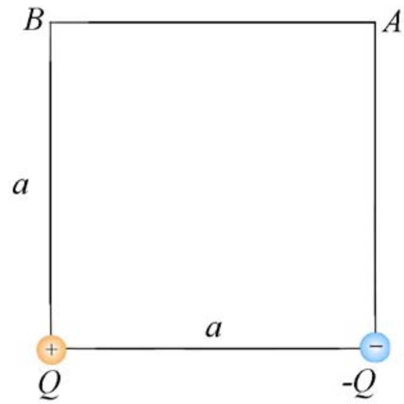
Electric Potential, Discrete and Continuous Distributions of Charge
Challenge Problem Solutions

Problem 1:

Two point-like charged objects with charges $+Q$ and $-Q$ are placed on the bottom corners of a square of side *a*, as shown in the figure.



You move an electron with charge $-e$ from the upper right corner marked A to the upper left corner marked B. Which of the following statements is true?

a) You do a negative amount of work on the electron equal to the amount of energy necessary to assemble the system of three charged objects with the electron at point B.

b) You do a positive amount of work on the electron equal to the amount of energy necessary to assemble the system of three charged objects with the electron at point B.

c) You do a positive amount of work on the electron and the potential energy of the system of three charged objects increases.

d) You do a negative amount of work on the electron and the potential energy of the system of three charged objects decreases.

e) You do a negative amount of work on the electron and the potential energy of the system of three charged objects increases.

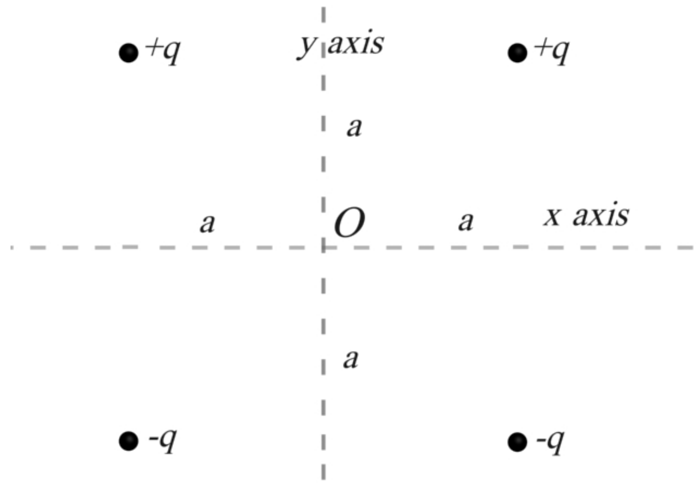
f) You do a positive amount of work on the electron and the potential energy of the system of three charged objects decreases.

Problem 1 Solution:
d. Because point B is closer to the positive charge than the point A, the electric potential difference $V(B) - V(A) > 0$. When you move an electron with charge $-e$ from the upper

right corner marked A to the upper left corner marked B, the potential energy difference is $U(B)-U(A)=-e(V(B)-V(A))<0$. This means that you do a negative amount of work and the potential energy of the system decreases.

Problem 2:

Four charged point-like objects, two of charge $+q$ and two of charge $-q$, are arranged on the vertices of a square with sides of length $2a$, as shown in the sketch.

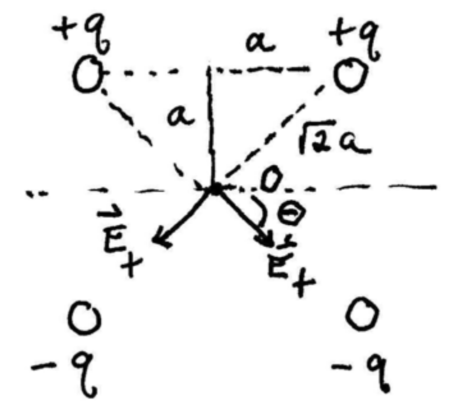


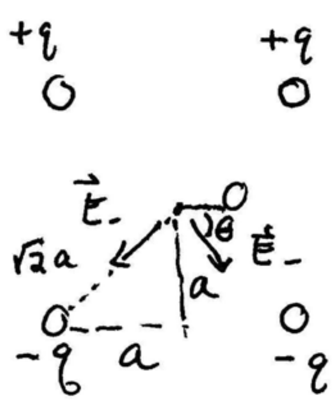
a) What is the electric field at point O, which is at the center of the square? Indicate the direction and the magnitude.

b) What is the electric potential V at point O, the center of the square, taking the potential at infinity to be zero?

Problem 2 Solution:

(a)When I add the contributions to the electric field at the origin from the two positive charges on the upper corners of the square, the horizontal component cancels and the vertical component points down.





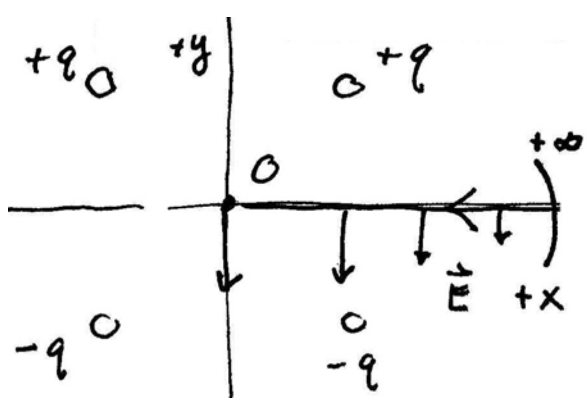
A similar argument holds for the contributions to the electric field at the origin from the two negative charges on the lower corners of the square. Therefore the electric field at the origin is

$$\vec{\mathbf{E}}_O = 4\left|\vec{\mathbf{E}}_{+q}\right|\sin\theta(-\hat{\mathbf{j}}) = 4k\frac{q}{2a^2}\left(\frac{1}{\sqrt{2}}\right)(-\hat{\mathbf{j}}) = \frac{1}{4\pi\varepsilon_0}\frac{\sqrt{2}q}{a^2}(-\hat{\mathbf{j}})$$

(b)The electric potential difference between infinity and the origin is just the sum

$$V(O) - V(\infty) = V(O) = k\frac{q}{(2a^2)^{1/2}} + k\frac{q}{(2a^2)^{1/2}} + k\frac{(-q)}{(2a^2)^{1/2}} + k\frac{(-q)}{(2a^2)^{1/2}} = 0\,.$$

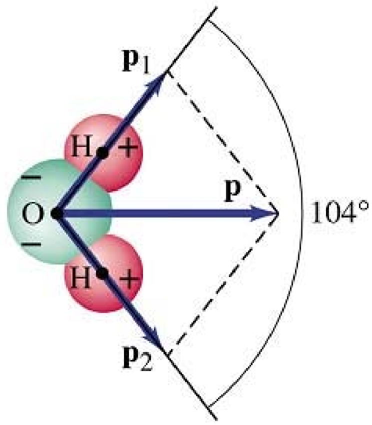
c) Sketch on the figure below one path leading from infinity to the origin at O where the integral $\int_\infty^O \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}$ is trivial to do by inspection. Does your answer here agree with your result in b)?



(c) The electric field at any point along the x-axis is points in the –y-direction. Therefore for a path from infinity to the origin at O along the x-axis, the dot product $\vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} = 0$ and hence the integral $\int_{\infty}^{O} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = 0$. Because by definition $\int_{\infty}^{O} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -(V(O) - V(\infty)) = 0$, and the integral is path independent, our answer for the above path along the x-axis

Problem 3:

The water molecule, (see figure below), consists of two hydrogen atoms and one oxygen arranged such that the lines joining the center of the O atom with each H atom make an angle of 104^0. A distance $d = 9.6\times10^{-11} m$ separates the centers of each hydrogen and oxygen pair. Each pair of hydrogen and oxygen atoms have dipole moments $\vec{\mathbf{p}}_1$ and $\vec{\mathbf{p}}_2$. The entire molecule has a dipole moment $\vec{\mathbf{p}}$, which is the sum of the dipole moments $\vec{\mathbf{p}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2$. The magnitude of the total dipole moment is $|\vec{\mathbf{p}}| = 6.1\times10^{-30} C-m$.



a) What is the effective charge on each hydrogen atom?

b) Set the electric potential at infinity to zero, $V(\infty) = 0$. Find an expression for the electric potential at a point P, a distance $r >> d$, from the center of the water molecule, due to each dipole, $\vec{\mathbf{p}}_1$ and $\vec{\mathbf{p}}_2$. The line connecting the origin at the center of the water molecule with the point P makes an angle θ with the direction of the total dipole moment $\vec{\mathbf{p}}$.

Problem 3 Solution:

(a) Let q be the effective charge on the H atom. The dipole moment of one H-O pair is then *qd*, where d is given as 9.6 x 10^{-11} m. The two dipoles are at an angle of 104 degrees or a half-angle of 72 degrees. The total dipole moment is thus

$$|\vec{\mathbf{p}}| = 2qd\cos(72) = 6.1x10^{-30} Cm \qquad (10)$$

If we use the value of d given above and solve (10) for *q*, we get 1.03 x 10^{-19} *C*, or about 2/3 of the charge of an electron. This gives a value of *qd* of 9.87 x 10^{-30}.

(b) We are far enough away that we can consider the dipoles to be perfect point dipoles, so a good approximation to the potential of either one is given by

$$\frac{1}{4\pi\varepsilon_o}\frac{\vec{\mathbf{p}}\cdot\hat{\mathbf{r}}}{r^2} \qquad (11)$$

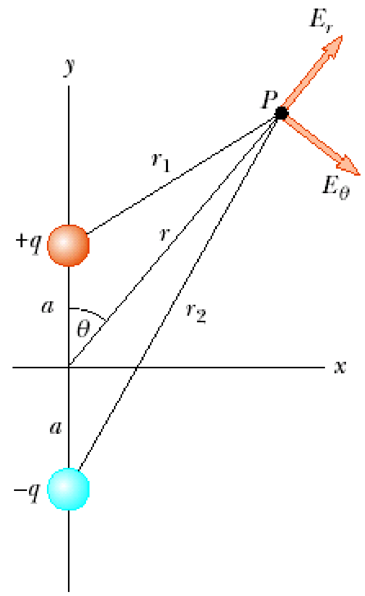
and since the angle each one makes is either $\theta+72$ or $\theta-72$, we have

$$V = \frac{qd}{4\pi\varepsilon_o}\frac{\cos(\theta-72)}{r^2} + \frac{qd}{4\pi\varepsilon_o}\frac{\cos(\theta+72)}{r^2} = \frac{qd}{4\pi\varepsilon_o}\frac{2\cos(72)\cos(\theta)}{r^2} \tag{12}$$

where we have used $\cos(A-B)+\cos(A+B)=2\cos(A)\cos(B)$. Using our results above, we have that

$$V = \frac{2qd\cos(72)}{4\pi\varepsilon_o}\frac{\cos(\theta)}{r^2} = \frac{6.1x10^{-30}}{4\pi\varepsilon_o}\frac{\cos(\theta)}{r^2} \tag{13}$$

Problem 4:



An electric dipole is located along the *y* axis as shown in the figure. The magnitude of its electric dipole moment is defined as $p = 2qa$.

(a) At a point *P*, which is far from the dipole ($r >> a$), show that the electric potential is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

(b) Calculate the radial component E_r and the perpendicular component E_θ of the associated electric field. Note that

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}$$

Do these results seem reasonable for $\theta = 90°$ and 0°? for $r = 0$?

(c) For the dipole arrangement shown, express *V* in terms of Cartesian coordinates using $r = (x^2 + y^2)^{1/2}$ and $\cos\theta = \frac{y}{(x^2+y^2)^{1/2}}$. Using these results and again taking $r >> a$, calculate the field components E_x and E_y.

Problem 4 Solutions:

(a)By superposition principle, the potential at *P* is given by

$$V = \sum_i V_i = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right)$$

where

$$r_1^2 = r^2 + a^2 - 2ra\cos\theta, \qquad r_2^2 = r^2 + a^2 + 2ra\cos\theta$$

If we take the limit where $r \gg a$, then

$$\frac{1}{r_1} = \frac{1}{r}\left[1 + (a/r)^2 - 2(a/r)\cos\theta\right]^{-1/2} = \frac{1}{r}\left[1 - \frac{1}{2}(a/r)^2 + (a/r)\cos\theta + \cdots\right]$$

$$\frac{1}{r_2} = \frac{1}{r}\left[1 + (a/r)^2 + 2(a/r)\cos\theta\right]^{-1/2} = \frac{1}{r}\left[1 - \frac{1}{2}(a/r)^2 - (a/r)\cos\theta + \cdots\right]$$

and the dipole potential can be approximated as

$$V = \frac{q}{4\pi\varepsilon_0 r}\left[1 - \frac{1}{2}(a/r)^2 + (a/r)\cos\theta - 1 + \frac{1}{2}(a/r)^2 + (a/r)\cos\theta + \cdots\right]$$
$$\approx \frac{q}{4\pi\varepsilon_0 r}\cdot\frac{2a\cos\theta}{r} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec{\mathbf{p}}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}$$

where $\vec{\mathbf{p}} = 2aq\,\hat{\mathbf{j}}$ is the electric dipole moment.

(b)In spherical polar coordinates, the gradient operator is

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}}$$

Since the potential is now a function of both r and θ, the electric field will have components along the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ directions. Using $\vec{\mathbf{E}} = -\nabla V$, we have

$$\boxed{E_r = -\frac{\partial V}{\partial r} = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}, \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}, \quad E_\phi = 0}$$

For $\theta = 90°$, we have

$$E_r = \frac{p\cos 90°}{2\pi\varepsilon_0 r^3} = 0, \qquad E_\theta = \frac{p\sin 90°}{4\pi\varepsilon_0 r^3} = \frac{p}{4\pi\varepsilon_0 r^3}$$

Similarly, for $\theta = 0$, we have

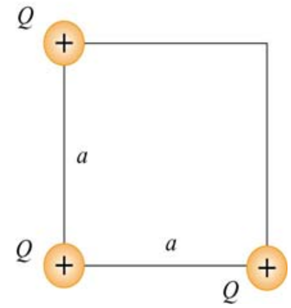
$$E_r = \frac{p\cos 0°}{2\pi\varepsilon_0 r^3} = \frac{p}{2\pi\varepsilon_0 r^3}, \qquad E_\theta = \frac{p\sin 0°}{4\pi\varepsilon_0 r^3} = 0$$

These results are reasonable for $r \gg a$. However, for $r \to 0$, the result is not reasonable. In deriving the above expression for the potential V, we have assumed that $r \gg a$.

(c)In Cartesian coordinates, the potential can be written as $V(x,y) = \frac{k_e p y}{(x^2+y^2)^{3/2}}$. Therefore, the x and y components of the electric field are

$$E_x = -\frac{\partial V}{\partial x} = \frac{3k_e pxy}{\left(x^2+y^2\right)^{5/2}}, \qquad E_y = -\frac{\partial V}{\partial y} = \frac{k_e p\left(2y^2 - x^2\right)}{\left(x^2+y^2\right)^{5/2}}$$

Problem 5:



Three identical charges $+Q$ are placed on the corners of a square of side a, as shown in the figure.

(a) What is the electric field at the fourth corner (the one missing a charge) due to the first three charges?

(b) What is the electric potential at that corner?

(c) How much work does it take to bring another charge, $+Q$, from infinity and place it at that corner?

(d) How much energy did it take to assemble the pictured configuration of three charges?

Problem 5 Solutions:

(a)We'll just use superposition:

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{a\hat{\mathbf{i}}}{a^3} + \frac{a\hat{\mathbf{i}} + a\hat{\mathbf{j}}}{\left(\sqrt{2}a\right)^3} + \frac{a\hat{\mathbf{j}}}{a^3}\right) = \frac{Q}{4\pi\varepsilon_0}\left(1 + 2^{-3/2}\right)\left(\hat{\mathbf{i}} + \hat{\mathbf{j}}\right)$$

(b)A common mistake in doing this kind of problem is to try to integrate the **E** field we just found to obtain the potential. Of course, we can't do that we only found the **E** field at a single point, not as a function of position. Instead, just sum the point charge potentials from the 3 points:

$$V = \frac{1}{4\pi\varepsilon_0}\sum_{i \neq j}\frac{q_i}{r_{ij}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{a} + \frac{Q}{\sqrt{2}a} + \frac{Q}{a}\right) = \frac{Q}{4\pi\varepsilon_0 a}\left(2 + \frac{1}{\sqrt{2}}\right)$$

(c)The work required to bring a charge $+Q$ from infinity (where the potential is 0) to the corner is:

$$W = Q\Delta V = \frac{Q^2}{4\pi\varepsilon_0 a}\left(2 + \frac{1}{\sqrt{2}}\right)$$

(d)The work done to assemble three charges as pictured is the same as the potential energy of the three charges already in such an arrangement. Now, there are two pairs of charges situated at a distance of a, and one pair of charges situated at a distance of $\sqrt{2}a$, thus we have

$$W = 2\left(\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{a}\right) + \left(\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{\sqrt{2}a}\right) = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{a}\left(2 + \frac{1}{\sqrt{2}}\right)$$

Alternatively we could have started with empty space, brought in the first charge for free, the second charge in the potential of the first and so forth. We'll get the same answer.

Problem 6:

A thin rod extends along the z-axis from $z=-d$ to $z=d$. The rod carries a charge Q uniformly distributed along its length $2d$ with linear charge density $\lambda = Q/2d$.

a) Find the electric potential at a point $z>d$ along the z-axis. Indicate clearly where you have chosen your zero reference point for your potential.

b) Use the result that $\vec{\mathbf{E}} = -\vec{\nabla}V$ to find the electric field at a point $z>d$ along the z-axis.

c) How much work is done to move a particle of mass m and positive charge q from the point $z=4d$ to the point $z=3d$?

Problem 6 Solutions:

(a) We choose our potential to be zero at infinity. Our expression for the potential is

$$V = \frac{\lambda}{4\pi\varepsilon_o}\int_{-d}^{d}\frac{dz'}{\sqrt{(z-z')^2}} = \frac{\lambda}{4\pi\varepsilon_o}\int_{-d}^{d}\frac{dz'}{(z-z')} = -\frac{\lambda}{4\pi\varepsilon_o}\ln(z-z')\Big|_{-d}^{d} = \frac{\lambda}{4\pi\varepsilon_o}\ln\frac{z+d}{z-d} \qquad (14)$$

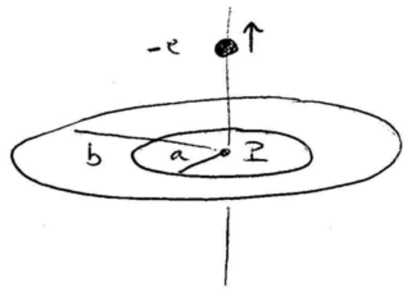
(b)

$$\mathbf{E} = -\frac{\partial V}{\partial z}\hat{\mathbf{z}} = -\frac{\lambda}{4\pi\varepsilon_o}\frac{\partial}{\partial z}\ln\frac{z+d}{z-d}\hat{\mathbf{z}} = \frac{2\lambda d}{4\pi\varepsilon_o}\frac{1}{z^2-d^2}\hat{\mathbf{z}} = \frac{Q}{4\pi\varepsilon_o}\frac{1}{z^2-d^2}\hat{\mathbf{z}} \qquad (15)$$

(c) The work done to move a particle of charge q from one point to another is just the potential difference between the points time q, by definition of potential. Thus we have

$$\Delta V = V_{z=3d} - V_{z=4d} = \frac{\lambda}{4\pi\varepsilon_o}\ln\frac{3d+d}{3d-d} - \frac{\lambda}{4\pi\varepsilon_o}\ln\frac{4d+d}{4d-d} = \frac{\lambda}{4\pi\varepsilon_o}\left[\ln 2 - \ln\frac{5}{3}\right] = \frac{\lambda}{4\pi\varepsilon_o}\ln\frac{6}{5} \qquad (16)$$

Problem 7:
A thin washer of outer radius b and inner radius a has a uniform negative surface charge density $-\sigma$ on the washer (note that $\sigma > 0$).



a) If we set $V(\infty) = 0$, what is the electric potential difference between a point at the center of the washer and infinity, $V(P) - V(\infty)$?

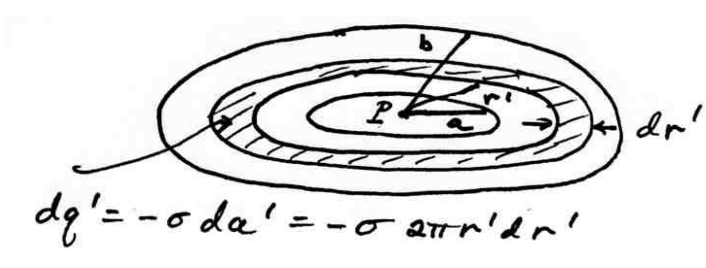
b) An electron of mass m and charge $q = -e$ is released with an initial speed v_0 from the center of the hole (at the origin) in the upward direction (along the perpendicular axis to the washer) experiencing no forces except repulsion by the charges on the washer. What speed does it ultimately obtain when it is very far away from the washer (i.e. at infinity)?

Problem 7 Solution:

(a)The potential difference $V(P) - V(\infty)$ between infinity and the point P at the center of the washer is given by

$$V(P) - V(\infty) = \int_{source} \frac{k(-\sigma)da'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|}$$

Choose as an integration element a ring of radius r' and width dr' with charge $dq' = (-\sigma)da'$ where $da' = 2\pi r' dr'$.



Because the field point P is at the origin $\vec{\mathbf{r}} = \vec{\mathbf{0}}$ and the vector from the origin to the any point on the ring is $\vec{\mathbf{r}}' = r'\hat{\mathbf{r}}$, therefore in the above expression the distance from the integration element, the ring, to the field point P is

$$\frac{1}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}=\frac{1}{r'}$$

So the integral becomes

$$V(P)-V(\infty)=\int_{source}\frac{k(-\sigma)da'}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}=\int_{r'=a}^{r'=b}\frac{k(-\sigma)2\pi r'dr'}{r'}=-k\sigma 2\pi(b-a)$$

(b)By conservation of energy (note that $V(\infty)-V(P)=k\sigma 2\pi(b-a)$ >0)

$$0=\Delta K+\Delta U=\Delta K+q(V(\infty)-V(P))=\Delta K-ek\sigma 2\pi(b-a)\ .$$

If we denote the initial speed of the electron by v_0 and the speed of the electron when it is very far away by v_f then $\Delta K=(1/2)mv_f^2-(1/2)mv_0^2$. Hence

$$(1/2)mv_f^2-(1/2)mv_0^2=ek\sigma 2\pi(b-a)>0\ .$$

We can now solve for the final speed of the electron when it is very far away from the washer

$$v_f=\sqrt{v_0^2+ek\sigma 4\pi(b-a)/m}\ .$$

8.02SC Physics II: Electricity and Magnetism
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