PROFESSOR: I will start today calculating for you the normal mode frequencies of a double pendulum. And then, I will drive that double pendulum. And then, we'll see very dramatic things that are changing.

So let us start here with the double pendulum. This is the equilibrium position if it hangs straight. Length is l, mass is m. This angle will be theta 1. And this angle, theta 2. I call this position x1, and I call this position x2.

I'm going to introduce a shorthand notation that omega 0 squared is g/l. I want to remind you that the sine of theta 1 is x1 divided by l and that the sine of theta 2 is going to be x2 minus x1 divided by l.

For small angle approximation, we do know the tensions. Here, we have a tension which I will call T2. That is here, of course, mg. So it has also a tension T2 here. Action equals minus reaction. And then, there is here the tangent, T1. And here is mg. Those are the only forces on these objects.

At small angles, I do know that T1 must be approximately 2mg because it's carrying both loads, so to speak. And we know that T2 is very close to mg. And we're going to use that in our approximation.

So let me first start with the bottom one that has only two forces on it, so that may be the easiest, m2x2 double dot. And so the only force that is driving it back to equilibrium in the small angle approximation is, then, the horizontal component of the T2. So that equals minus T2 times the sine of theta 2.

And for that, I can write minus mg divided by l times theta 2 x2 minus x1. We divide out m. And we use our shorthand notation, so we get that x2 double dot. And now we have here minus mg over lx2 So that comes in. That becomes a plus. So I got plus omega 0 squared times x2. I have here plus x1. When that comes in, that becomes a minus. So I get minus omega 0 squared x1. And that equals 0.

So this is my first differential equation for the second object. So now, I'm going to my first object. So now, we get mx1 double dot. And now, there is one force that is driving it back to equilibrium. That is the horizontal component of T1. But the horizontal component of T2 is driving it away from equilibrium in the drawing that I have made here.

So you get minus T1 times the sine of theta 1 plus T2 times the sine of theta 2. And so I'm going to substitute in there now the sines. And I'm going to substitute in there T2 equals 2mg and T2 equals mg. So this becomes equal to minus 2mg times the sine of theta 1, which is x1 divided by l. And then, I get plus T2, which is mg times the sine of x2 minus x1 divided by l.

And I'm going to divide by m. And I'm going to use my shorthand notation. And when I do that, I will come out here. x1 double dot is this one. Then, the m goes. g/l becomes omega 0 squared. But now, look closely. There is here a 2 times x1 and here there is a 1 times x1. And both have a minus sign.

So when they come out, I get 3 omega 0 squared. So I get plus 3 omega 0 squared times x1. And then, I have to bring x2 out. That becomes minus omega 0 squared times x2. And that is a 0.

And so we have to solve now this differential equation coupled with x1 and x2 together with this one. My solution has to satisfy both. I want to go over that one to make sure that it is correct. It probably pays off because one sign wrong and you hang. The whole thing falls apart. You're dead in the water. So it pays off to think about it again.

So we have x2 double dot. I can live with that, plus omega 0 squared x2. That has the right smell for me. Minus omega 0 squared x1. I can live with that differential equation.

And I go to this one. Of course, the 3 is well known in a system like this that you get the 3. I have an x1 double dot plus 3 omega 0 squared x1. I think we are in good shape.

And so now, we're going to put in our trial solutions, x1 is C1 times cosine omega t and x2 is C2 times cosine omega t. And we're going to search for the frequencies. So this omega, we have to solve for this omega. This is not a given.

Since we're looking for normal mode solutions, these two omegas must be the same-- can't write down omega 1 and omega 2. And I don't have to worry about any phase angles because since we have no damping, either they're in phase or they're 180 degrees out of phase. And 180 degrees out of phase simply means a minus sign.

So that's the great thing about this. The signs will take care, then, of the possible phase angles. So now, we're going to substitute this solution in these two differential equations.

And I'm going to write it down in the form that I put the C1 to the left and the C2 to the right. And so I'm first going through this one that is my object number one. I take the second derivative. So I get C1 times minus omega squared because the second derivative here gives me a minus omega squared.

I ditch the cosine omega t terms because I'm going to have a cosine omega t everywhere. So I'm not going to write down the cosine omega t. I have here plus 3 omega 0 squared times C1. I have here minus omega 0 squared times C2, and that is 0. Notice I put the C1's here and the C2 there.

I'm going to the other equation. I'm going to put this C1 on the left. So we get minus omega 0 squared times C1. That's the term you have here. And then, we have plus C2 times omega 0 squared minus omega squared. And this minus omega squared comes from the second derivative of this one, second derivative of this one, minus omega squared.

You get omega squared comes out. And then, the cosine goes to a sine. That gives you the minus sign. So now, we have here two equations with three unknowns. That is typical for normal mode solutions of a system with two oscillations.

We don't know C1. We don't know C2. And we do not know omega. And so we remember from last time that you can always solve for the ratio C1/C2, and you can solve for omega. You can only find C1 if you also know the initial conditions, which I have not given. Instead of solving it the high school way that I did last week, the simple way, the fast way, I'm going to do it now using Cramer's Rule.

And the reason why I want to do this once, even though in this case it really is not necessary, when you have three coupled oscillators or four, there's just no way that the high school method will do it for you. And you have to have a more general approach. I've sent this to you by email, all of you. And I also assume that you have worked on this a little bit since I requested that you would prepare for that.

So I'm first going to write down what D is, which is the determinant of these columns that you see. There are a, b, and c. Of course, we only have here two columns, the C1's and we have the C2's. So let me write down here what D is.

So I'm going to get 3 omega 0 squared. That is the C1. Then, I get here minus omega squared. That's the C2.

And then, I get here minus omega 0 squared. And here I get omega 0 squared minus omega squared. And the determinant of this, that's D. And I presume you all know how to get the determinant of this very simple matrix. We'll do that together, of course.

So following Cramer's Rule, then, my C1 which is x there, that's the one I want to solve for. So my C1, I now have to take this 0-- this is also a 0, by the way. I forgot to mention that this is a 0. This equation is a 0.

So this column now, 0, 0, has to come first. So I get here 0, 0. And then, the second column is the same as it was here, minus omega 0 squared. And then I get omega 0 squared minus omega squared. And that's divided by D. So we now know that that is C1.

And then, we go to C2. So now the 0, 0 column shifts towards the right, goes here. And the first column is unchanged. So we have here 3 omega 0 squared minus omega squared minus omega 0 squared. 0, 0, divided by D.

You've got to admit that the upstairs here, the determinant of this matrix, is 0 because you have two 0's here. And it's also 0 for this one. But clearly, zero solutions for C1 and C2 are meaningless. They're not incorrect because you will get, in this differential equation, that 0 is 0, which is rather obvious. So we don't want the solutions that C1 and C2 are 0.

And the only way that we can avoid that is to demand that D becomes 0. Because now, you get 0 divided by 0. And that's a whole different story. That's not necessarily 0. And that, then, is the idea behind getting the solutions to the searched for normal mode frequencies.

So we must demand in this case that this becomes 0. Otherwise, you get trivial solutions which are of no interest. So when we make D equal 0, we do get that the determinant of that matrix becomes three omega 0 squared minus omega squared times omega 0 squared minus omega squared, and then minus the product of that becomes a plus-- that becomes a minus, not a plus, becomes a minus. Because minus minus is already plus, so I get minus omega 0 to the 4 equals 0.

And this equation is an equation in omega to the 4. You can solve that. You can solve that for omega squared. I will leave you with that solution. This is utterly trivial. And out comes, then, that omega minus squared, which is the lowest frequency of the two, is 2 minus the square root of 2 times omega zero squared. And omega plus squared is 2 plus the square root of 2 times omega 0 squared.

So this step for you will take you no more than maybe half a minute. But be careful, because you can easily slip up of course. So we now have a solution that omega minus is approximately, if I calculate the 2 minus square root of 2, it's approximately 0.77, 0.76 omega 0. Not intuitive at all, so it's lower than the resonance frequency of a single pendulum. And then, I can substitute this value for omega either back into my equations, or I substitute it back in this if you want to.

You get 0 divided by 0, which is not going to be 0. And you're going to find, then, that C2 divided by C1 in that minus mode, in that lowest possible mode, you will find that it is 1 plus the square root of 2, which is approximately plus 2.4.

And the omega plus solution gives you a frequency which is about 0.85 omega 0. That solution, you can put back into your differential equations here. Or not the differential equations, I mean into this one. Or you put it back into this if you want to. And you will find now that C2/C1 for that plus mode-- I call this the plus mode, that's my shorthand notation-- is going to be minus 1 divided by 1 plus the square root of 2.

So it's going to be minus 1 divided by 2.4. So it's going to be roughly minus 0.41. So those are, then, the formal solutions for the normal modes. And if I gave you the initial conditions, then of course you could calculate C1. And then, you know everything because you know the ratio C2/C1. But without the initial conditions, you cannot do that.

Each of these normal modes solutions satisfies both differential equations. So therefore, a linear superposition of the two normal mode solutions is a general solution. So when you start that system off at time t equals 0, you specify x1. You specify the velocity of object one. You specify x2, and you specify the velocity of that object. Then, it's going to oscillate in the superposition, the linear superposition of two normal modes solutions.

One you see here has omega minus. And this will be the ratio of the amplitudes. And this is the omega plus. And that'll be the ratio of the amplitudes. And that is non-negotiable. That's what the system will do.

Now, we're going to make a dramatic change. Now, we're going to drive this system. So now, this is the equilibrium position now of the whole system. And I am going to drive it back and forth, holding it in my hand like this. And I'm going to shake it. I call the position of my hand eta equals eta 0 times the cosine omega t.

So eta 0 is the amplitude of my motion of my hand. That is what I'm going to do. And so when I look now at this equation, at this figure, not the equation but at the figure, I call this now x1. I call this now x2 because you should always call x1 and x2 the distance from equilibrium. And this is now equilibrium.

If I don't shake at all, I have the pendulum here. And so it's hanging straight down. So this is now equilibrium. And this location here at a random moment in time is now eta.

What has changed? Well, at first sight, very little has changed. The only thing that has changed now is that the sine of theta 1 is now x1 minus eta divided by l. So I have here a minus eta. That looks rather innocent. But the consequences will be unbelievable.

So I can leave everything on the blackboard the way I have it. All I have to do is to put instead of the sine theta x1 divided by l, I have to put in x1 minus eta divided by l. See, nature was very kind to me. It already left some space there. You see that? Nature anticipated that I was going to do that.

So you get a minus eta there. Well, if now you divide m out and you're going to use the shorthand notation, then leaving eta on the right side, which is nice to do, you get minus minus eta. So this 0 now becomes 2 omega 0 squared times eta 0 times the cosine of omega t. It becomes 2 omega 0 squared times eta. You see?

So it is no longer a 0. And so when we now substitute in there these trial functions, they now have a completely different meaning. Omega is no longer negotiable. Omega, as set by me, is a driver. So we're not going to solve for omega. Omega is a given.

That means, if omega is a given, that we're going to end up not with two equations with three unknowns-- C1, C2, and omega-- but we're going to end up with two equations with two unknowns, only C1 and C2 because omega is a given.

And so in the steady state solution, you get a number for C1. And you get a number for C2 in terms of eta 0, of course. You will see how that works.

So when we put in these functions now, these omegas are fixed, are no longer something that we search for. They're my omegas. I can make them 0. I can make them infinite. I can make them anything I want to. And that's what we want to study.

So if we want to go back now to these two equations, what changes here since we have put in this trial function, the only thing that disappears is this cosine omega t. So we end up here with 2 omega 0 squared times eta 0. And nothing else changes. But now, we're looking now at our solution for C1. And we're going to look at our solution for C2 in steady state.

It's easy, right? We apply Cramer's Rule. And the only thing that changes is this one, which has to be replaced by this, 2 omega 0 squared times eta 0. Eta 0, remember, is the amplitude of my hand. Eta is the displacement of my hand at any moment in time. Eta 0 is the amplitude.

And so here, we get then also 2 omega 0 squared times eta 0. And so now, we can solve for C1 and C2. We get an answer-- not just a ratio only, we get an answer. We know exactly what C1 is going to be. And we know exactly what C2 is going to be because omega is non-negotiable. Omega is now a known. When we solved this, we were searching for omega. And out came these omegas. That's not the case anymore. We know omega. It's called omega, and that's it.

So I can write down now C1. So I take the determinant there of the upstairs. So that gives me 2 omega 0 squared times eta 0 times omega 0 squared minus omega squared. That is this diagonal. And this one is 0.

And I have to divide that by D. Now, I could write D in that form. And I could do that. There's nothing wrong with that. But I can write it in a form which is a little bit more transparent.

We do know that omega minus and omega plus will make D 0. So you can write this then in the following way. You can write here omega squared minus omega minus squared times omega squared minus omega plus squared. That must be the same as what I have there because you see that this one becomes omega minus, then this goes to 0. And when this one becomes omega plus, that is also 0. So it's a different way of writing. Gives you a little bit more insight.

It reminds you that the downstairs will go to 0 when you hit those resonance frequencies. And so I will also write down C2, then. This one is 0. I get minus this one. So I get plus 2 omega 0 to the 4 times eta 0 divided by that same D. You can write this for it.

So let me check that, see whether I'm happy with that. 2 omega 0 squared, eta 0, omega 0 squared, right? Is omega squared? That looks good. This looks good. This looks good. And I have here-- I'm happy.

Our task now is not to look at these equations but to see through them. And they are by no means trivial, what they're going to do as a function of omega. It's an extraordinarily complicated dependence on omega.

And I have plotted for you these values of C1 and C2 for which you get an answer now. You also know C1/C2 of course because you know C1 and you know C2. And I'm going to show you this as a function of omega. And then, we will try to digest that together.

And this plot, like the other plots that I will show you later today, will be on the 803 website. They will be part of lecture notes. So you have to click on Lecture Notes. And then, you will see these plots. This is the first one, which is the double pendulum.

What is plotted here horizontally is omega divided by omega 0. So that's the omega 0 that we mentioned there. And so you see the first resonance here is about at 0.76 omega 0. And the second resonance here is at that 1.85 omega 0.

And what we plot here is the amplitude, C, divided by eta 0. Because obviously if you make eta 0 larger, you expect that has an effect on the C's, of course. If you drive it to the larger amplitude, of course the object will also respond accordingly. And so therefore, we have it as a function of eta 0. You see the eta 0 here? C1 is linearly proportional with eta 0. C2 is linearly proportional to eta 0. That's no surprise, of course. So we divided by eta 0.

When we plot it upstairs, it means that the amplitude is in phase with the driver. That's the way we have written it. And if it is below the zero line, it means that the amplitude of the object is out of phase with the driver. That's all it means. That's the sign convention.

So let us now look at this and try to digest it and use, to some degree, our intuition and see whether that agrees with our intuition. And let us start when omega goes to 0. And don't look now at the solutions. If omega is 0, I have a double pendulum in my hand. And I'm going to move it to the left. And 25 years from now, I'm going to go back. And I move it to the right again.

So the pendulum is always straight. And it is clear that C1 and C2 to both must be eta 0. And you better believe it that if you substitute in there omega 0, that's what you will find.

So C1 must be C2 and must be eta 0. And it must even be plus eta 0. It must be in phase with the driver. And you see that the ratio-- not the ratio, which is C divided by eta 0. That is a ratio that is plotted at plus 1. And so both are in phase with the driver.

So that's a trivial result. It means that the pendulum which is here, 25 years from now is here. That's omega 0, almost 0. So notice that we now know the ratio C2/C1, which is plus 1. The ratio C1/C2 or C2/C1 are now entirely dictated by this. Nothing to do with normal modes anymore, so don't be surprised that C1/C2 is now plus 1.

Now, look what happens. I'm going to increase my omega. And what you see is that the red curve, which is the second object, the lowest one of the two, is going to have a larger amplitude than the top one. You see, it already begins to grow very rapidly.

And when you reach resonance, the ratio is going to be plus 2.4, of course. That's obvious. Now at resonance, you get an infinite amplitude for each. That, of course, is nonsense. It has physically no meaning. So you should think of it as going to infinity. For one thing, if C1 became anything, that means if this point here ends up there in the hole, it's hard to argue that it's a small angle, right?

So in any case, the solution wouldn't even hold apart from the fact, of course, it has always some damping. And so we never go completely off scale, so to speak.

However, you will see that when you drive the system, the double pendulum, and you approach resonance, that you will very quickly see that the ratio of the amplitude of the second one over the top one will grow, will become 1 and 1/2, will become 2. And then, in the limiting case, you will hit plus 2.4.

So as omega goes up, you're going to see that C2/C1 is going to be larger than one, right there, this one. And then ultimately, it will reach that 2.4. But that is that extreme case of resonance.

And when you look at the motion of the pendulum, if you drive it somewhere here, notice that they are both in phase with the driver. And C2 is larger than C1. So what you will see is this is C1. And then, this is C2. And so C2 is larger than C1. And they are in phase with the driver. And so when they return, you will see the pendulum like this. So that's the sweeping that you will see.

There's something truly bizarre. I go a little higher in frequency. I go over the resonance. And I see here a point whereby the frequency happens to be exactly the frequency of a single pendulum-- omega divided by omega 0 is 1-- and the top one refuses to move. But the bottom one does.

The bottom one here has an amplitude which is twice the amplitude of the driver. You look, you see a 2 here. And it is out of phase with the driver. That is unimaginable. It's unimaginable what you're going to see.

So when omega is omega 0, C1 becomes 0. C2 becomes a minus 2 eta 0. And so this pendulum is going to look, then, as follows. I'll make a drawing here. I have a little bit more space.

So if my hand is here at eta 0, then number 1 will stand still, won't do anything. But number two will have an amplitude which is twice eta 0. But it's on the other side. So this is 2 eta 0.

And then, if you look half a period later, it will look like this. So this is eta 0. And then, this is 2 eta 0. And this one doesn't move. That's what it tells you.

How on Earth can the lower pendulum oscillate if the upper one stands still? I want you to think about that. I'm still having sleepless nights about it. And so maybe you will have some tool. And if you have some clever ideas, come to see me.

But the logical consequence of what we did is that this one would stand still. And the other one will still oscillate and be driven by my hand. I have to keep moving this. Otherwise, this will go to pieces. I must be doing this all the time at that frequency, which is exactly the resonance frequency of a single pendulum. It is that omega 0. I must keep doing this. And this one does nothing. And this one has double the swing of this and is out of phase.

If, then, you go even higher, then you will see that the two objects will go out of phase. The upper one will be in phase with the driver. And the lower one will be out of phase with the driver. And there, you hit that second resonance when things will get out of hand. And you get that ratio, minus 0.42, back of course.

I will want to demonstrate to you this situation here and this situation to see whether they make sense. And so now, I'm going to use a double pendulum and drive it with my own frequency, which I determine. I'm the boss. I determine omega, not looking for normal mode solutions. I determine omega.

And I'm going to first drive it with omega 0 with omega is about 0. In other words, I'm going to drive it here. And what you're going to see is, of course, fantastic, absolutely fantastic. It's hanging straight down now. And I'm moving it over a distance of one foot. So eta 0 is one foot. And C1 is one foot, and C2 is one foot. And they're in phase with the driver. Physics works.

If I go a little higher up here, so I go somewhere here approaching resonance, then you will see that C2 becomes larger than C1. This is C2 and this is C1. And you get to see a picture which is very much like this. And I'll try that.

So I'll drive it below resonance but not too far below. And there you see it. You see? C1 is smaller than C2. No longer 1 plus 1. If this was 1, your C1/C2 is 1. That's no longer the case now. You really see that C2 is getting ahead-- not in terms of phase ahead, but in terms of amplitude, very clear.

Now, I'm going to attempt to do the impossible. And the impossible is to try to hit this point, the point whereby the upper one stands still and whereby the lower one will have an amplitude which is twice that of my hand but out of phase with my hand. How on Earth can I ever drive this system with that frequency, omega 0, which is the frequency of a single pendulum?

Well, maybe I can't. But I will try. And the way that I'm going to try this is the following. I know what the resonance frequency of a single pendulum is. That's this. I can feel it in my hands. I can feel it in my stomach. I can feel it in my brain. I feel it all over my body.

I can burn this frequency into my chips here. And then, I can close my eyes while you are looking and generate that frequency which was burned here and drive the system as a double pendulum. If I succeed, you will see the upper one stand still and the bottom one will have twice the amplitude of my hands. So the success of this depends exclusively on how accurately I can burn this frequency into my chips.

So you have to be quiet. So I'm going to count. One, two, three-- I'm burning now-- one, two, three, four, one, two, three, four. I'm closing my eyes. One, two, three, four. One, two, three, four. One, two, three, four. One, two, three, four. One-- you're not saying anything! Didn't you see this one standing still? Did you see it? Did you also see that the other one had twice the amplitude of my hands and out of phase with my hands? You didn't see that, right? Admit it. You didn't see it because you were not looking for it. Especially for you, I'll do it again.

So you really have to see number one that this one will practically standstill. Number two, that this one has double the amplitude of my hands but out of phase. The decay time of burning is only one minute. So I have to burn in again. One, two, three, four, five. One, two, three, four, five. One-- uh-oh. These things happen. You have to start all over with the burning.

There we go. One, two, three, four, five. One, two, three, four, five. One, two, three, four, five. One, two, three--- are you seeing it?

AUDIENCE: Yes!

PROFESSOR: All right.

[APPLAUSE]

PROFESSOR: This is an ideal moment for the break. I'm going to hand out the mini-quiz. And we will reconvene. I'll give you six minutes this time, so you can even stretch your legs. So I would like some help handing this out. And then, you bring it back. And I'll put some boxes out there.

So if you can help me handing it out here, you can start right away. Hand this out here. For those of you who have no seats, come forward and get some seats. Nicole, we are still friends, right? So why don't you hand that out. And why don't you hand this out here? You can also give it to people here, here, if you do that. You can start right away.

I'm now going to do something perhaps even more ambitious. And I'm going to now couple three oscillators-- not pendulums yet, but I'm going to couple three oscillators which I connect with four springs. I'm going to work on this. Three masses, equal masses, four springs, spring constant, k. And the spring constants are the same.

And I'm going to drive that system one, two, three, four. And this is the end. In other words, I have here a spring. And here's the first mass, second mass, third mass. And here, it is fixed. And I'm going to drive it here with a displacement, eta, which is eta 0 times cosine omega t.

So at a random moment in time, this is where my hands will be. So this is eta. This is where the first mass will be. Remember, you always call the displacement x1 from its equilibrium, that is from its dotted line.

So here's the spring. This one is here. So I call this x2. So here is the spring. And this one is here. So this is x3. So here is the spring, and here is a spring. You may have noticed more than once now that I have a certain discipline that I always offset them in the same direction. Do you have to do that? No.

If you don't do it, your chance of a mistake on a sign slip is much larger than if you always set them off in the same direction. You'll see shortly why. So that is certainly something that is not a must, but it's a smart thing to do. I define this as my positive direction, but that, of course, is a complete free choice.

Now, let at this situation at this moment in time, let x1 be larger than eta. Let x2 be larger than x1. And let x3 be larger than x2. And this assumption will have no consequences for what follows, at least for the differential equations.

If x1 is larger than eta, that first spring is longer than it wants to be because I've assumed that x1 is larger than eta. And so that means there will be a force in this direction because this spring is longer than it wants to be.

If x2 is larger than x1, this spring is also longer than it wants to be. So it will contract. So there's a force in this direction. And so I can write down now the differential equation for my first object.

So that's going to be nx1 double dot, that equals minus k times x1 minus eta because that's the amount by which it is longer than it wants to be. So times x1 minus eta, that is this force. And this force is now in the plus direction, is plus k times-- this spring here is longer than it wants to be by an amount x2 minus x1. Not omega 1, but x1.

That's my differential equation for the first object. And this one is always correct, even if x1 is not larger than eta because if x1 is not larger than eta, then this force flips over. Well, this will also flip over. So that's why it's always kosher and advisable to make that assumption to start with because, again, it reduces the probability of making mistakes. That's all. There's nothing else to it, just reduce the chance of slipping up.

So let's now go to this object. If this spring is longer than it wants to be, it wants to contract. So this object will see a force to contract. But if this spring is longer than it wants to be because x3 is larger than x2, it will experience a force to the right. So I can write down now the differential equation for object number two.

mx2 double dot, notice that the one that is here to the left is the same one that is here to the right, right? Action equals minus reaction. This pull is the same as this pull. So it is going to be this term which now has a minus sign. And you always see that in coupled oscillators that what was a plus here is going to come out here as a minus sign. You see, that comes out nicely because this spring is longer than it wants to be by an amount x2 minus x1. And the force is in the minus direction.

And this one is now going to be plus k times x3 minus x2. So now, I go to the next spring, to the next object. So this object here will experience a force to the left because this thing is longer than it wants to be. So it wants to contract. But this one is pushing. So therefore, the force due to this spring is now also in this direction because the end is fixed.

And so we get for the third object mx3 double dot equals minus k times x3 minus x2, which is this term, but it switches signs. And then in addition, I get minus k times x3.

When you reach this point on an exam, you pause, take a deep breath, and you go over every term and every sign. If you slip up on one sign, one casual mistake that you just even though you know it you casually write here for instance a 1 instead of a 3, it's all over. You're dead in the water. The problem will fall apart. And it may not even oscillate in a simple harmonic way.

So therefore, let's look at it. mx1 double dot-- x1 is larger than eta. Therefore, force is in this direction. I love it. The other one is in this direction, perfect, x2 minus x1. That same force here is going to pull on the second one. So if this is correct, this is also correct. This one is driving it away from equilibrium, x3 minus x2, got to be right. This term shows up here with a minus sign, can't go wrong there. And since this spring is always shorter here, if it's pushed to the right, I am happy with my differential equations.

So now, you're going to substitute in here x1 is C1 cosine omega t, trial functions. X2 is C2 cosine of omega t. And x3 is C3 cosine omega t. Are we looking for omegas? Oh, no, oh, no. Omega is given by me. I am telling you what omega is. You're not going to negotiate that with me.

We are only solving for C1, C2, and C3. And in the steady state, you will be able to do that because omega is nonnegotiable. You're going to get three equations with three unknowns--- C1, C2, C3. You don't have to settle to only calculate the ratios of the amplitude. No, you're going to get a real answer for C1, for C2, and for C3 which, of course, will depend on eta 0-- sure, if you know eta 0.

Do we worry about phase angles here? No, because there's no damping. And if there's no damping, either the objects are in phase or they're out of phase because it is the damping that gives these phase angles in between. And 180 degrees out of phase is a minus sign. So we have the power to introduce 180 phase changes and 0 phase. For that, we have plus and minus signs.

Now, you are going to do some grinding. I did all the grinding on every detail of the double pendulum. Now, you're going to do the grinding. However, I want to make sure that if you go through that effort to make the grinding that you indeed end up with the right solution. So in that sense, I'm going to help you a little bit by giving you the D, which is the D that we have here.

But you have to bring me to the D. And so the D is going to be-- so we have to divide by m. You have to also-- omega s squared, k/m, shorthand notation. Some of you may want to call it omega 0. That's fine because there's only one-- only springs, there are no pendulums and springs. But I still call it an s to remind you that it is the resonance frequency or a single spring.

Then, my D becomes minus omega squared. That's always the result of that second derivative, remember? You always get that minus omega squared out. Then you get plus 2 omega s squared. Then, you get in the second column minus omega s squared. And in the third column, you get a 0.

No surprise that you get a 0 in the third column because the first differential equation has no connection with x3 at all. And so you never see anything in the third column that will be a 0. But if you look at the second differential equation, that has an x1, an x2, and an x3 in it. So now, you don't see 0's. So what you're going to see is minus omega s squared. That's going to be the C1 term.

And then, you get here minus omega squared. I think it is plus 2 omega squared. And your last column is going to be minus omega s squared.

Now, the third differentiable equation, there's no x1. Therefore, that is a 0 here. And then, you get minus omega s squared. And then here, you get minus omega squared plus 2 omega s squared. And you have to take the determinant of this matrix. That is D.

Let me check it to make sure that I didn't slip up with a minus sign so when you get home that you don't wonder, why didn't you get that result? And I think that looks good to me. Remember, all those omega squareds always come from those second derivatives because you have to take the second derivative of cosine omega t. That always brings out the minus omega squared. So no surprise that this is a minus, that this is a minus, and that that is a minus.

So now, we want to know what C1 is. And the first column will reflect this eta because the right side now is not going to be 0, remember, like the double pendulum? So you're going to get here in the first column, you're going to get omega s squared times eta 0. And there, you're going to get a 0 and a 0.

And then, this column comes here. And this column comes here. That is the determinant of the upstairs. And you divide it by D.

That, then, is C1. And of course, I will only go one step further to go to C2. But that becomes a little boring now. C2, then, you get that's this one. And then, the second column according to Cramer's Rule is going to be omega s squared, eta 0, 0, 0. And then, the third one is going to be this. And then, you go on, and we divide it by D, of course. And then, you can write down C3. You're on your own. I'll help you.

So when you do this, you could, if you wanted to, first solve for what we call the resonance frequencies. The resonance frequencies are the ones which are the normal mode frequencies. That's a resonance. And so you may want to put in D equals 0. So you make that determinant equal 0, which gives you, then, the three resonance frequencies, which earlier we would have called normal modes in case we are not driving.

And so for those of you who worked this out, the lowest frequency, resonance frequency which was a normal mode, is 2 minus the square root of 2 times omega s squared. The one that follows is 2 omega 0 squared. And the one that follows then, which I will call omega plus plus, is going to be 2 plus the square root of 2 times omega s squared. None of these are, of course, intuitive. But none of the resonance frequencies of our coupled double pendulum were intuitive.

There's no way that you could even look at this and say oh yeah, of course. Excuse me? What did I do wrong?

AUDIENCE: Omega 0, or--

PROFESSOR: Yes, thank you very much. I have an omega 0 there. You deserve extra credit. I've called that omega s. If you want to change all the omega s's and omega 0's, that's fine. But you cannot have one omega 0 and the other omega s. Yes, so this is the square of the frequency or a single spring oscillating mass m. Thank you very much for pointing that out.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Ah, boy, you also deserve extra credit. I really set you up with this, didn't I? I wanted two people to get extra credit. You deserve one. Come see me later. And who was the other one? You did one, thank you. Yeah, there's a square here. Oh, there are more people than one who claim this one. All right, thank you very much.

OK, so now for any given value of omega, for any given value of omega you find three answers. You get C1. You get C2. And you get C3 in the steady state solution.

Of course, you get infinite amplitude, which is physically meaningless. So you've got to stay away in your solutions from the infinities. But you'll see if you don't come to close to infinities that the results you get are quite accurate for high q systems. And so now I will show you the three amplitudes for this system, which will also be put on the web this afternoon. And so that is the three car system which we have set up there.

The only difference with the previous one is that we don't plot here omega divided by omega 0 but the squares. This plot was provided to me by Professor Wyslouch who lectured 803 a year ago. I was a recitation instructor at the time. And it was very kind of him that he gave me this plot. He even added this at my request, which is very nice.

Car 1 is the first car, green. And then, the car 2 is the red one, is the second car. And this is supposed to be blue. You don't see it. But in any case, if you think it's black, that's fine. That's then the black line. So horizontally, it is the ratio of the frequency squared. So you see that the second resonance is indeed at a 2 here. You notice that 2 that I have there on the blackboard.

If we plot it above the 0, it means that it is in phase with the driver. If we plot it below the 0, it means it's out of phase with the driver. If now you look at this, then already at omega 0 you see something that is by no means intuitive.

Notice that C1, C2, and C3 are all substantially lower than eta 0 because this is in units of eta 0. And they're not even the same. They're all three different. Would I have anticipated that? No, I wouldn't.

Maybe two would be the same for me, but not all three different. And that's the case. They're all three different. When we approach resonance, things go out of hand. All three are in phase. And when you just cross over the first resonance, they are all three again in phase but out of phase with the driver. That's not so surprising all by itself.

But now, look at this ridiculous point. If I drive that system with the resonance frequency of the individual spring with one mass on it-- because remember, if this squared is 1, then omega divided by omega 0 is also 1, but that's exactly the square root of k/m-- then this one will stand still. And these two have roughly the same amplitude. You could eyeball it here. It looks like it almost crosses over there. And it's about minus eta 0 because it's about minus 1.

Let me write that on the blackboard. That's a quite bizarre situation. So at that one frequency, so omega equals the square root of k/m, I get C1 equals 0. And my C2 is about C3, maybe even exactly C3. I never checked that. And that is roughly minus eta 0. You can just see that there.

So it means that this car will stand still. It's closest to the driver. These two are in phase with each other, have the same amplitude. But they're out of phase with the driver. Crazy! Got to be wrong, right? How on Earth can this one stand still, and these happily go hand in hand and go--? It can't be right. But I will demonstrate to you that it is right at least to a high degree of accuracy.

I'm going to concentrate on two points in that graph. The first point that I want to concentrate on is this one, the hardest one of the two. I'm going to drive this system at a frequency that we think is almost there. I'm going to turn it on right now because it will take three to four minutes for the transience to die out. And then in the meantime, I'll explain what you're going to see.

I'm driving it now with that frequency. So don't look at it now. It looks chaotic. It's going into a ridiculous transient mode. This one is also oscillating. It's not supposed to oscillate. Look there. It's oscillating, and it looks chaotic. The amplitudes are changing. Just let it cook.

Now, what are you going to see? If the transients die out which can really take three minutes, you will see that this is my eta 0. This is 2 eta 0. You see this arrow? That is 2 eta 0. That's the driver. So you can calibrate the amplitude eta 0.

You will see, then, that these two cars have the same displacement but out of phase. So when this one goes to the right for me, they go to the left for me and vice versa. They have an amplitude which is very closely the same. And this one is going to stand still.

What you're going to see is not something truly spectacular. But it's extremely subtle. You have never seen it before and I don't think this has ever been demonstrated in any lecture hall. It's extremely subtle, first of all, to have the patience to wait four minutes for this to happen, number one. And then, number two, to make such a daring prediction that this one will almost come to a halt and that the other will have an amplitude which is the same as the driver but 180 degrees out of phase.

So be patient. And it will pay off. Don't fall asleep now. Let's look at this middle one to see whether its amplitude is becoming constant. As long as the amplitudes are not constant, there is still transience. Well, these two are already going hand in hand. You see that? And I would say very much with the same amplitude.

And look at this. Hey, [INAUDIBLE]. They are already out of phase with each other. You see that? Now this one, oh boy. That amplitude is nowhere nearly as large as this one. And yet, when we started it, it was even larger than this one because of the transient phenomenon.

Look at this one. It's almost standing still already. Can we find that precise frequency? No. We have to set a dial somehow. We do the best we can. I think it's fantastic. It's growing my mind.

I can't believe it. Physics is working. This one is practically standing still. If I had to estimate this amplitude, I would say maybe 1/10 of eta 0. And these, you can mark, you see the marks here. This one is really eta 0. For those of you who see these marks, that is the same as this. It's really eta 0. And boy, they go hand in hand. Aren't you thrilled?

[APPLAUSE]

PROFESSOR: Now, I'm going to try to make you see this point. I'm going to start it. And then, I'll explain to you what is so special about that point. Because again, we have to be patient. And we have to wait for the transience to die out.

I'm going to drive it very close to resonance right below to resonance. You get a huge amplitude of C1 and C3. But look what C2 is going to do. C2 is going to have an amplitude which you and I may have thought is 0, namely the outer ones do this and the middle one stands still. No, no, look at 1 and 3, by the way. They go nuts already. You see? That is this. This is 1 and this is 3.

Now you see the transient phenomenon. Now, it's picking up again. So we have to be patient. But there's something very special about this C2. Why is this C2 not 0, which is what you would expect? You would expect that the outer two go like this and the middle one would just stand still.

AUDIENCE: [INAUDIBLE]

PROFESSOR: That's one answer, yes, of course. But isn't it so that this must be 0? You don't believe that this upstairs is going to be 0? Just say you don't believe it.

AUDIENCE: No.

PROFESSOR: It is 0! Why now, even though the upstairs is 0, why do we get an amplitude which is half eta 0 but with a minus sign? Because we get 0 divided by 0 because right at that frequency here, that is a resonance frequency. So D is 0. So you get an amazing coincidence that the upstairs is 0, the downstairs is 0. But that's what we have 1801 for. The ratio is not 0. The ratio is minus 1/2 eta 0. And that's what you see, it's minus 1/2 eta 0.

Now look. 1 and 3 are happy. They oscillate happily out of phase with each other. Number 1 has to be in phase with the driver. It's doing that.

Number 3, out of phase with the driver, it's doing that. And look at the middle one. The middle one, for those of you who can see, the arrows are given eta 0 to eta 0, its right half of it. Just walk up to it and you can see. It's amazing, this.

We hit that right on the nose. The reason why this is easy is look, my omega doesn't have to be precisely here. Even if it's a little bit to the left, I'm still OK. That amplitude of number 2 is not changing very much. You see that? This is so nicely horizontal. So this, for me, was a piece of cake. This was hard. Piece of cake. So you see there?

Now, I'm going to move to the triple pendulum. In other words, now you've seen the structure of my talk. We first did the double pendulum. The double pendulum was-- I worked out all the way for you. Then, we did the three cars with the four springs. I set it up for you so that you can't go wrong anymore. I gave you the final solution. And I demonstrated that indeed it is working the way we calculated.

Now, I'm going to simply show you the results of a triple pendulum. And the plots that I'm going to show you will be on the web, but no calculations at all. So that triple pendulum, or that-- let me put these down so that we have not so much shadow on this board.

Triple pendulum, this is a triple pendulum. And we're going to drive it here, eta equals eta 0 times the cosine of omega t. The top one is going to be green. The middle one is going to red. And the bottom one is going to be blue even though it may look black there, but I'll make it blue. That is the color code that we have on our plot, which will be put on the web this afternoon.

So here it comes. Horizontally, we plot again omega divided by omega 0 where omega 0 is the square root of g/l, the resonance frequency of a single pendulum. And vertically, we do the same thing. We plot C divided by eta 0. Everything has the same meaning, namely plus 1 means in phase with the driver and the same amplitude as eta 0. And minus is out of phase with the driver.

Now, if you take this pendulum and you move it with 0 frequency, you don't have to know any differential equations that it will be here. And 20 years from now, it will be there. And this separation is, then, eta 0. And so you expect that C1, C2, and C3 will all three be eta 0. And they will be plus eta 0-- not minus, but plus, in phase with the driver. And look, that is what you see there. So that is by no means a surprise.

You go closer to resonance. And you see that the bottom one, which is the blue one, black here, is picking up an amplitude which is larger than the middle one and the red one. And the red one, larger than the top one.

So you're going then into the domain where you're going to see something like this. So this one, C1, C2, C3. And then finally, when you hit resonance, I do not know what the ratios are. But everything gets out of hand anyhow.

And now, there comes a point which boggles the mind right here. The top one is not moving, stands still. And there's nothing really so special about that frequency. That frequency for which the top one stands still, so C1 goes to 0, that omega if I try to eyeball it, I would say it's about 0.75, 0.77. 0.77 times omega 0.

And somehow at that frequency, the top one will not move. The other two will move. They have an amplitude which is not stunning. It's nothing to write your mother about. But it is about eta 0. It is a minus sign, so it's out of phase with the driver. And this one is a little more. Bizarre.

Could I demonstrate this? No way on Earth can I generate 0.77 times omega 0. I can burn into my chip omega 0, as I did. But I cannot do 0.77. There's no way. So forget it. I have to disappoint you.

But now comes the good news. Look at this point here. That is a point where the middle one will stand still. And that is truly amazing. The middle one stands still. The top one has an amplitude of about 0.7 eta 0. I just eyeball that. And the bottom one has an amplitude of about minus 1.5 eta 0. And let me try to make a drawing of that.

So this now is at omega equals omega 0. So my hand is here, which is eta 0. The top one has an amplitude which is about 0.7 eta 0, this one. So it is roughly here. In phase, so this is connected. So this is roughly 0.7 eta 0.

The next one stands still, believe it or not. It's here. And then, the next one is out of phase with me and has roughly an amplitude of 1 and 1/2 times this. So that's 1 and 1/2. So I call that minus 1.5 times eta 0.

And then half a period later, my hand is here. And this object is here. And this one stands still. And this one is here.

Truly, these are almost not believable. And then, there's another frequency whereby the top one would stand still. I will attempt-- I will make the daring attempt to aim for this solution. And the reason why I can try that is because the frequency is omega 0, which I can burn into my chips. And that's the last attempt I will make today.

So this is the triple pendulum. So I have to go through the exercise of learning again the period. And then, as I close my eyes and drive it with this frequency, the idea then is that you would see the top one move in phase with my hand, the one below that will stand still. And then, the one below that will have an even larger amplitude.

So one, two, three, four, five, six, seven, one, two, three, four, five, one, two, three, four, five, one-- you see it?

AUDIENCE: Yes.

PROFESSOR: Three. You see it? Did you see that one stand still? Isn't it amazing that physics works? OK, see you Thursday.