Massachusetts Institute of Technology

Physics 8.03 Fall 2004

Final Exam

Thursday, December 16, 2004

- You have 3 hours
- Do all eight problems
- You may use calculators
- This is a closed-book exam; no notes are allowed

Useful Formulae

General differential equation for oscillators

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

has solutions

$$\begin{aligned}
x(t) &= A\,\mathrm{e}^{-\frac{\gamma t}{2}} \cos\left(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\; t + \alpha\right) + x_{ss}(t) && \omega_0 > \frac{\gamma}{2} \\
x(t) &= (A + B\,t)\;\mathrm{e}^{-\frac{\gamma t}{2}} + x_{ss}(t) && \omega_0 = \frac{\gamma}{2} \\
x(t) &= A\,\mathrm{e}^{-\Gamma_1 t} + B\,\mathrm{e}^{-\Gamma_2 t} + x_{ss}(t) && \omega_0 < \frac{\gamma}{2}
\end{aligned}$$

where

$$\Gamma_{\frac{1}{2}} = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

and the steady-state solution is

$$x_{ss}(t) = A(\omega)\,\cos(\omega t - \delta(\omega))$$

$$A(\omega) = \frac{f_0}{\left[(\omega_0^2 - \omega^2)^2 + \gamma^2\,\omega^2\right]^{1/2}} \qquad \tan\delta(\omega) = \frac{\gamma\,\omega}{\omega_0^2 - \omega^2}$$

Non-dispersive wave equation

$$\frac{\partial^2}{\partial x^2} y(x,t) = \frac{1}{v^2}\,\frac{\partial^2}{\partial t^2} y(x,t)$$

where $v = \sqrt{T/\mu}$ for a string or $v = \sqrt{\kappa/\rho}$ for a gas.

Kinetic, potential energy and power

$$\frac{dK}{dx} = \frac{1}{2}\,\mu\left(\frac{\partial y}{\partial t}\right)^2 \qquad \frac{dU}{dx} = \frac{1}{2}\,T\left(\frac{\partial y}{\partial x}\right)^2 \qquad P(t) = -T\left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial y}{\partial x}\right)$$

Reflection and transmission coefficients

$$R = \frac{v_2 - v_1}{v_2 + v_1} \qquad T = \frac{2v_2}{v_2 + v_1}$$

Fourier series for a function $f(\theta) = f(\theta + 2\pi)$

$$\begin{aligned} f(\theta) &= \sum_{m=1}^{\infty}\left[\frac{A_0}{2} + A_m\,\cos(m\,\theta) + B_m\,\sin(m\,\theta)\right] \\ A_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,\cos(m\theta)\,d\theta \qquad m = 0, 1, 2, \ldots \\ B_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,\sin(m\theta)\,d\theta \qquad m = 1, 2, 3, \ldots \end{aligned}$$

Dispersion

$$v_{phase} = \frac{\omega}{k} \qquad \text{and} \qquad v_{group} = \frac{\mathrm{d}\omega}{\mathrm{d}k}$$

Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\,\epsilon_0\,\frac{\partial\vec{E}}{\partial t} + \mu_0\,\vec{J}$$

EM force, flux, energy, intensity

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \qquad \vec{S} = \tfrac{1}{\mu_0}\,\vec{E}\times\vec{B}$$

$$U_E = \frac{\epsilon_0}{2}\left|\vec{E}\right|^2 \qquad U_M = \frac{1}{2\,\mu_0}\left|\vec{B}\right|^2$$

Dipole approximation

$$\begin{aligned} \vec{E}_{rad}(\vec{r},t) &= \frac{-q\vec{a}_{\perp}(t - r/c)}{4\pi\epsilon_0 c^2 r} && \text{Volt/m} \\ \vec{B}_{rad}(\vec{r},t) &= \frac{1}{c}\,\hat{r}\times\vec{E}_{rad}(t) && \text{Tesla} \\ \vec{S}_{rad}(\vec{r},t) &= \frac{1}{\mu_0}\vec{E}_{rad}\times\vec{B}_{rad} && \text{Watt/m}^2 \\ P &= \frac{q^2 a^2}{6\pi\epsilon_0 c^3} && \text{Watt} \end{aligned}$$

Boundary conditions at the surface of a perfect conductor

$$E_{//} = 0 \qquad \left|B_{//}\right| = \mu_0 \, |J_S|$$
$$E_{\perp} = \frac{\rho_S}{\epsilon_0} \qquad B_{\perp} = 0$$

Transmission lines

$$\frac{\partial V}{\partial z} = -L_0 \, \frac{\partial I}{\partial t} \qquad v_p = \frac{1}{\sqrt{L_0 \, C_0}} \qquad \frac{V_r}{V_i} = \frac{Z_L - Z_0}{Z_L + Z_0}$$
$$\frac{\partial I}{\partial z} = -C_0 \, \frac{\partial V}{\partial t} \qquad Z_0 = \sqrt{\frac{L_0}{C_0}} \qquad \frac{I_r}{I_i} = \frac{Z_0 - Z_L}{Z_L + Z_0}$$

Boundary conditions at the surface of a perfect dielectric

$$E_{//}^{(1)} = E_{//}^{(2)} \qquad \frac{B_{//}^{(1)}}{\mu_1} = \frac{B_{//}^{(2)}}{\mu_2}$$
$$\kappa_{e1} \, E_{\perp}^{(1)} - \kappa_{e2} \, E_{\perp}^{(2)} = \frac{\rho_s}{\epsilon_0} \qquad B_{\perp}^{(1)} = B_{\perp}^{(2)}$$

$$n_1 \, \sin\theta_1 = n_2 \, \sin\theta_2$$

Fresnel equations

$$r_{\parallel} = E_{0r\parallel}/E_{0i\parallel} = \frac{n_1 \, \cos\theta_2 - n_2 \, \cos\theta_1}{n_1 \, \cos\theta_2 + n_2 \, \cos\theta_1} = -\frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}$$
$$r_{\perp} = E_{0r\perp}/E_{0i\perp} = \frac{n_1 \, \cos\theta_1 - n_2 \, \cos\theta_2}{n_1 \, \cos\theta_1 + n_2 \, \cos\theta_2} = -\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)}$$
$$t_{\parallel} = E_{0t\parallel}/E_{0i\parallel} = \frac{2n_1 \, \cos\theta_1}{n_1 \, \cos\theta_2 + n_2 \, \cos\theta_1} = \frac{2 \, \sin\theta_2 \, \cos\theta_1}{\sin(\theta_1 + \theta_2) \, \cos(\theta_1 - \theta_2)}$$
$$t_{\perp} = E_{0t\perp}/E_{0i\perp} = \frac{2n_1 \, \cos\theta_1}{n_1 \, \cos\theta_1 + n_2 \, \cos\theta_2} = \frac{2 \, \sin\theta_2 \, \cos\theta_1}{\sin(\theta_1 + \theta_2)}$$

Special case of normal incidence ($\theta_1 = \theta_2 = 0$)

$$r_{\parallel,\perp} = \frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2} \qquad t_{\parallel,\perp} = \frac{E_t}{E_i} = \frac{2\, n_1}{n_1 + n_2}$$

Doppler Effect

$$\frac{\lambda'}{\lambda} = \frac{1 - \beta \, \cos\theta}{\sqrt{1 - \beta^2}} \qquad \text{for EM waves}$$
$$\frac{f'}{f} = \frac{v_s + v_r \, \cos\theta_r}{v_s - v_t \, \cos\theta_t} \qquad \text{for sound waves}$$

N source interference and diffraction:

Interference $$I = I_0 \left[\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right]^2 \qquad \delta = \frac{2\pi}{\lambda}\, d\, \sin\theta$$

Diffraction $$I = I_0 \left[\frac{\sin\beta}{\beta}\right]^2 \qquad \beta = \frac{\pi}{\lambda}\, D\, \sin\theta$$

Diffraction gratings:

$$I = I_0 \left[\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right]^2 \left[\frac{\sin\beta}{\beta}\right]^2$$

Physical constants

Speed of light	c	3×10^8	$\mathrm{m\ s^{-1}}$
Vacuum permeability	μ_0	1.26×10^{-6}	$\left(\mathrm{V\ m^{-1}}\right)/\mathrm{A}$
Vacuum permittivity	ϵ_0	8.85×10^{-12}	$\mathrm{C}/\left(\mathrm{V\,m^{-1}}\right)$
Electron rest mass	m	9.1×10^{-31}	kg
Elementary charge	e	1.6×10^{-19}	C
Gravitational constant	G	6.7×10^{-11}	$\mathrm{N\,m^2/kg^2}$

Trigonometric Formulae

$$\begin{aligned}
\sin(a+b) &= \sin a \cos b + \cos a \sin b \\
\cos(a+b) &= \cos a \cos b - \sin a \sin b \\
\sin a + \sin b &= 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \\
\sin a - \sin b &= 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right) \\
\cos a + \cos b &= 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \\
\cos a - \cos b &= -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)
\end{aligned}$$

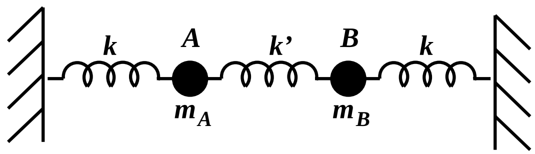
$$\begin{aligned}
\sin(\theta \pm \frac{\pi}{2}) &= \pm\cos\theta \\
\cos(\theta \pm \frac{\pi}{2}) &= \mp\sin\theta \\
\sin(\theta \pm \pi) &= -\sin\theta \\
\cos(\theta \pm \pi) &= -\cos\theta
\end{aligned}$$

Complex exponentials

$$e^{j\,\theta} = \cos\theta + j\,\sin\theta \qquad \sin\theta = \frac{e^{j\,\theta} - e^{-j\,\theta}}{2\,j} \qquad \cos\theta = \frac{e^{j\,\theta} + e^{-j\,\theta}}{2}$$

Problem 1 (16 pts): Coupled Oscillators

Two objects, A and B, with masses m_A and m_B, respectively, are connected by springs as shown in the figure. The spring constant of the spring on the left and on the right are both k; the spring constant of the spring in the middle is k'.



In parts (a), (b), and (c) you are allowed to give the answers without any calculations.

a. (2 pts) If $m_A = \infty$, what are the normal mode frequencies of the system?

b. (3 pts) If $k = 0$, what are the normal mode frequencies of the system?

c. (2 pts) If $k' = 0$, what are the normal mode frequencies of the system?

d. (4 pts) For the general situation, write down the coupled equations of motion.

e. (5 pts) Find the normal mode frequencies.

Problem 2 (15 pts): Dispersive string

The dispersion relation for oscillations of a realistic piano string with mass density μ and under tension T is given by

$$\omega = k\sqrt{\frac{T}{\mu} + \alpha\, k^2}$$

where α is a positive constant that depends on the stiffness of the string; $k(= 2\pi/\lambda)$ is the wave number. The string is firmly clamped at $x = 0$ and at $x = L$. At $t = 0$, the string is at rest and its displacement in the y-direction is given by:

$$y(x, 0) = \sin\left(\frac{\pi\, x}{L}\right) + 4\,\sin\left(\frac{2\,\pi\, x}{L}\right) + 9\,\sin\left(\frac{3\,\pi\, x}{L}\right)$$

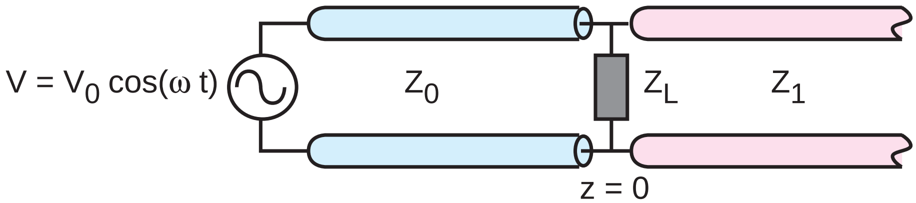
a. (3 pts) What is the phase velocity on the string? Express your answer only in terms of k, μ, α, and T.

b. (4 pts) What is the group velocity on the string? Express your answer only in terms of k, μ, α, and T.

c. (5 pts) The string is released with zero velocity at $t = 0$. What is the displacement of the string at time t?

d. (3 pts) In the absence of any damping or any other form of loss of energy, at what time t will the string for the first time have exactly the same shape as it did at time $t = 0$? Or will this never happen? Give your reasons.

Problem 3 (15 pts): Transmission line



Two transmission lines of characteristic impedance $Z_0 = 50\,\Omega$ and $Z_1 = 100\,\Omega$ are connected by a load impedance Z_L. A power supply drives the transmission line with characteristic impedance Z_0 harmonically at frequency ω. The resulting voltage wave propagates along the z-axis and is incident on the junction with the load impedance and the other transmission line of characteristic impedance Z_1. Take the junction to be at the position $z = 0$, with increasing z to the right of the junction. Consider the transmitted and reflected voltages and currents for the following different situations. Assume that there is no reflection from the far end of the second transmission line (with characteristic impedance Z_1).

a. (2 pts) Write down an expression for the voltage wave approaching the junction from the left $V_i(z < 0)$ in terms of its initial amplitude, V_0, ω, $k = \omega/v$ and z. Here v is the velocity of the wave.

b. (3 pts) What are the voltage amplitudes (relative to V_i) of the reflected and transmitted waves for $Z_L = 0$.

In what follows (in all 3 questions), $Z_L = 100\ \Omega$.

c. (2 pts) What is the net impedance which terminates the first transmission line at $z = 0$?

d. (4 pts) What are the amplitudes of the reflected and transmitted voltages?

e. (4 pts) If the maximum current in the upper wire for $z < 0$ is 10 A, what then is the maximum current in
(i) the lower wire for $z < 0$
(ii) the load
(iii) the upper wire for $z > 0$
(iv) the lower wire for $z > 0$?

Problem 4 (5 pts): Design your own pinhole camera

You are being asked to design a pinhole camera. Your box is a cube of 70 cm on the side. You drill a small circular hole in one side, and use the opposite inside wall as the screen where the photographic film is placed. You have to optimize the resolution of the camera.

Derive the approximate diameter (in mm) of the hole that will give you the best resolution at a wavelength of 500 nm.

Problem 5 (8 pts): Reflection of light

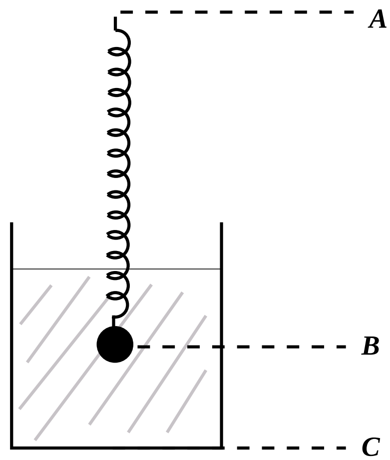
A beam of unpolarized light of 500 nm in air is incident on a plate of glass. The angle of incidence is 40^o (this is the angle between the direction of the incoming light and the normal to the glass). The index of refraction of the glass is 1.5.

a. (5 pts) Which fraction of the incoming 10 kW is reflected off the front face of the glass.

b. (3 pts) What is the degree of linear polarization of the reflected light?

Problem 6 (15 pts): Oscillator in viscous medium

A mass m is held by a spring with spring constant k. The mass is immersed in a cup of water (see the figure). The water exerts a viscous force $-b\vec{v}$ on the mass; $\vec{v}$ is the velocity of the mass **relative to the liquid**, and b is a positive constant. A indicates the position of the suspension point of the spring, B the equilibrium position of the mass m, and C the position of the bottom of the cup.



a. (2 pts) The mass is displaced vertically from its equilibrium (B); it is then released. Find the differential equation of vertical motion of the mass m. The cup is at rest.

The cup is now moved up and down at an angular frequency ω. The position of the bottom of the cup (C) is given by $d_1(t) = D_1 \ cos(\omega t)$.

b. (3 pts) Find the differential equation for the position $x(t)$ of the mass m. Give your answer in terms of m, k, b, D_1, and ω. **Remember that $\vec{v}$ in the viscous force is the velocity relative to the liquid.**

c. (3 pts) What is the *steady state* amplitude of the mass m? Give your answer in terms of m, k, b, D_1, and ω.

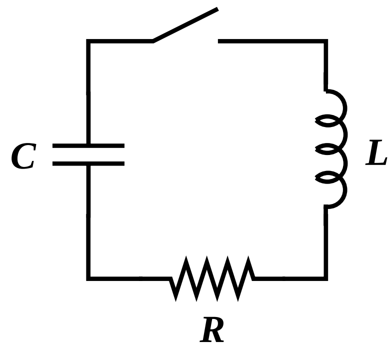
In addition to driving the cup, we now also drive the mass by moving the suspension end (A) of the spring up and down with the same frequency ω. The position of the suspension end is given by $d_2(t) = D_2 \ cos(\omega t + \phi)$.

d. (3 pts) Write down the differential equation for $x(t)$ in the case that both the cup and the spring are driven.

e. (4 pts) Find D_2 and ϕ, for which the *steady state solution* is $x(t) = 0$ at all times when both the cup and the spring are driven.

Problem 7 (12 pts): Discharging a capacitor

A capacitor of capacitance C in an LRC circuit (see the figure) is initially charged and the switch is open; the charge on the plates of the capacitor is q_o. At time $t = 0$ the switch is closed and the capacitor is discharged.



a. (3 pts) Write down the differential equation for the charge $q(t)$ during the discharge.

b. (2 pts) What are the initial conditions for the discharge?

c. (2 pts) What should the value of the resistor R be to obtain critical damping?

d. (3 pts) Write down the analytic expression of the charge $q(t)$ in terms of q_o, L, and R for the case of critical damping.

e. (2 pts) Make a sketch of $q(t)$ for the case of critical damping. Mark your time axis in units of T, where T is the period of undamped oscillations (i.e., in the case that $R = 0$).

Problem 8 (13 pts): Interferometric Radio Telescope

An interferometric radio telescope is comprised of 10 telescopes separated from one another by 800 m. The telescopes are aligned in the direction East-West. Each radio dish has a diameter of 25 m. The interferometer is operating at a wavelength of 6 cm. The telescopes are all pointing towards the South at the same elevation in the sky. A bright radio source is in the South and it is moving from East to West (thus horizontally) in the sky due to the Earth rotation. During the observations, none of the telescopes move, they all remain pointed towards the same direction in the sky.

a. (4 pts) Imagine first that all 10 telescopes are operating like radio transmitters; they emit spherical waves ($\lambda = 6$ cm), and that all these ten sources (telescopes) of radiation are in phase. Sketch the intensity pattern of this array of telescopes as a function of θ to a fictitious observer who is light years away. Since we are dealing here with very small angles, $sin\theta \approx \theta$.

b. (3 pts) We now reverse the scenario. The radio source is emitting EM radiation, and the interferometer is receiving. The radio interferometer detects the source loud and clear as the source is at the peak of the zero order maximum. How many radians will the source have to move in the sky for the interferometer to detect the radio source at the peak of the first order maximum?
Hint: This interferometer behaves just like a multiple source array. Instead of EM waves being emitted by each of the telescopes, they are receiving EM waves. The E fields, as received by the individual telescopes, are vectorially added by the interferometer. When all E-fields are in phase, a maximum signal is received.

c. (4 pts) What is the approximate width (in radians) of the zero order maximum and what is the width of the second order maximum? (The narrower the width, the higher is the angular resolution of the array.)

d. (2 pts) If the diameter of the radio dishes were 100 m (instead of 25 m), would that increase the angular resolution of the array, and if so, by how much? Give your reasons.