Maxwell's Equations and Electromagnetic Waves
Challenge Problem Solutions

Problem 1:

The magnetic field of a plane electromagnetic wave is described as follows:

$$\vec{B} = B_0 \sin(kx - \omega t)\,\hat{j}$$

a) What is the wavelength λ of the wave?

b) Write an expression for the electric field $\vec{E}$ associated to this magnetic field. Be sure to indicate the direction with a unit vector and an appropriate sign (+ or –).

c) What is the direction and magnitude Poynting vector associated with this wave? Give appropriate units, as well as magnitude.

d) This wave is totally reflected by a thin conducting sheet lying in the *y-z* plane at $x = 0$. What is the resulting radiation pressure on the sheet? Give appropriate units, as well as magnitude.

f) The component of an electric field parallel to the surface of an ideal conductor must be zero. Using this fact, find expressions for the electric and magnetic fields for the reflected wave? What are the total electric and magnetic fields at the conducting sheet, $x = 0$. Check that your answer satisfies the condition on the electric field at the conducting sheet, $x = 0$.

g) An oscillating surface current $\vec{K}$ flows in the thin conducting sheet as a result of this reflection. Along which axis does it oscillates? What is the amplitude of oscillation?

Problem 1 Solutions:

Solution:

(a) $\lambda = \frac{2\pi}{k}$

(b) $\vec{E} = cB_0 \sin(kx - \omega t)(-\hat{k})$

Since $\text{dir } \vec{E} \times \text{dir } \vec{B} = \text{dir (propagation)}$

$$\pm(-\hat{k}) \times \pm(\hat{j}) = \hat{i}$$

(c) $\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{cB_0^2 \sin^2(kx - \omega t)}{\mu_0}\hat{i}$

(d) $\langle P_{rad} \rangle = \frac{2|\langle \vec{S} \rangle|}{c}$

time averaged $|\langle \vec{S} \rangle| = \frac{cB_0^2}{2\mu_0}$

$$\langle P_{rad} \rangle = \frac{cB_0^2}{\mu_0 c}$$

Boundary Condition:

$$\vec{E}_{reflected}(x=0) + \vec{E}_{incident}(x=0) = 0$$

$$\vec{E}_{reflected}(x=0) = -\vec{E}_{incident}(x=0)$$

Let $\vec{E}_{reflected}(x=0) = E_{r,0}\sin(kx + \omega t)\Big|_{x=0}(-\hat{k})$

$$= E_{r,0}\sin\omega t \,(-\hat{k})$$

$$\vec{E}_{incident}(x=0) = cB_0 \sin(-\omega t)(-\hat{k})$$

$$= cB_0 \sin \omega t \ (\hat{k})$$

$$\vec{E}_{reflected}(x=0) = -\vec{E}_{incident}(x=0)$$

$$\Rightarrow \quad E_{r,0} \sin \omega t (-\hat{k}) = -cB_0 \sin \omega t (\hat{k})$$
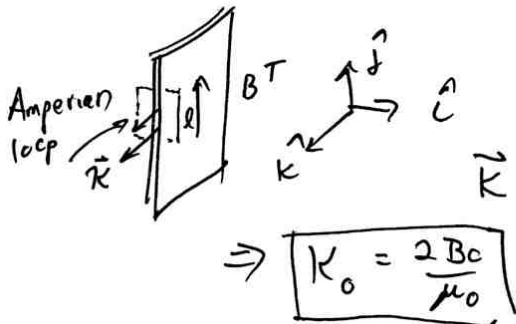
$$\Rightarrow \quad E_{r,0} = +cB_0$$

$$\Rightarrow \vec{E}_{reflected} = +cB_0 \sin(kx+\omega t)(-\hat{k})$$

$$= -cB_0 \sin(kx+\omega t)\hat{k}$$

$$\Rightarrow \vec{B}_{reflected} = -B_0 \sin(kx+\omega t)\hat{j}$$

$$\vec{E}_{total} = \vec{E}_{reflected} + \vec{E}_{incident}$$

$$= -cB_0 \sin(kx+\omega t)\hat{k} + cB_0 \sin(kx-\omega t)(-\hat{k})$$

$$\vec{B}_{total} = -B_0 \sin(kx+\omega t)\hat{j} + B_0 \sin(kx-\omega t)\hat{j}$$

$$\text{at} \quad x=0 \quad \vec{B}_{Total} = -2B_0 \sin(\omega t)\hat{j}$$

$$\oint \vec{B}\cdot d\vec{r} = \mu_0 \iint \vec{J}\cdot d\vec{a}$$

$$B^T \ell = \mu_0 K \ell$$

$$\vec{K} = \frac{B^T}{\mu_0}\hat{k} = \frac{2B_0 \sin(\omega t)}{\mu_0}\hat{k}$$

$$\Rightarrow \boxed{K_0 = \frac{2B_0}{\mu_0}}$$

Problem 2:

The electric field of an electromagnetic wave is given by the superposition of two waves

$$\vec{E} = E_0 \cos(kz - \omega t)\hat{i} + E_0 \cos(kz + \omega t)\hat{i}.$$

You may find the following identities and definitions useful

$$\cos(kz + \omega t) = \cos(kz)\cos(\omega t) - \sin(kz)\sin(\omega t)$$
$$\sin(kz + \omega t) = \sin(kz)\cos(\omega t) + \cos(kz)\sin(\omega t)$$

a) What is the associated magnetic field $\vec{B}(x, y, z, t)$.
b) What is the energy per unit area per unit time (the Poynting vector $\vec{S}$) transported by this wave?
c) What is the time average of the Poynting $\langle\vec{S}\rangle$ vector? Briefly explain your answer.

   Note the time average is given by

$$\langle\vec{S}\rangle \equiv \frac{1}{T}\int_0^T \vec{S}dt.$$

Problem 2 Solutions:

The electric field is given by

$$\vec{E} = E_0 \cos(kz - \omega t)\hat{i} + E_0 \cos(kz + \omega t)\hat{i}$$

The first term describes an electric field propagating in the $+x$-direction. The associated magnetic field is given by

$$\vec{B}_1 = E_0 \cos(kz - \omega t)\hat{i} + E_0 \cos(kz + \omega t)\hat{i}$$

$$\vec{E} = E_0 \cos(kz - \omega t)\, \hat{\imath} + E_0 \cos(kz + \omega t)\, \hat{\imath}$$

$$\vec{B} = -\int (\vec{\nabla} \times \vec{E})\, dt$$

$$= -\int \frac{\partial E_x}{\partial z}(+\hat{\jmath})\, dt$$

$$= -\int -kE_0 \sin(kz - \omega t)\, dt\, \hat{\jmath} + -\int -kE_0 \sin(kz + \omega t)\, dt\, \hat{\jmath}$$

$$= -E_0 k \frac{\cos(kz - \omega t)}{(-\omega)}\, \hat{\jmath} - E_0 \frac{\cos(kz + \omega t)}{(+\omega)}\, \hat{\jmath}$$

$$= \frac{k}{\omega} E_0 \cos(kz - \omega t)\, \hat{\jmath} - \frac{k}{\omega} E_0 \cos(kz + \omega t)\, \hat{\jmath}$$

$$\frac{\partial^2 E_x}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} \quad\Rightarrow\quad -k^2 E_x = -\frac{\omega^2}{c^2} E_x$$

$$\Rightarrow \quad \frac{\omega}{k} = c$$

$$\vec{B} = \frac{E_0}{c}\cos(kz - \omega t)\, \hat{\jmath} - \frac{E_0}{c}\cos(kz + \omega t)\, \hat{\jmath}$$

note: $\vec{E} = E_0(\cos(kz)\cos\omega t) - \sin(kz)\sin(-\omega t)\,\hat{i}$

$\quad + E_0(\cos(kz)\cos(\omega t) - \sin(kz)\sin(\omega t))\,\hat{i}$

$\quad = 2E_0\cos(kz)\cos(\omega t)\,\hat{i}$

$\vec{B} = \dfrac{2E_0}{c}\sin(kz)\sin(\omega t)\,\hat{j}$

$\vec{S} = \dfrac{\vec{E}\times\vec{B}}{\mu_0} = \dfrac{4E_0^2}{c}\sin(kz)\cos(kz)\cos(\omega t)\sin(\omega t)\,\hat{k}$

$\vec{S} = \dfrac{E_0^2}{c}\sin(2kz)\cos(2\omega t)\,\hat{k}$

$\langle\vec{S}\rangle = 0 \qquad\qquad \text{since}$

$\dfrac{1}{T}\displaystyle\int_0^T \cos(2\omega t)\,dt = 0$

this is a standing wave so it
does not transport power.

Problem 3:

The magnetic vector of a plane electromagnetic wave is described as follows:

$$\vec{B} = \hat{k}\, B_o \cos\left[\left(10\, meter^{-1}\right) y + \left(3\times10^9\, s^{-1}\right) t\right]$$

where $\hat{k}$ is a unit vector in the z-direction, y is in meters, and t is in seconds.

(a) What is the wavelength λ of the wave? Your answer can contain π. Indicate units!

(b) What is the period T of the wave? Your answer can contain π. Indicate units!

(c) In which direction does this wave propagate? Be sure to indicate the direction of propagation with a unit vector ($\hat{i}, \hat{j}$, or $\hat{k}$) and an appropriate sign (+ or –). *Briefly explain why you choose this direction.*

(d) Write an expression for the electric vector $\vec{E}$ of the wave in terms of the quantities given and the speed of light c. Be sure to indicate the direction of the electric vector with a unit vector ($\hat{i}, \hat{j}$, or $\hat{k}$) and an appropriate sign (+ or –).

(e) What is the time-dependent Poynting vector associated with this wave? Do not time-average. Be sure to give both magnitude and direction, and *write the magnitude in terms the speed of light c, B_o, and μ_o*, in addition to the numerical constants given. *Do not use ε_o in your final expression.*

Problem 3 Solutions:

(a) We have a magnetic field of the form $\vec{B} = \hat{k}\, B_o \cos\left(ky + \omega t\right)$, where $k = 10\,\mathrm{m^{-1}}$ is the wave vector and $\omega = 3\times10^9\,/\mathrm{s}$ is the angular frequency. The wavelength of the wave is given by

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{10\,\mathrm{m^{-1}}} = \frac{\pi}{5}\,\mathrm{m}$$

(b) The period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{3\times10^9\,/\mathrm{s}} = \frac{2\pi}{3}\times10^{-9}\,\mathrm{s}$$

(c) A *positive* sign between the y and t terms in the argument $ky + \omega t$ of the cosine function means that the direction of propagation is $-y$ (with a unit vector $-\hat{j}$).

(d) The direction of propagation, $-\hat{j}$, of the electromagnetic waves corresponds to the

direction of the cross product $\vec{E} \times \vec{B}$ (recall that $\vec{S} = \vec{E} \times \vec{B} / \mu_0$). With $\vec{B}$ pointing in the $+\hat{k}$ direction, the electric field must point in the $+\hat{i}$ direction, so that $\hat{i} \times \hat{k} = -\hat{j}$. On the other hand, from Maxwell's equations, the amplitudes of $\vec{E}$ and $\vec{B}$ are related by $E_0 = cB_0$. Thus, the electric field can be written as

$$\vec{E} = \hat{i}\, cB_o \cos\left[\left(10\ meter^{-1}\right) y + \left(3 \times 10^9\ s^{-1}\right) t \right]$$

(e) The Poynting vector is

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} \left(\hat{i}\, cB_o\right) \times \left(\hat{k}\, B_o\right) \cos^2\left[\left(10\ meter^{-1}\right) y + \left(3 \times 10^9\ s^{-1}\right) t \right]$$

$$= -\hat{j}\left(\frac{cB_0^2}{\mu_0} \right) \cos^2\left[\left(10\ meter^{-1}\right) y + \left(3 \times 10^9\ s^{-1}\right) t \right]$$

Problem 4:

In this question you are going to derive the wave equation – that is, prove that electromagnetic radiation as you have studied it in class is a natural outcome of Maxwell's equations.

Consider a wave traveling along the x-axis, where the magnetic field is polarized along the z-axis and the electric field along the y-axis.

(a) Use the Ampere-Maxwell law (with the rectangular loop pictured at right) to calculate a relationship between the spatial derivative of the magnetic field and the time derivative of the electric field.
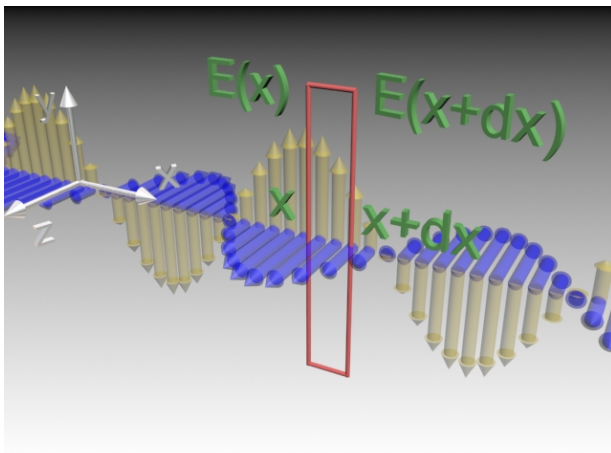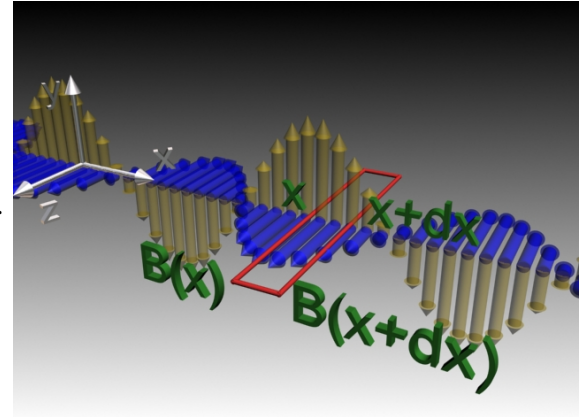
b) Use Faraday's Law (with the rectangular loop pictured at left) to calculate a relationship between the spatial derivative of the electric field and the time derivative of the magnetic field.

(c) Now you have two partial differential equations, relating spatial derivatives and time derivatives of E and B. Unfortunately they are coupled (that is, E depends on B and B depends on E). To solve these types of equations we need to decouple them – to get an equation that only has E in is, and another that only has B in it.

To do this, take an x-derivative of one of your equations and a time derivative of the other. Since the derivatives commute (that is, taking the space derivative followed by the time derivative is the same as taking the time derivative followed by the space derivative) you will find that you can substitute one equation into the other and get an equation that only has E in it or only has B in it. If you change which equation you do which derivative to you will get the other.

These two new equations are Wave Equations! They are second order differential equations (two spatial derivatives on one side, two time derivatives on the other).

(d) Finally, let's prove that the pictured wave is actually a solution to the equations you just derived. We'll just do the electric field. Write an equation (looking at the picture) for what **E** is as a function of time and position. Remember that it is a traveling wave (you should be able to tell from the picture in what direction it is traveling). You don't know the amplitude, period or wavelength of the wave, just use variables for these, but make sure that your wave equation is satisfied (it should force a well-known relationship between your period and wavelength).

Problem 4 Solutions:

(a)

Ampere's Law: $\displaystyle\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \varepsilon_0 \frac{d}{dt}\int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$

$LHS: \displaystyle\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B_z(x,t)l - B_z(x+dx,t)l$

$RHS: \dfrac{d}{dt}\int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \dfrac{\partial}{\partial t}\left(E_y \cdot l\, dx\right) = l\, dx \dfrac{\partial E_y}{\partial t}$

Equating the left and right hand sides and dividing through by $l\, dx$:

$$-\frac{B_z(x+dx,t) - B_z(x,t)}{dx} = \boxed{-\frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}}$$

(b)

Faraday's Law: $\displaystyle\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d}{dt}\int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$

$LHS: \displaystyle\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = E_y(x+dx,t)l - E_y(x,t)l$

$RHS: \dfrac{d}{dt}\int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = ldx\dfrac{\partial B_z}{\partial t}$

Equating the left and right hand sides and dividing through by $l\, dx$:

$$\frac{E_y(x+dx,t) - E_y(x,t)}{dx} = \boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}}$$

(c)
Start with the two equations that we derived in (a) and (b):

$$-\frac{\partial B_z}{\partial x} = \mu_0\varepsilon_0\frac{\partial E_y}{\partial t} \qquad \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial x}\right) = \underline{\frac{\partial^2 E_y}{\partial x^2}} = \frac{\partial}{\partial x}\left(-\frac{\partial B_z}{\partial t}\right) = \frac{\partial}{\partial t}\left(-\frac{\partial B_z}{\partial x}\right) = \mu_0\varepsilon_0\underline{\frac{\partial^2 E_y}{\partial t^2}}$$

$$\boxed{\frac{\partial^2 E_y}{\partial x^2} = \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2}}$$

If we instead take the spatial derivative of the first equation we get our wave eqn for B:

$$\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial t}\right) = \underline{\frac{\partial^2 B_z}{\partial t^2}} = \frac{\partial}{\partial t}\left(-\frac{\partial E_y}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial t}\right) = \underline{\frac{1}{\mu_0\varepsilon_0}\frac{\partial^2 B_z}{\partial x^2}}$$

$$\boxed{\frac{\partial^2 B_z}{\partial x^2} = \mu_0\varepsilon_0\frac{\partial^2 B_z}{\partial t^2}}$$

Note that these two equations are essentially "the same" – they only differ in which components of the electric/magnetic fields they show propagating.

(d)
Looking at the electric field, it is a sinusoidal traveling wave polarized along the y-axis and traveling along the +x direction (we know it is +x because the cross product of **E** and **B** give you the direction of travel):

$$\vec{E} = \hat{j}f\left(x - vt\right) = \hat{j}E_0\sin\left(kx - \omega t\right) = \hat{j}E_0\sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right)$$

You could use any of these forms (or cosines). The first (with $f(x\text{-}vt)$) is completely generic while the next two assume a sinusoidal variation. Since no indication of the time of the picture was given, the phase (whether it is sin, cos or something in between) is undetermined. Since the question mentioned wavelength and period I'll use the last form, substituting it into the wave equation for E we derived in (c):

$$\vec{E} = \hat{j}f\left(x - vt\right) = \hat{j}E_0\sin\left(kx - \omega t\right) = \hat{j}E_0\sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right)$$

$$LHS: \frac{\partial^2 E_y}{\partial x^2} = \frac{\partial^2}{\partial x^2}E_0\sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right) = -\left(\frac{2\pi}{\lambda}\right)^2 E_0\sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right) = -\left(\frac{2\pi}{\lambda}\right)^2 E_y$$

Similarly, $RHS: \mu_0\varepsilon_0\dfrac{\partial^2 E_y}{\partial t^2} = -\mu_0\varepsilon_0\left(\dfrac{2\pi}{T}\right)^2 E_y$

For the equation to be satisfied then we have:

$$-\left(\frac{2\pi}{\lambda}\right)^2 E_y = -\mu_0\varepsilon_0\left(\frac{2\pi}{T}\right)^2 E_y \Rightarrow \boxed{\frac{\lambda}{T} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c}$$

The ratio of wavelength to period is the speed of the wave which is the speed of light.

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