8.03 Fall 2004
Problem Set 7 Solutions

Solution 7.1: Polarized radiation

Part (a)

For angle $\alpha = \pi/4$ from the $+y$ direction

$$\vec{E}_{\pi/4} = \frac{E_0}{\sqrt{2}} \cos(\omega t - kx)(\hat{y} + \hat{z}) \tag{1}$$

$$\begin{aligned} \vec{B}_{\pi/4} &= \frac{1}{\omega} k \times \vec{E} = \frac{E_0}{c\sqrt{2}} \hat{x} \times (\hat{y} + \hat{z}) \cos(\omega t - kx) \\ &= \frac{E_0}{c\sqrt{2}} \cos(\omega t - kx)(\hat{z} - \hat{y}) \end{aligned} \tag{2}$$

For angle $\alpha = -\pi/4$ from the $+y$ direction

$$\vec{E}_{-\pi/4} = \frac{E_0}{\sqrt{2}} \cos(\omega t - kx)(\hat{y} - \hat{z}) \tag{3}$$

$$\begin{aligned} \vec{B}_{-\pi/4} &= \frac{1}{\omega} k \times \vec{E} = \frac{E_0}{c\sqrt{2}} \hat{x} \times (\hat{y} - \hat{z}) \cos(\omega t - kx) \\ &= \frac{E_0}{c\sqrt{2}} \cos(\omega t - kx)(\hat{y} + \hat{z}) \end{aligned} \tag{4}$$

Part (b)

$$\vec{E} = E_0[\cos(\omega t - kx)\hat{y} + \cos(\omega t - kx + \pi/2)\hat{z}] \tag{5}$$

$$\begin{aligned} \vec{B} &= \frac{1}{\omega} k \times \vec{E} = \frac{E_0}{c} \hat{x} \times [\cos(\omega t - kx)\hat{y} + \cos(\omega t - kx + \pi/2)\hat{z}] \\ &= \frac{E_0}{c}[\cos(\omega t - kx)\hat{z} - \cos(\omega t - kx + \pi/2)\hat{y}] \end{aligned} \tag{6}$$

This is called left handed circular polarization by many authors, though it is called right handed by Bekefi and Barrett.

$$\vec{E} = E_0[\cos(\omega t - kx)\hat{y} + \cos(\omega t - kx - \pi/2)\hat{z}] \tag{7}$$

$$\begin{aligned} \vec{B} &= \frac{1}{\omega} k \times \vec{E} = \frac{E_0}{c} \hat{x} \times [\cos(\omega t - kx)\hat{y} + \cos(\omega t - kx - \pi/2)\hat{z}] \\ &= \frac{E_0}{c}[\cos(\omega t - kx)\hat{z} - \cos(\omega t - kx - \pi/2)\hat{y}] \end{aligned} \tag{8}$$

This is called right handed circular polarization by many authors, though it is called left handed by Bekefi and Barrett.

Solution 7.2 — Linear polarizers – Malus' law + absorption

The E-field amplitude of the transmitted part of an EM wave through a linear polarizer is $E_T = E\cos\theta$, where E_T and E are the E-field amplitudes of the transmitted and incident waves, respectively, and θ is the angle between the polarization of the incident wave and the direction of polarization of the polarizer. Thus, the intensity is reduced by $\cos^2\theta$. Since $\langle\cos^2\theta\rangle = 1/2$, half of unpolarized light passes through a perfect polarizer.

Furthermore, since the polarizers are HN30, the transmitted intensity through one polarizer is $I = (0.5 \times 0.7)I_u$, where I_u is the intensity of the unpolarized light. This I is I_0 in our problem. The intensity through two polarizers is $I = I_0(0.7\cos^2\theta_{12})$, where θ_{12} is the angle between the polarization axes of the first and second polarizers. Similarly, the intensity through three polarizers is $I = I_0(0.7\cos^2\theta_{12})(0.7\cos^2\theta_{23})$, where θ_{23} is the angle between the polarization axes of the second and the third polarizers.

Let's examine each case individually.

F The unpolarized light passes through only one polarizer, so $I = (0.5 \times 0.7)I_u = I_0$.

G The light passes through two polarizers at right angles, so $I = I_0(0.7\cos^2\pi/2) = 0$.

H Two polarizers: $\theta_{12} = \pi/6$. Hence, $I = I_0(0.7\cos^2\pi/6) = 0.525I_0$.

K Three polarizers: $\theta_{12} = \pi/6$ and $\theta_{23} = \pi/3$. Hence, $I = I_0(0.7\cos^2\pi/6)(0.7\cos^2\pi/3) \approx 0.368I_0$;

L Note that this case is physically identical to H so $I = 0.525I_0$.

M Two polarizers: $\theta_{12} = \pi/3$. Hence, $I = I_0(0.7\cos^2\pi/3) = 1.4I_0$.

N The light passes through only one polarizer, so $I = I_0$.

Solution 7.3: (Bekefi & Barrett 4-1) Radiation from an accelerated charge

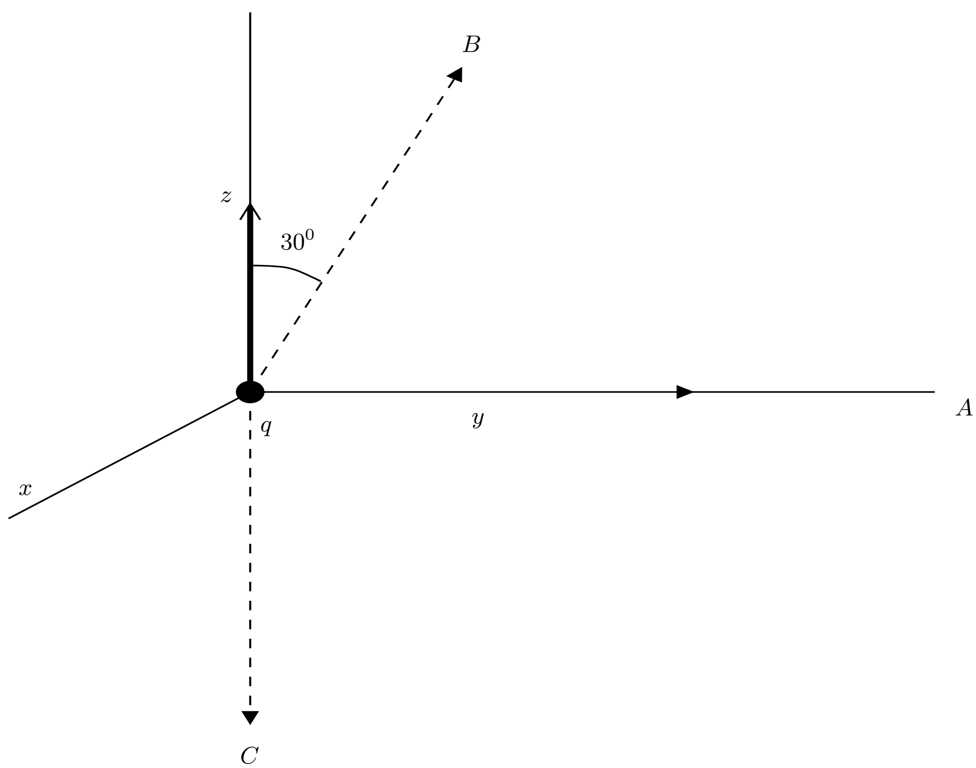

FIG. 1: Problem 7.3 Radiation from an accelerated charge

$\vec{a}_\perp$ is the component of the acceleration of the charge in a direction perpendicular to the position vector of the observer. θ is the angle between the direction of acceleration and the position vector of the observer.

$$
\begin{aligned}
t' &= t - \frac{r}{c} \\
\vec{E}(\vec{r},t) &= \frac{-q\vec{a}_\perp(t')}{4\pi\epsilon_0 r c^2} \quad Vm^{-1} \\
\vec{B}(\vec{r},t) &= \frac{\hat{r}}{c} \times E(\vec{r},t) \quad Wm^{-2} \\
\vec{S} &= \frac{1}{\mu_0}\vec{E} \times \vec{B} \ Wm^{-2} \\
E(\vec{r},t) &= \frac{-qa(t')\sin\theta}{4\pi\epsilon_0 r c^2} \\
|\vec{S}(\vec{r},t)| &= \frac{q^2 a^2(t')\sin^2\theta}{16\pi^2\epsilon_0 r^2 c^3} \\
P(t) &= \int_0^\pi |\vec{S}(\vec{r},t)| 2\pi r^2 \sin\theta d\theta = \frac{q^2 a^2(t')}{6\pi\epsilon_0 c^3} \quad Watt
\end{aligned}
$$

Part (a)

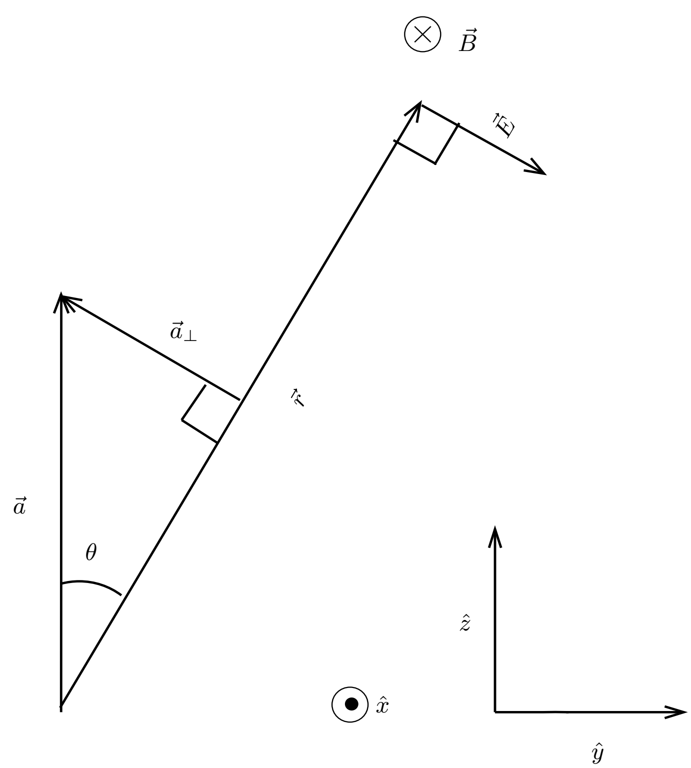

FIG. 2: Orientation of radiated electric and magnetic fields

Arrival time at all the three observers A, B and C is $t_{arrival} = R/c$. The direction of the electric field at the point of observation is anti-parallel to the component of the acceleration perpendicular to the position vector. The direction of the magnetic field is cross product of the position vector with the electric field. Fig. 2 shows the directions.

- Observer A

$$\begin{aligned}\vec{E}_A &= \frac{qa(t')}{4\pi\epsilon_0 Rc^2}\sin\theta_A(\vec{r}_A\times\hat{x}) \qquad \theta_A=\frac{\pi}{2}\\ &= \frac{-qa(t')}{4\pi\epsilon_0 Rc^2}\,\hat{z} \end{aligned}\tag{9}$$

- Observer B

$$\begin{aligned}\vec{E}_B &= \frac{qa(t')}{4\pi\epsilon_0 Rc^2}\sin\theta_B(\vec{r}_B\times\hat{x}) \qquad \theta_B=\frac{\pi}{6}\\ &= \frac{1}{2}\frac{qa(t')}{4\pi\epsilon_0 Rc^2}(\frac{\sqrt{3}}{2}\hat{y}-\frac{1}{2}\hat{z}) \end{aligned}\tag{10}$$

- Observer C

$$\begin{aligned}\vec{E}_C &= \frac{qa(t')}{4\pi\epsilon_0 Rc^2}\sin\theta_C(\vec{r}_C\times\hat{x}) \qquad \theta_C=0\\ &= 0 \end{aligned}\tag{11}$$

Part (b)
As $|B| = |E|/c$, hence the relative strengths of the magnetic field B are same as the relative strengths of the electric field E in Part(a) at the three observation points. The arrival time of the magnetic field at the three observers A, B and C is $t_{arrival} = R/c$. The direction of the induced magnetic field at the three points is in the - x direction.

Solution 7.4: (Bekefi & Barrett 4-2) Radiation from an accelerated charge

Part (a)
A point charge $+q$ from time interval $t = t_0$ to $t = t_0 + \Delta t$ feels a force perpendicular to its trajectory, and moves along a new trajectory without changing its speed $|\vec{w}|$. Since the angle $\Delta\alpha$ is small, the acceleration along x axis is negligible and does not effect the answer. The only significant acceleration of the point charge is along the $-y$ direction.

$$\begin{aligned}\vec{a} &= \frac{\Delta V_y}{\Delta t}\hat{y} = \frac{w\sin\Delta\alpha}{\Delta t}\hat{y} \simeq w\frac{\Delta\alpha}{\Delta t}\hat{y}\\ \Rightarrow \quad a_\perp &= a_y\sin\theta = w\frac{\Delta\alpha}{\Delta t}\sin\theta\end{aligned}$$

here $a_\perp$ is the component of acceleration perpendicular to the position vector of the distant point P_1. Then the electric field at point P_1 is anti-parallel to $a_\perp$ and is oriented as shown in Fig. 3.

$$E = \frac{q}{4\pi\epsilon_0 r}\frac{a_\perp}{c^2}(\hat{r}_{P1}\times\hat{z}) = \frac{q}{4\pi\epsilon_0 r}\frac{v}{c^2}\frac{\Delta\alpha}{\Delta t}\sin\theta(\cos\theta\hat{x}+\sin\theta\hat{y})\tag{12}$$

So at a distant point P_1 the electric field caused by the acceleration has the direction $(\cos\theta\hat{x} + \sin\theta y)$ where θ is as shown in Fig. 3.

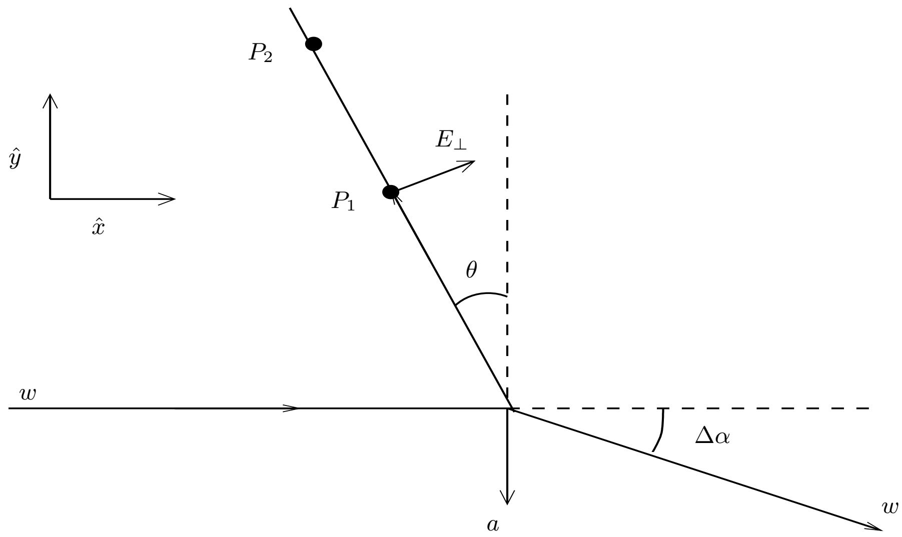

FIG. 3: Problem 7.4 Radiation from an accelerated charge

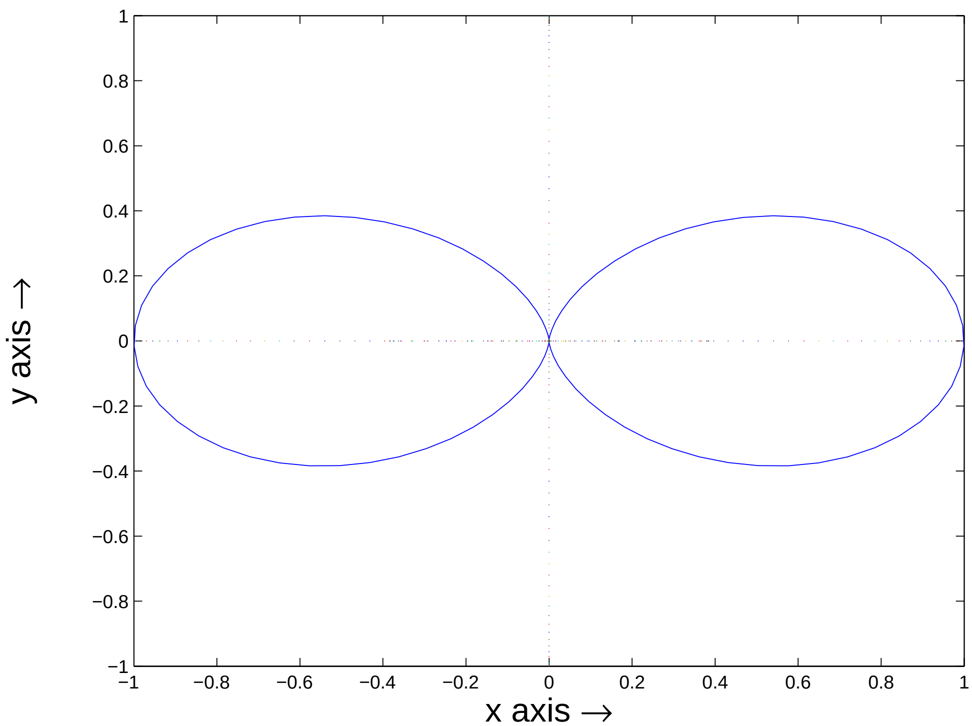

FIG. 4: Problem 7.4(b) Radial plot of intensity versus angle θ

Part (b)

The radiation intensity $\propto |\vec{E}_\perp|^2 \propto \sin^2\theta$. So it is most intense in the $x - z$ plane.

Part (c)

The least intense direction is along the y axis. Fig. 4 shows the radial plot of variation of intensity with angle θ from along the $+y$ direction.

Part (d)

$$\vec{B}(\vec{r},t) = \hat{r} \times \frac{\vec{E}(\vec{r},t)}{c} \quad \Rightarrow \quad \vec{B} \simeq \frac{E_\perp}{c} \propto \frac{1}{r} \tag{13}$$

so, the amplitude decreases by a factor of 2.

Part (e)

$$\Delta E_{radiated} = P\Delta t = \frac{q^2 a^2 \Delta t}{6\pi\epsilon_0 c^3} = \frac{q^2 w^2}{6\pi\epsilon_0 c^3}\left(\frac{\Delta\alpha}{\Delta t}\right)^2 \Delta t \tag{14}$$

Solution 7.5: Speed checked by radar

$$f'(received\ by\ moving\ car) \simeq f(1+\beta) \tag{15}$$

$\beta = v/c$ is positive for approaching car

$$f''(received\ by\ police) \simeq f'(1+\beta) \simeq f(1+\beta)^2 \simeq f(1+2\beta)$$

Part (a)

$$\lambda = c/f \quad \Rightarrow \quad f = 3\times 10^8/3\times 10^{-2} = 10^{10}\ Hz \tag{16}$$

Part (b)

See above.

Solution 7.6: Can't you hear the whistle blowing

Part (a)

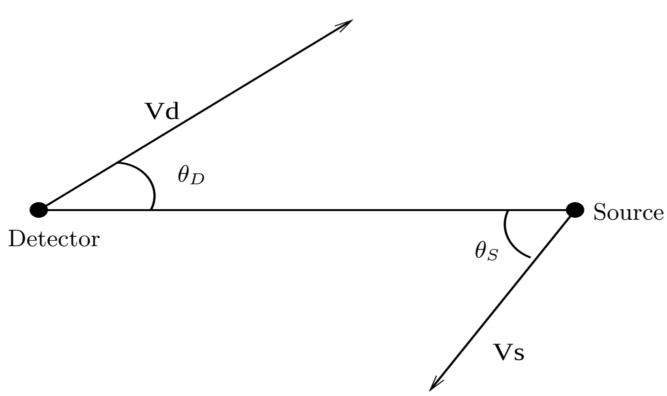

FIG. 5: Problem 7.5 General Doppler Effect

The original Doppler expression for sound is

$$f' = \left(\frac{v + v_D \cos\theta_D}{v - v_S \cos\theta_S} \right) f$$

where v, v_S and v_D are the speeds of sound, the source and the detector with respect to the medium, respectively. θ_D and θ_S are angles as shown in Fig. 5.

Since $v_S/v \sim 20/340 \sim 0.06 \ll 1$ and $v_D = 0$, the Doppler expression can thus be simplified as

$$f' = \left(\frac{v}{v - v_S \cos\theta_S} \right) f \approx \left(1 + \frac{v_S}{v} \cos\theta_S\right) f$$

As the train passes the detector, the angle θ_S goes from 0 to π.

- Far away approaching $\theta \sim 0$, so $f' \approx (1 + 0.059) f$

$$f'_{far\ approach} = 1059\ Hz \tag{17}$$

- Far away receding $\theta \sim \pi$, so $f' \approx (1 - 0.059) f$

$$f'_{far\ recede} = (1 - 0.059) f = 941\ Hz \tag{18}$$

- Closest approach $\theta = \pi/2$

$$f'_{t=0} = (1 - 0.059 \cos\pi/2) f = f = 1000\ Hz \qquad (\cos\theta = 0\ !) \tag{19}$$

Part (b)

At $t = -10$ sec, $\cos\theta = 200/100\sqrt{5} = 0.89$

$$f'_{t=-10} = (1 + 0.059 \times 0.89) f = 1053\ Hz \tag{20}$$

At $t = -5$ sec, $\cos\theta = 100/100\sqrt{2} = 0.71$

$$f'_{t=-5} = (1 + 0.059 \times 0.71) f = 1042\ Hz \tag{21}$$

Part (c)

Fig. 6 shows the plot of heard frequency versus time for the train whistle.

Solution 7.7: Our expanding universe - simplified

Part (a)

$s = R\theta$ so

$$\theta = s/R = \frac{s + \Delta s}{R + \Delta R} \tag{22}$$

Dividing by Δt

$$s\left(\frac{\Delta R}{\Delta t}\right) = R\frac{\Delta s}{\Delta t} \quad \Rightarrow \quad \left(\frac{\Delta s}{\Delta t}\right) = \left[\frac{1}{R}\frac{\Delta R}{\Delta t}\right] s \tag{23}$$

Part (b) Hubble's Law: $v = Hd$, here v is the recession velocity of a galaxy, and d is the distance between us and that galaxy. For the balloon universe, $ds/dt = v$, and $s = d$. Thus $H = (1/R)(\Delta R/\Delta t)$. The units for H are 1/time. $\Delta R/\Delta t$ is the expansion rate of the balloon.

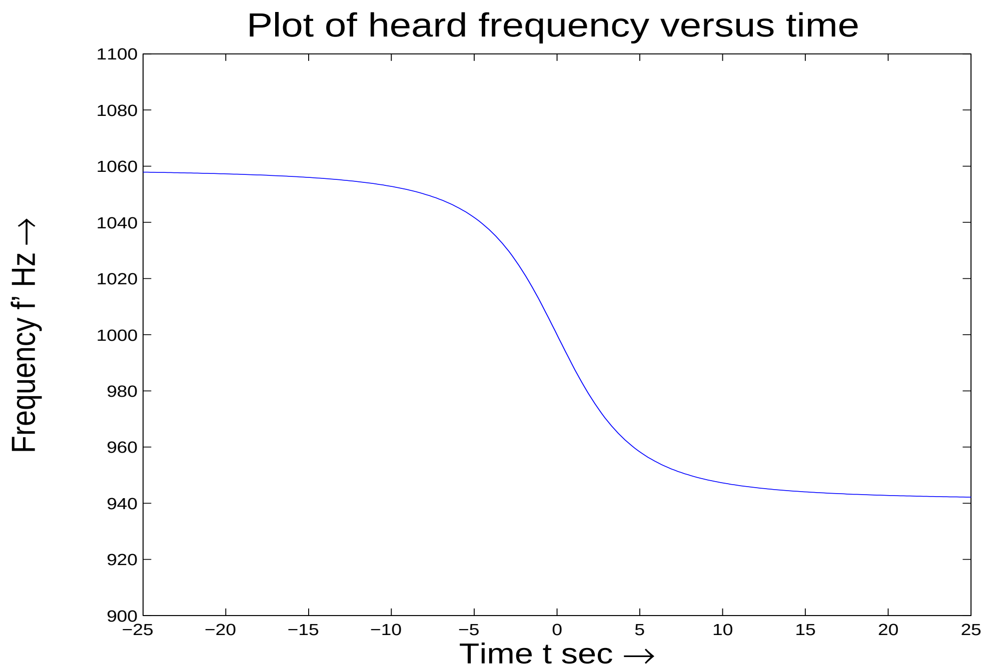

FIG. 6: Problem 7.6 Plot of heard frequency versus time

Part (c)

If $\Delta s/\Delta t = l$, the recession velocity at the horizon equals the maximum velocity of the ants. For constant $\Delta R/\Delta t$ this occurs at a distance

$$s_{max} = Rl\left(\frac{\Delta R}{\Delta t}\right)^{-1} \tag{24}$$

Note: R is the radius of the universe (Here, the radius of the balloon.)

Part (d)

Multiply Eq.(3) from the Problem Set 7.7, by $2/R^2$ and introduce $V = \Delta R/\Delta t$ and $\rho = 3M/4\pi R^3$. Then

$$\left(\frac{1}{R}\frac{\Delta R}{\Delta t}\right)^2 = \frac{8\pi G\rho}{3} + \frac{2(constant)}{R^2} \tag{25}$$

Part (e)

For a flat universe, the constant= 0, so

$$H^2 = \left(\frac{1}{R}\frac{\Delta R}{\Delta t}\right)^2 = \frac{8\pi G\rho}{3} \quad \Rightarrow \quad \rho_0 = \left(\frac{3}{8\pi G}\right) H_0^2 \sim 10^{-26}\ kg/m^3 \sim 10^{-29}\ g/cm^3\ \ for\ H_0 = 70\ km/sec\ per\ Mpc. \tag{26}$$

Part (f)

$$\left(\frac{1}{R}\frac{\Delta R}{\Delta t}\right)^2 = \frac{2MG}{R^3} + \frac{2(constant)}{R^2}$$

As before the constant=0, so $\Delta R/\Delta t = \sqrt{2MG/R}$ and $\Delta t = \Delta R\sqrt{R/2MG}$. Integrating gives

$$t = \frac{2R^{3/2}}{3\sqrt{2MG}} \qquad \Rightarrow \qquad R(t) \propto t^{2/3} \tag{27}$$

Now, since $H = \Delta R/(R\Delta t)$ and $R \propto t^{2/3}$ one can find an expression for H in terms of t. Let $R = ct^{2/3}$ then

$$\frac{\Delta R}{\Delta t} = \frac{2}{3}ct^{-1/3} \qquad H = \frac{1}{ct^{2/3}}\left(\frac{2}{3}ct^{-1/3}\right) = \frac{2}{3t} \tag{28}$$

Age of the universe is now equal to $t_0 = 2/3H_0 \sim 9.3 \times 10^9$ years.

Part (g)

Combining Eqs. 27 and 28 shows that H is inversely proportional to $R^{3/2}$ (no time dependence!). Since R was smaller in the past, H must have been larger.

Part (h)

H becomes negative if $\Delta R/\Delta t$ becomes negative. This can occur for a closed universe. It means that the universe is collapsing, and thus the redshifted galaxies and QSO's would become blueshifted.