot product that is (a) positive, (b) zero or (c) negative.

The result for the dot product can be generalized easily for arbitrary vectors

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \quad (\text{A.2.15})$$

and

$$\vec{\mathbf{B}} = B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}} \quad (\text{A.2.16})$$

to yield

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_x B_x + A_y B_y + A_z B_z \quad (\text{A.2.17})$$

A.3 Cross Product [\(link\)](#)

We shall now introduce our second vector operation, called the “cross product” that takes any two vectors and generates a new vector. The cross product is a type of “multiplication” law that turns our vector space (law for addition of vectors) into a vector algebra (laws for addition and multiplication of vectors). The first application of the cross product will be the physical concept of torque about a point P which can be described mathematically by the cross product of a vector from P to where the force acts, and the force vector.

A.3.1 Definition: Cross Product

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Since any two vectors form a plane, we define the angle θ to be the angle between the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as shown in Figure A.3.2.1. The magnitude of the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the sine of the angle θ between the two vectors,

$$|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = AB \sin \theta \quad (\text{A.3.1})$$

where A and B denote the magnitudes of \vec{A} and \vec{B} , respectively. The angle θ between the vectors is limited to the values $0 \leq \theta \leq \pi$ insuring that $\sin \theta \geq 0$.

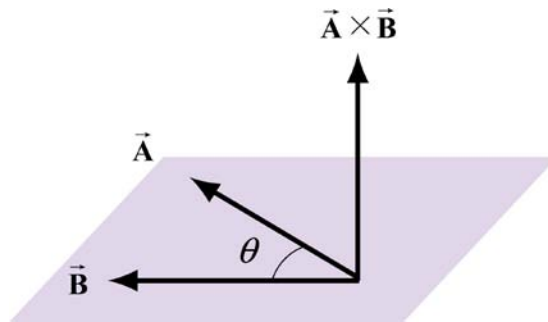


Figure A.3.1 Cross product geometry.

The direction of the cross product is defined as follows. The vectors \vec{A} and \vec{B} form a plane. Consider the direction perpendicular to this plane. There are two possibilities, as shown in Figure A.3.1. We shall choose one of these two for the direction of the cross product $\vec{A} \times \vec{B}$ using a convention that is commonly called the “*right-hand rule*”.

A.3.2 Right-hand Rule for the Direction of Cross Product

The first step is to redraw the vectors \vec{A} and \vec{B} so that their tails are touching. Then draw an arc starting from the vector \vec{A} and finishing on the vector \vec{B} . Curl your right fingers the same way as the arc. Your right thumb points in the direction of the cross product $\vec{A} \times \vec{B}$ (Figure A.3.2).

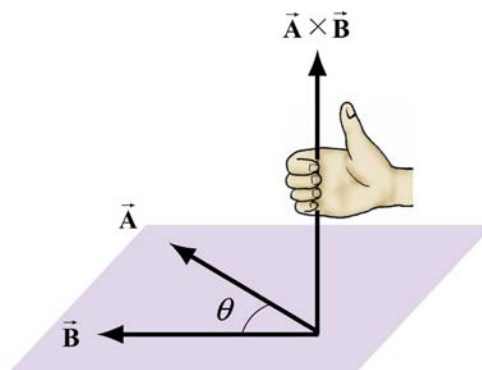


Figure A.3.2 Right-Hand Rule.

You should remember that the direction of the cross product $\vec{A} \times \vec{B}$ is perpendicular to the plane formed by \vec{A} and \vec{B} .

We can give a geometric interpretation to the magnitude of the cross product by writing the definition as

$$|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = A(B \sin \theta) \quad (\text{A.3.2})$$

The vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ form a parallelogram. The area of the parallelogram equals the height times the base, which is the magnitude of the cross product. In Figure A.3.3, two different representations of the height and base of a parallelogram are illustrated. As depicted in Figure A.3.3(a), the term $B \sin \theta$ is the projection of the vector $\vec{\mathbf{B}}$ in the direction perpendicular to the vector $\vec{\mathbf{A}}$. We could also write the magnitude of the cross product as

$$|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = (A \sin \theta)B \quad (\text{A.3.3})$$

Now the term $A \sin \theta$ is the projection of the vector $\vec{\mathbf{A}}$ in the direction perpendicular to the vector $\vec{\mathbf{B}}$ as shown in Figure A.3.3(b).

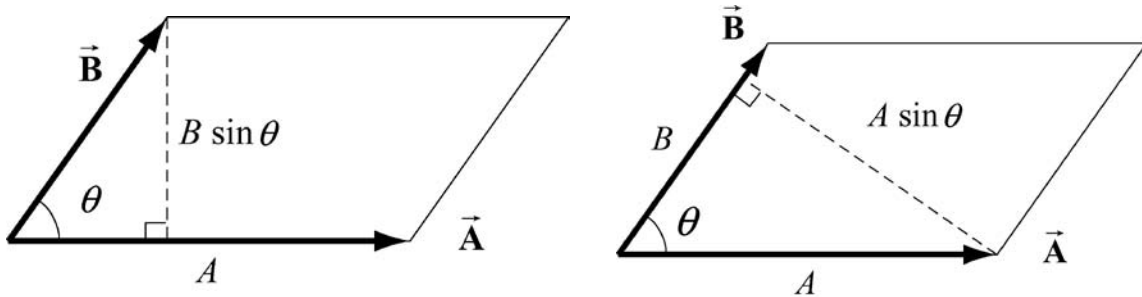


Figure A.3.3 Projection of vectors and the cross product

The cross product of two vectors that are parallel (or anti-parallel) to each other is zero since the angle between the vectors is 0 (or π) and $\sin(0) = 0$ (or $\sin(\pi) = 0$). Geometrically, two parallel vectors do not have any component perpendicular to their common direction.

A.3.3 Properties of the Cross Product

- (1) The cross product is anti-commutative since changing the order of the vectors cross product changes the direction of the cross product vector by the right hand rule:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}} \quad (\text{A.3.4})$$

- (2) The cross product between a vector $c\vec{\mathbf{A}}$ where c is a scalar and a vector $\vec{\mathbf{B}}$ is

$$c\vec{\mathbf{A}} \times \vec{\mathbf{B}} = c(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \quad (\text{A.3.5})$$

Similarly,

$$\vec{\mathbf{A}} \times c\vec{\mathbf{B}} = c(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \quad (\text{A.3.6})$$

(3) The cross product between the sum of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with a vector $\vec{\mathbf{C}}$ is

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \times \vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{C}} + \vec{\mathbf{B}} \times \vec{\mathbf{C}} \quad (\text{A.3.7})$$

Similarly,

$$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}} \quad (\text{A.3.8})$$

A.3.4 Vector Decomposition and the Cross Product

We first calculate that the magnitude of cross product of the unit vector $\hat{\mathbf{i}}$ with $\hat{\mathbf{j}}$:

$$|\hat{\mathbf{i}} \times \hat{\mathbf{j}}| = |\hat{\mathbf{i}}| |\hat{\mathbf{j}}| \sin\left(\frac{\pi}{2}\right) = 1 \quad (\text{A.3.9})$$

since the unit vector has magnitude $|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = 1$ and $\sin(\pi/2) = 1$. By the right hand rule, the direction of $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is in the $+\hat{\mathbf{k}}$ as shown in Figure A.3.4. Thus $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$.

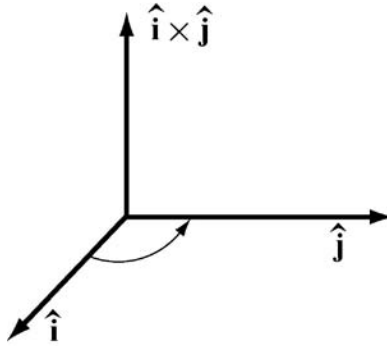


Figure A.3.4 Cross product of $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$

We note that the same rule applies for the unit vectors in the y and z directions,

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \quad (\text{A.3.10})$$

Note that by the anti-commutativity property (1) of the cross product,

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \quad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}} \quad (\text{A.3.11})$$

The cross product of the unit vector $\hat{\mathbf{i}}$ with itself is zero because the two unit vectors are parallel to each other, ($\sin(0) = 0$),

$$|\hat{\mathbf{i}} \times \hat{\mathbf{i}}| = |\hat{\mathbf{i}}| |\hat{\mathbf{i}}| \sin(0) = 0 \quad (\text{A.3.12})$$

The cross product of the unit vector $\hat{\mathbf{j}}$ with itself and the unit vector $\hat{\mathbf{k}}$ with itself, are also zero for the same reason.

$$|\hat{\mathbf{j}} \times \hat{\mathbf{j}}| = 0, \quad |\hat{\mathbf{k}} \times \hat{\mathbf{k}}| = 0 \quad (\text{A.3.13})$$

With these properties in mind we can now develop an algebraic expression for the cross product in terms of components. Let's choose a Cartesian coordinate system with the vector $\vec{\mathbf{B}}$ pointing along the positive x -axis with positive x -component B_x . Then the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ can be written as

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \quad (\text{A.3.14})$$

and

$$\vec{\mathbf{B}} = B_x \hat{\mathbf{i}} \quad (\text{A.3.15})$$

respectively. The cross product in vector components is

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}) \times B_x \hat{\mathbf{i}} \quad (\text{A.3.16})$$

This becomes, using properties (3) and (2),

$$\begin{aligned} \vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (A_x \hat{\mathbf{i}} \times B_x \hat{\mathbf{i}}) + (A_y \hat{\mathbf{j}} \times B_x \hat{\mathbf{i}}) + (A_z \hat{\mathbf{k}} \times B_x \hat{\mathbf{i}}) \\ &= A_x B_x (\hat{\mathbf{i}} \times \hat{\mathbf{i}}) + A_y B_x (\hat{\mathbf{j}} \times \hat{\mathbf{i}}) + A_z B_x (\hat{\mathbf{k}} \times \hat{\mathbf{i}}) \\ &= -A_y B_x \hat{\mathbf{k}} + A_z B_x \hat{\mathbf{j}} \end{aligned} \quad (\text{A.3.17})$$

The vector component expression for the cross product easily generalizes for arbitrary vectors

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \quad (\text{A.3.18})$$

and

$$\vec{\mathbf{B}} = B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}} \quad (\text{A.3.19})$$

to yield

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_y B_z - A_z B_y) \hat{\mathbf{i}} + (A_z B_x - A_x B_z) \hat{\mathbf{j}} + (A_x B_y - A_y B_x) \hat{\mathbf{k}}. \quad (\text{A.3.20})$$

B. Coordinate Systems [\(link\)](#)

B.1 Cartesian Coordinates

A coordinate system consists of four basic elements:

- (1) Choice of origin
- (2) Choice of axes
- (3) Choice of positive direction for each axis
- (4) Choice of unit vectors for each axis

We illustrate these elements below using Cartesian coordinates.

(1) Choice of Origin

Choose an origin O . If you are given an object, then your choice of origin may coincide with a special point in the body. For example, you may choose the mid-point of a straight piece of wire.

(2) Choice of Axis

Now we shall choose a set of axes. The simplest set of axes are known as the Cartesian axes, x -axis, y -axis, and the z -axis. Once again, we adapt our choices to the physical object. For example, we select the x -axis so that the wire lies on the x -axis, as shown in Figure B.1.1:

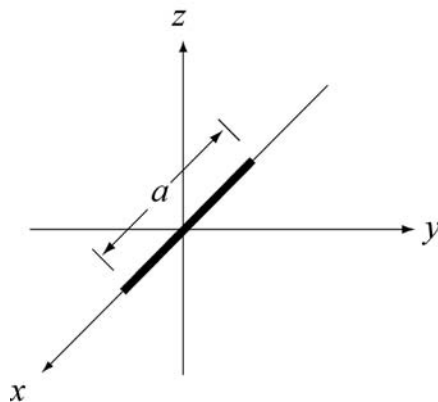


Figure B.1.1 A wire lying along the x -axis of Cartesian coordinates.

Then each point P in space our S can be assigned a triplet of values (x_p, y_p, z_p) , the Cartesian coordinates of the point P . The ranges of these values are: $-\infty < x_p < +\infty$, $-\infty < y_p < +\infty$, $-\infty < z_p < +\infty$.

The collection of points that have the same the coordinate y_p is called a level surface. Suppose we ask what collection of points in our space S have the same value of $y = y_p$. This is the set of points $S_{y_p} = \{(x, y, z) \in S \text{ such that } y = y_p\}$. This set S_{y_p} is a plane, the x - z plane (Figure B.1.2), called a level set for constant y_p . Thus, the y -coordinate of any point actually describes a plane of points perpendicular to the y -axis.

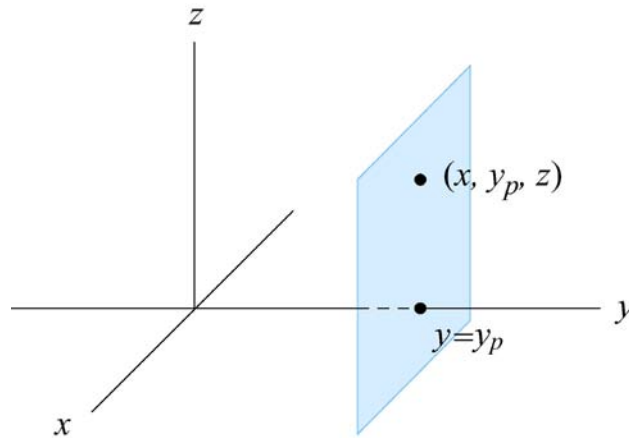


Figure B.1.2 Level surface set for constant value y_p .

(3) Choice of Positive Direction

Our third choice is an assignment of positive direction for each coordinate axis. We shall denote this choice by the symbol $+$ along the positive axis. Conventionally, Cartesian coordinates are drawn with the x - y plane corresponding to the plane of the paper. The horizontal direction from left to right is taken as the positive x -axis, and the vertical direction from bottom to top is taken as the positive y -axis. In physics problems we are free to choose our axes and positive directions any way that we decide best fits a given problem. Problems that are very difficult using the conventional choices may turn out to be much easier to solve by making a thoughtful choice of axes. The endpoints of the wire now have the coordinates $(a/2, 0, 0)$ and $(-a/2, 0, 0)$.

(4) Choice of Unit Vectors

We now associate to each point P in space, a set of three unit directions vectors $(\hat{\mathbf{i}}_p, \hat{\mathbf{j}}_p, \hat{\mathbf{k}}_p)$. A unit vector has magnitude one: $|\hat{\mathbf{i}}_p| = 1$, $|\hat{\mathbf{j}}_p| = 1$, and $|\hat{\mathbf{k}}_p| = 1$. We assign the direction of $\hat{\mathbf{i}}_p$ to point in the direction of the increasing x -coordinate at the

point P . We define the directions for $\hat{\mathbf{j}}_p$ and $\hat{\mathbf{k}}_p$ in the direction of the increasing y -coordinate and z -coordinate respectively. (Figure B.1.3).

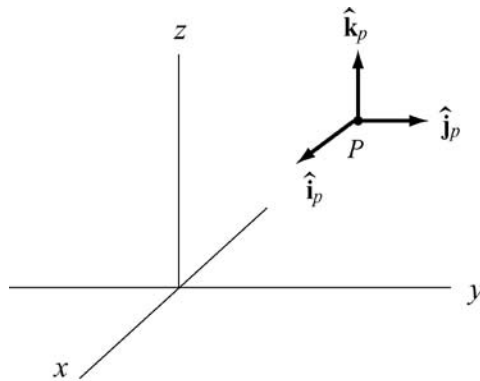


Figure B.1.3 Choice of unit vectors.

B.1.1 Infinitesimal Line Element

Consider a small infinitesimal displacement $d\vec{s}$ between two points P_1 and P_2 (Figure B.1.4). In Cartesian coordinates this vector can be decomposed into

$$d\vec{s} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}} \quad (\text{B.1.1})$$

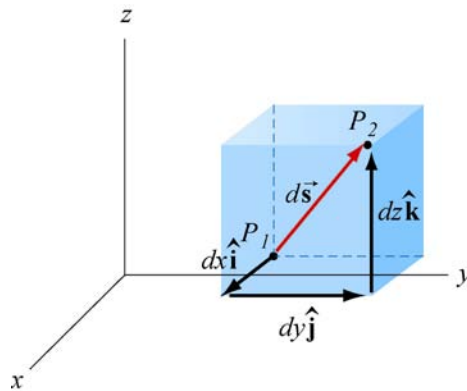


Figure B.1.4 Displacement between two points

B.1.2 Infinitesimal Area Element

An infinitesimal area element of the surface of a small cube (Figure B.1.5) is given by

$$dA = (dx)(dy) \quad (\text{B.1.2})$$

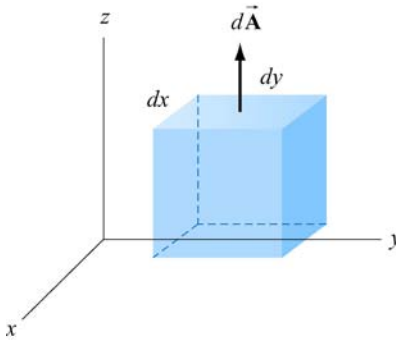


Figure B.1.5 Area element for one face of a small cube

Area elements are actually vectors where the direction of the vector $d\vec{\mathbf{A}}$ is perpendicular to the plane defined by the area. Since there is a choice of direction, we shall choose the area vector to always point outwards from a closed surface, defined by the right-hand rule. So for the above, the infinitesimal area vector is

$$d\vec{\mathbf{A}} = dx dy \hat{\mathbf{k}} \quad (\text{B.1.3})$$

B.1.3 Infinitesimal Volume Element

An infinitesimal volume element (Figure B.1.6) in Cartesian coordinates is given by

$$dV = dx dy dz \quad (\text{B.1.4})$$

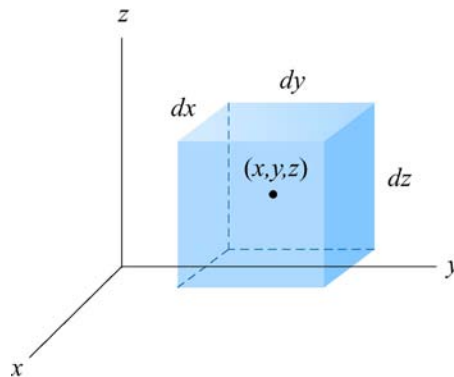


Figure B.1.6 Volume element in Cartesian coordinates.

B.2 Cylindrical Coordinates

We first choose an origin and an axis we call the z -axis with unit vector $\hat{\mathbf{z}}$ pointing in the increasing z -direction. The level surface of points such that $z = z_p$ define a plane. We shall choose coordinates for a point P in the plane $z = z_p$ as follows.

The coordinate ρ measures the distance from the z -axis to the point P . Its value ranges from $0 \leq \rho < \infty$. In Figure B.2.1 we draw a few contours that have constant values of ρ . These “level contours” are circles. On the other hand, if z were not restricted to $z = z_p$, as in Figure B.2.1, the level surfaces for constant values of ρ would be cylinders coaxial with the z -axis.

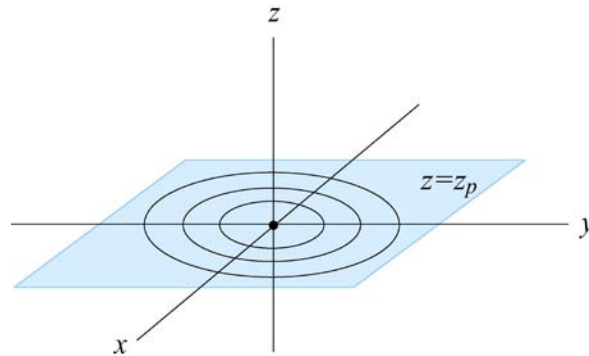


Figure B.2.1 Level surfaces for the coordinate ρ .

Our second coordinate measures an angular distance along the circle. We need to choose some reference point to define the angular coordinate. We choose a “reference ray,” a horizontal ray starting from the origin and extending to $+\infty$ along the horizontal direction to the right. (In a typical Cartesian coordinate system, our reference ray is the positive x -direction). We define the angle coordinate for the point P as follows. We draw a ray from the origin to P . We define ϕ as the angle in the counterclockwise direction between our horizontal reference ray and the ray from the origin to the point P , (see Figure B.2.2):

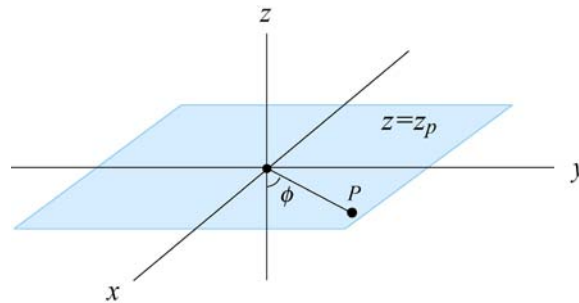


Figure B.2.2 The angular coordinate

All the other points that lie on a ray from the origin to infinity passing through P have the same value of ϕ . For any arbitrary point, ϕ can take on values from $0 \leq \phi < 2\pi$. In Figure B.2.3 we depict other “level surfaces” for the angular coordinate.

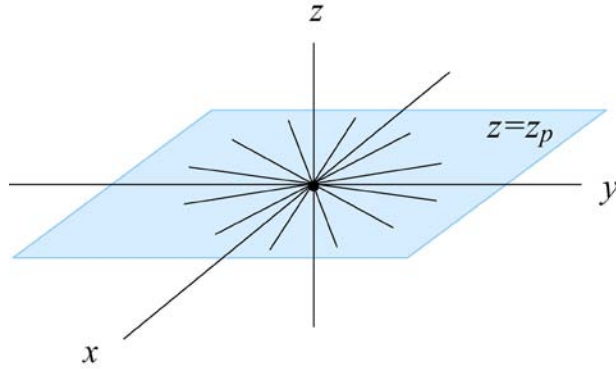


Figure B.2.3 Level surfaces for the angle coordinate.

The coordinates (ρ, ϕ) in the plane $z = z_p$ are called *plane polar coordinates*. We choose two unit vectors in the plane at the point P as follows. We choose $\hat{\rho}$ to point in the direction of increasing ρ , radially away from the z -axis. We choose $\hat{\phi}$ to point in the direction of increasing ϕ . This unit vector points in the counterclockwise direction, tangent to the circle. Our complete coordinate system is shown in Figure B.2.4. This coordinate system is called a “cylindrical coordinate system.” Essentially we have chosen two directions, radial and tangential in the plane and a perpendicular direction to the plane.

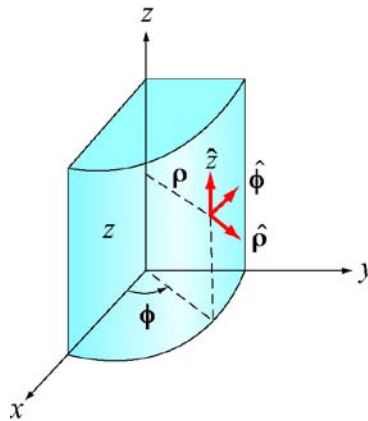


Figure B.2.4 Cylindrical coordinates

When referring to any arbitrary point in the plane, we write the unit vectors as $\hat{\rho}$ and $\hat{\phi}$, keeping in mind that they may change in direction as we move around the plane, keeping \hat{z} unchanged. If we need to make a reference to this time changing property, we will write the unit vectors as explicit functions of time, $\hat{\rho}(t)$ and $\hat{\phi}(t)$.

If you are given polar coordinates (ρ, ϕ) of a point in the plane, the Cartesian coordinates (x, y) can be determined from the coordinate transformations:

$$x = \rho \cos \phi, \quad y = \rho \sin \phi \tag{B.2.1}$$

Conversely, if you are given the Cartesian coordinates (x, y) , the polar coordinates (ρ, ϕ) may be represented as

$$\rho = +(x^2 + y^2)^{1/2}, \quad \phi = \tan^{-1}(y/x) \quad (\text{B.2.2})$$

Note that $\rho \geq 0$ so you always need to take the positive square root. Note also that $\tan \phi = \tan(\phi + \pi)$. Suppose that $0 \leq \phi \leq \pi/2$, then $x \geq 0$ and $y \geq 0$. Then the point $(-x, -y)$ will correspond to the angle $\phi + \pi$.

The unit vectors also are related by the coordinate transformations

$$\hat{\rho} = \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}}, \quad \hat{\phi} = -\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}} \quad (\text{B.2.3})$$

Similarly,

$$\hat{\mathbf{i}} = \cos \phi \hat{\rho} - \sin \phi \hat{\phi}, \quad \hat{\mathbf{j}} = \sin \phi \hat{\rho} + \cos \phi \hat{\phi} \quad (\text{B.2.4})$$

The crucial difference between cylindrical coordinates and Cartesian coordinates involves the choice of unit vectors. Suppose we consider a different point P_1 in the plane. The unit vectors in Cartesian coordinates $(\hat{\mathbf{i}}_1, \hat{\mathbf{j}}_1)$ at the point P_1 have the same magnitude and point in the same direction as the unit vectors $(\hat{\mathbf{i}}_2, \hat{\mathbf{j}}_2)$ at P_2 . Any two vectors that are equal in magnitude and point in the same direction are equal; therefore

$$\hat{\mathbf{i}}_1 = \hat{\mathbf{i}}_2, \quad \hat{\mathbf{j}}_1 = \hat{\mathbf{j}}_2 \quad (\text{B.2.5})$$

A Cartesian coordinate system is the unique coordinate system in which the set of unit vectors at different points in space are equal. In polar coordinates, the unit vectors at two different points are not equal because they point in different directions. We show this in Figure B.2.5.

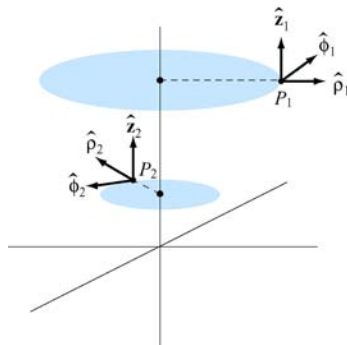


Figure B.2.5 Unit vectors at two different points in polar coordinates.

B.2.1 Infinitesimal Line Element

Consider a small infinitesimal displacement $d\vec{s}$ between two points P_1 and P_2 (Figure B.2.6). This vector can be decomposed into

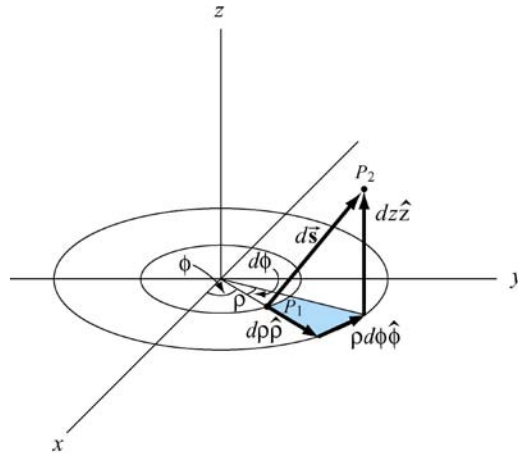


Figure B.2.6 displacement vector $d\vec{s}$ between two points

$$d\vec{s} = d\rho \hat{\rho} + \rho d\phi \hat{\phi} + dz \hat{z} \quad (\text{B.2.6})$$

B.2.2 Infinitesimal Area Element

Consider an infinitesimal area element on the surface of a cylinder of radius ρ (Figure B.2.7).

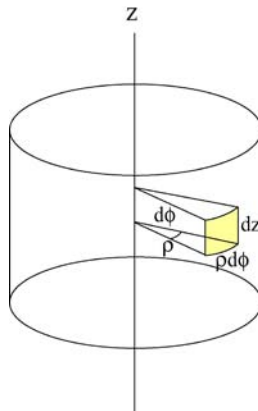


Figure B.2.7 Area element for a cylinder

The area of this element has magnitude

$$dA = \rho d\phi dz \quad (\text{B.2.7})$$

Area elements are actually vectors where the direction of the vector $d\vec{A}$ points perpendicular to the plane defined by the area. Since there is a choice of direction, we shall choose the area vector to always point outwards from a closed surface. So for the surface of the cylinder, the infinitesimal area vector is

$$d\vec{A} = \rho d\phi dz \hat{k} \quad (\text{B.2.8})$$

Consider an infinitesimal area element on the surface of a disc (Figure B.2.8) in the x - y plane.

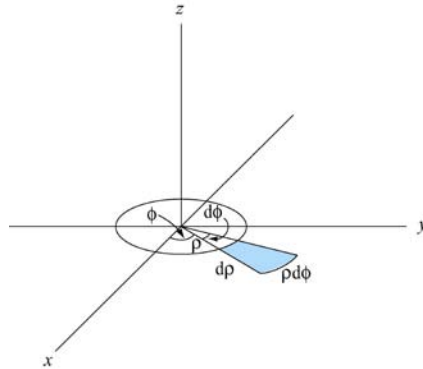


Figure B.2.8 Area element for a disc.

This area element is given by the vector

$$d\vec{A} = \rho d\phi d\rho \hat{k} \quad (\text{B.2.9})$$

B.2.3 Infinitesimal Volume Element

An infinitesimal volume element (Figure B.2.9) is given by

$$dV = \rho d\phi d\rho dz \quad (\text{B.2.10})$$

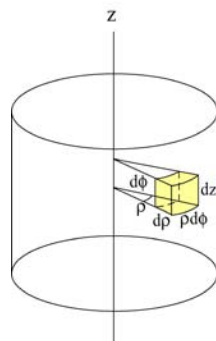


Figure B.2.9 Volume element for a cylinder.

B.3 Spherical Coordinates

We first choose an origin. Then we choose a coordinate, r , that measures the radial distance from the origin to the point P . The coordinate r ranges in value from $0 \leq r < \infty$. The set of points that have constant value for r are spheres (“level surfaces”).

Any point on the sphere can be defined by two angles (θ, ϕ) and r . We will define these angles with respect to a choice of Cartesian coordinates (x, y, z) . The angle θ is defined to be the angle between the positive z -axis and the ray from the origin to the point P . Note that the values of θ only range from $0 \leq \theta \leq \pi$. The angle ϕ is defined (in a similar fashion to polar coordinates) as the angle in the between the positive x -axis and the projection in the x - y plane of the ray from the origin to the point P . The coordinate angle ϕ can take on values from $0 \leq \phi < 2\pi$.

The spherical coordinates (r, θ, ϕ) for the point P are shown in Figure B.3.1. We choose the unit vectors $(\hat{r}, \hat{\theta}, \hat{\phi})$ at the point P as follows. Let \hat{r} point radially away from the origin, and $\hat{\theta}$ point tangent to a circle in the positive θ direction in the plane formed by the z -axis and the ray from the origin to the point P . Note that $\hat{\theta}$ points in the direction of increasing θ . We choose $\hat{\phi}$ to point in the direction of increasing ϕ . This unit vector points tangent to a circle in the xy -plane centered on the z -axis. These unit vectors are also shown in Figure B.3.1.

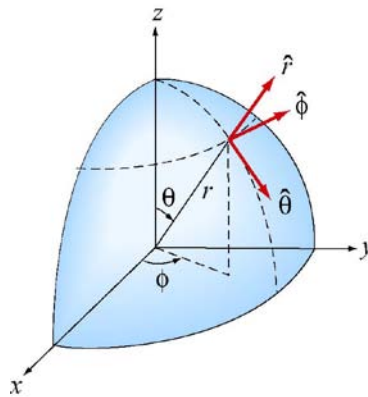


Figure B.3.1 Spherical coordinates

If you are given spherical coordinates (r, θ, ϕ) of a point in the plane, the Cartesian coordinates (x, y, z) can be determined from the coordinate transformations

$$\begin{aligned}x &= r \sin \theta \cos \phi \\y &= r \sin \theta \sin \phi \\z &= r \cos \theta\end{aligned}\tag{B.3.1}$$

Conversely, if you are given the Cartesian coordinates (x, y, z) , the spherical coordinates (r, θ, ϕ) can be determined from the coordinate transformations

$$\begin{aligned} r &= +(x^2 + y^2 + z^2)^{1/2} \\ \theta &= \cos^{-1}\left(\frac{z}{(x^2 + y^2 + z^2)^{1/2}}\right) \\ \phi &= \tan^{-1}(y/x) \end{aligned} \quad (\text{B.3.2})$$

The unit vectors also are related by the coordinate transformations

$$\begin{aligned} \hat{\mathbf{r}} &= \sin \theta \cos \phi \hat{\mathbf{i}} + \sin \theta \sin \phi \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}} \\ \hat{\boldsymbol{\theta}} &= \cos \theta \cos \phi \hat{\mathbf{i}} + \cos \theta \sin \phi \hat{\mathbf{j}} - \sin \theta \hat{\mathbf{k}} \\ \hat{\boldsymbol{\phi}} &= -\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}} \end{aligned} \quad (\text{B.3.3})$$

These results can be understood by considering the projection of $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ into the unit vectors $(\hat{\boldsymbol{\rho}}, \hat{\mathbf{k}})$, where $\hat{\boldsymbol{\rho}}$ is the unit vector from cylindrical coordinates (Figure B.3.2),

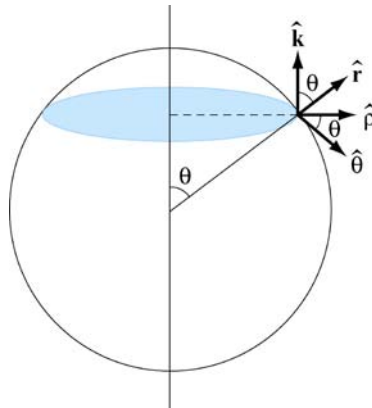


Figure B.3.2 Cylindrical and spherical unit vectors

$$\begin{aligned} \hat{\mathbf{r}} &= \sin \theta \hat{\boldsymbol{\rho}} + \cos \theta \hat{\mathbf{k}} \\ \hat{\boldsymbol{\theta}} &= \cos \theta \hat{\boldsymbol{\rho}} - \sin \theta \hat{\mathbf{k}} \end{aligned} \quad (\text{B.3.4})$$

We can use the vector decomposition of $\hat{\boldsymbol{\rho}}$ into the Cartesian unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}})$:

$$\hat{\boldsymbol{\rho}} = \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}} \quad (\text{B.3.5})$$

To find the inverse transformations we can use the fact that

$$\hat{\rho} = \sin \theta \hat{r} + \cos \theta \hat{\theta} \quad (\text{B.3.6})$$

to express

$$\begin{aligned} \hat{i} &= \cos \phi \hat{\phi} - \sin \phi \hat{\theta} \\ \hat{j} &= \sin \phi \hat{\phi} + \cos \phi \hat{\theta} \end{aligned} \quad (\text{B.3.7})$$

as

$$\begin{aligned} \hat{i} &= \cos \phi \sin \theta \hat{r} + \cos \phi \cos \theta \hat{\theta} - \sin \phi \hat{\phi} \\ \hat{j} &= \sin \phi \sin \theta \hat{r} + \sin \phi \cos \theta \hat{\theta} + \cos \phi \hat{\phi} \end{aligned} \quad (\text{B.3.8})$$

The unit vector \hat{k} can be decomposed directly into $(\hat{r}, \hat{\theta})$ with the result that

$$\hat{k} = \cos \theta \hat{r} - \sin \theta \hat{\theta} \quad (\text{B.3.9})$$

B.3.1 Infinitesimal Line Element

Consider a small infinitesimal displacement $d\vec{s}$ between two points (Figure B.3.3). This vector can be decomposed into

$$d\vec{s} = dr\hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi} \quad (\text{B.3.10})$$

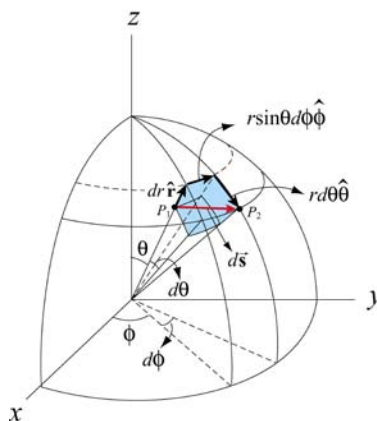


Figure B.3.3 Infinitesimal displacement vector $d\vec{s}$.

B.3.2 Infinitesimal Area Element

Consider an infinitesimal area element on the surface of a sphere of radius r (Figure B.3.4).

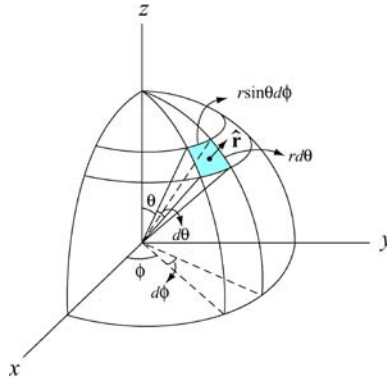


Figure B.3.4 Area element for a sphere.

The area of this element has magnitude

$$dA = (rd\theta)(r \sin \theta d\phi) = r^2 \sin \theta d\theta d\phi \quad (\text{B.3.11})$$

points in the radially direction (outward from the surface of the sphere). So for the surface of the sphere, the infinitesimal area vector is

$$d\vec{A} = r^2 \sin \theta d\theta d\phi \hat{r} \quad (\text{B.3.12})$$

B.3.3 Infinitesimal Volume Element

An infinitesimal volume element (Figure B.3.5) is given by

$$dV = r^2 \sin \theta d\theta d\phi dr \quad (\text{B.3.13})$$

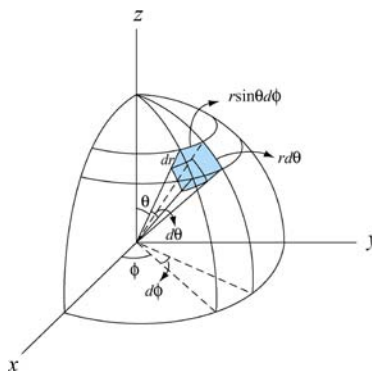


Figure B.3.5 Infinitesimal volume element.

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