MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.033 November 17, 2006

Problem Set 9

Due: December 8, at 4:00PM. Please deposit the problem set in the appropriate 8.033 bin, **labeled with name and recitation section number and stapled** as needed (3 points).

Reading: Chapters 2, 3, 4, 5, and Project D in the Taylor & Wheeler book - "Exploring Black Holes, Introduction to General Relativity". You will be responsible only for the corresponding material that was actually covered in the lectures. Project E should also be understandable, but this topic will be mentioned only very briefly in lecture.

Problem 1

Concept questions

1. According to the no hair theorem, which three physical quantities uniquely characterize a black hole? (2 points)
2. To explain why most astrophysicists now believe that black holes really exist, briefly give one piece of evidence for the existence of supermassive black holes and one piece of evidence for the existence of stellar-mass black holes. (**Hint:** see the black hole section in the *Science* handout.)(3 points)
3. Due to a miscalculation, your friend falls into the supermassive black hole at the center of our Galaxy. Assuming that this black hole is neither charged nor rotating, indicate whether each of the following statements is true or false.(10 points)
   (a) He will die just as he enters the event horizon.
   (b) He will die only after he has entered the event horizon.
   (c) He will be killed by tidal forces.
   (d) He will be killed by the singularity.
   (e) He will emerge unscathed from a while hole in another Universe.
   (f) You will see him disappear through the event horizon.
   (g) You will see him forever, seemingly frozen on the event horizon.
   (h) You will see him seemingly frozen on the event horizon until he redshifts out of sight.
   (i) Most physicists now believe that this black hole will last forever, since nothing can come out from it.

(j) The concepts of space and time as we know them are no longer valid inside the event horizon.

Problem 2 (9 points)

Show that the Gullstrand-Painlevé (GP) metric

$$d\tau^2 = dt_{\text{ff}}^2 - (dr + \beta_r dt_{\text{ff}})^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

and the standard Schwarzschild metric

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

are equivalent. Here

$$\beta_r \equiv \left(\frac{2M}{r}\right)^{1/2}$$

is the escape velocity.

1. first rewrite the Schwarzschild metric as

$$d\tau^2 = \gamma_r^{-2} dt^2 - \gamma_r^2 dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

where

$$\gamma_r \equiv \frac{1}{\sqrt{1-\beta_r^2}}.$$

2. Start with the relation

$$dt_{\text{ff}} = dt + \beta_r \gamma_r^2 dr$$

from the lecture notes which defines the time coordinate t_{ff} and show that

$$dr + \beta_r dt_{\text{ff}} = \beta_r dt + \gamma_r^2 dr.$$

3. Show that when you plug these two relations into the GP metric, you get the Schwarzschild metric.

Problem 3(6 points)

"Why tidal forces make you tall and slim"

For simplicity, use classical mechanics and Newtonian gravity for this problem.
Consider a closed capsule that is freely falling radially in the gravitational field outside a spherically symmetric body of mass M. At a given time, the center of the capsule is at a distance r from the center of the mass, and the capsule size is very small compared with r. Define a local Cartesian coordinate system with an origin fixed at the center of the capsule. The coordinate y is then the distance from the origin along the radial direction ($\boldsymbol{r}$), while x is the distance from the origin in a direction perpendicular to the radial direction.

Show that for $|x| \ll r$ and $|y| \ll r$, the accelerations of free test particles, relative to an observer at (0,0), are given by:

$$\boldsymbol{a} = \frac{GM}{r^3}(-x\,\hat{x} + 2y\,\hat{y})$$

Consider 4 test particles at locations $(0, +y)$, $(0, -y)$, $(x, 0)$, and $(-x, 0)$. Compute the initial motions (before the particle has moved very far) for each particle relative to $(0, 0)$, and sketch their trajectories.
If fall feet first into a Schwarzschild black hole, why would you become tall and slim before you die?

Problem 4(6 points)

"Comparison of r_{shell} and r"

Compute and plot the shell radius, r_{shell}, vs. the coordinate radius r. Follow the integration steps outlined in Taylor & Wheeler, Sample Problem 2, page 2-28. Integrate r_{shell} from $r_0 \to r_s$, where r_0 is an arbitrary starting value. Make a plot of $(r_s - r_0)$ vs. r, starting from $r = 2M$ to a sufficiently large value of r to be able to discern the asymptotic behavior. [Note: we use units where $G = 1$ and $c = 1$.]

Problem 5

"Gravitational Redhift"(3+3+3 points)

A radioactive Fe source emits a 6 keV X-ray line in its rest frame. Suppose such a source is located on the surface of a neutron star of mass $M = 2.8 \times 10^{30}$ kg, and radius 10 km. Assume that the atomic transition which gives rise to the X-ray line is not significantly physically altered by the strong gravity or the possible presence of an intense magnetic field. Ignore any rotation of the neutron star.

(a) Compute the energy of these X-rays as they would be seen by a very distant observer, O_∞, i.e., an astronomer on Earth. (Neglect the gravitational potential of the Earth itself.)

(b) Suppose that another stationary observer, O_r, is fixed at radius $r > 10$ km from the center of the neutron star. Find an expression for the energy, E_r, of the X-ray line (coming from the surface) that would be detected by such an observer.

(c) If the observer at O_r sends X-rays of energy E_r to the observer back on Earth, as a report of what he/she has seen, what energy would be detected at O_∞?

Problem 6

"Global Positioning Satellite System (*GPS*)" (9×3 points)

Start with equation (3) of Project A in Taylor & Wheeler, page A–3, which we will derive in lecture. Proceed to answer "Queries" 1 through 9.

Problem 7

"A Dilute Black Hole" (3 points)
Taylor & Wheeler, Problem 2–5, page 2–46.

Problem 8

"Orbital Periods Around Black Holes" (6 points)

Consider two black holes of mass $M_1 = 1\ M_\odot$ and $M_2 = 10^6\ M_\odot$ (where $M_\odot = 2 \times 10^{30}$ kg).
a. Find the Schwarzschild radius ($R_S \equiv 2GM/c^2$) for each object.
b. Kepler's 3rd law (for circular orbits, at least) works exactly for orbits in the Schwarzschild metric if the bookkeeper's coordinates r and t are used: $(2\pi/P)^2 = GMr^{-3}$, where P is the orbital period. Find the orbital period in seconds for a circular orbit at r just outside the Schwarzschild radius for an arbitrary mass (expressed in units of $M_\odot$). What are the corresponding orbital periods for the two black holes given in this problem?

Problem 9 (6×3 points)

"Falling into a Black Hole"

You fall radially into a black hole with $\tilde{E} = E/m = 1$, *i.e.*, starting with negligible velocity far away.

1. Write down your total aging $\Delta\tau$ using the GP metric from Problem 2 as an integral along an arbitrary trajectory $r(t_{\rm ff})$.
2. Prove that the motion given by
$$\frac{dr}{dt_{\rm ff}} = -\beta_r \qquad (1)$$
is a geodesic, *i.e.*, maximizes your aging.
Hint: a simple argument suffices — no need to use the variational calculus.
3. What is the relation between $\Delta\tau$ and the free-fall time interval $\Delta t_{\rm ff}$?
4. Integrate equation (1) above to compute the time elapsed on your wristwatch between passing a radius r and when it gets destroyed at $r \approx 0$.
5. What is the bookkeeper time interval Δt between the start of your fall and your crossing the event horizon? (No calculation needed.)

6. To get used to working with the orbital equations of motion, use them to rederive equation (1) above. Specifically, use $d\tau = dt_{\text{ff}}$ and these two equations as your starting point:

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \tilde{V}(\tilde{L}, r)^2,$$

$$\tilde{V}(\tilde{L}, r)^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right).$$

Problem 10(9 points)

"Dropping in on a Black Hole"

Exercise 7, Chapter 3, page 3–30 of Taylor & Wheeler.

Problem 11(6 points)

"Orbit of a Satellite With Same Clock Speed As One On Earth"

Find the orbital radius, r, of a satellite whose clock will be found to run at the same rate as that for an Earth-bound clock. Take the orbital speed to be $v = (GM/r)^{1/2}$, where M is the mass of the Earth. Why would you guess that the GPS satellites are placed in higher orbits?

Other possibly useful pieces of information: $R_{\text{Earth}} = 6378$ km; $M_{\text{Earth}} = 6 \times 10^{24}$ kg; rotation speed of the Earth's surface equals $\sim 6\%$ of the orbital speed at R_{Earth}.

Problem 12

"Time Travel Using the Black Hole"(6+3+3 points)

1. Do exercise 7, Chapter 4, page 4–32 of Taylor & Wheeler.

2. A more realistic circular orbit to use is the "tourist" orbit with $E/m = 1$ that can be reached with essentially no use expenditure of rocket fuel. Derive r for this orbit using Taylor & Wheeler equations [30] and [43], then compute the time dilation factor $d\tau/dt$ using the same formula as you did above for the $r = 6M$ case.
Hint: If you've found the correct r-value in the lecture notes, you can simply verify that it satisfies equations [30] and [43] for an appropriate L-value.

3. Compute $d\tau/dt$ for the $r = 3M$ orbit. How much energy is required to reach this orbit?

Optional Problem A

Without knowing it, you have almost learned how to do advanced classical mechanics with Lagrangeans, where a particle moves along a trajectory $x(t)$ such that the "action"

$$S \equiv \int_{t_0}^{t_1} (T - V) dt$$

is minimized. This is called the *principle of least action*. Here $V = V(x(t))$ is the potential energy and $T = \frac{1}{2} m \dot{x}(t)^2$ is the kinetic energy. Use the Euler-Lagrange equation to derive the law of motion $F = ma$, *i.e.*,

$$m\ddot{x} = -V'(x).$$

Optional Problem B

"Heuristic Derivation of the Schwarzschild Metric"

(a) Read the first 4 pages of the article by Matt Visser – on the 8.033 web site.
(b) Follow Visser's simple derivation of the invariant interval leading to:

$$d\tau^2 = \left[1 - \frac{2M}{r}\right] dt^2 - 2\sqrt{\frac{2M}{r}} dr dt + dr^2 - r^2 \left(d\theta^2 + \sin\theta^2 d\phi^2\right)$$

where we have set $c = G = 1$.
(c) To transform this form of the metric to the usual Schwarzschild form, make the following transformation of variables:

$$\begin{aligned} dr &= dr' \\ dt &= dt' + \alpha dr' \quad , \end{aligned}$$

where α will turn out to be a function of r'.
(d) Set the coefficient of the $dt'dr'$ term equal to zero to find α.
(e) Finally, use the functional form of α to derive the coefficients in front of the dr'^2 and dt'^2 terms, and thereby show that ds^2 takes the standard form of the Schwarzschild metric.
Big hint: The metric above is simply the GP metric from problem 2 if you rename the time coordinate $t_{\rm ff}$.

Optional problem C

"More rigorous derivation of the Schwarzschild metric"

As indicated in the lecture notes and optional handout, the Einstein field equations for GR are

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}, \tag{1}$$

where $G_{\alpha\beta}$ is the Einstein tensor and $T_{\alpha\beta}$ is the "stress–energy" tensor, and G (with no indices) is Newton's gravitational constant. For the case of trying to find the space-time metric in the region outside of a spherically symmetric mass distribution (e.g., a neutron star or black hole), the Einstein tensor reduces to:

$$\begin{aligned} G_{00} &= \frac{1}{r^2} e^{2\Phi} \frac{d}{dr}[r(1-e^{-2\Lambda})] \ ; \\ G_{11} &= -\frac{1}{r^2} e^{2\Lambda}(1-e^{-2\Lambda}) + \frac{2}{r}\frac{d\Phi}{dr} \ , \end{aligned} \tag{2}$$

where the metric has been taken to be of the form:

$$ds^2 = -e^{2\Phi} d(ct)^2 + e^{2\Lambda} dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2 \tag{3}$$

and $e^{2\Phi}$ and $e^{2\Lambda}$ are convenient ways of writing the two unknown functions of r (only). In the region outside the mass distribution, take the stress–energy tensor to be equal to zero, and use G_{00} and G_{11} to solve for $e^{2\Phi}$ and $e^{2\Lambda}$. Take the constant of integration from the G_{00} equation to be $2GM/c^2$, and take the constant from the G_{11} equation to be 0.

Optional problem D

"Non-Relativistic Keplerian Orbits"

In the very weak-field limit (the Kepler problem), the equation for the conserved energy becomes:

$$E = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2}, \tag{4}$$

where L is the angular momentum constant associated with the orbit, E is a negative quantity for a bound orbit, and m is the mass of the orbiting particle. The distances of closest and farthest approach of the orbiting body occur when $dr/dt = 0$. These are the points where a line of constant E intersects the effective potential curve. Solve the resulting quadratic equation to find:

$$r_{\mathrm{max,min}} = -\frac{GMm}{2E}\left[1 \pm \left(1 + \frac{2L^2E}{G^2M^2m^3}\right)^{1/2}\right]. \tag{5}$$

The leading term is defined as the "semimajor axis", a, of the binary orbit, while the square root term is the "orbital eccentricity", i.e., $r_{\mathrm{max}} = a(1+e)$ and $r_{\mathrm{min}} = a(1-e)$. Show that:

$$L^2 = a(1-e^2)GMm^2 \tag{6}$$

and

$$E = -\frac{GMm}{2a}. \tag{7}$$

Note that the energy of a Keplerian orbit depends only on the semimajor axis, and not on the orbital eccentricity. The physical parameters L and E thereby uniquely determine the orbital shape – which turns out to be an ellipse.

Optional problem E

"Simple Gravitational Lens System"

Consider an astronomical object, S, sufficiently distant ($D_{\rm S}$) that it appears pointlike to an observer at O. Now introduce a pointlike gravitational lens at a distance $D_{\rm L}$ from the observer. The unperturbed angular separation between the source and the lens is β as indicated in the sketch. In the presence of the gravitational lens, light from the source can travel the heavy-line path shown in the sketch, pass a distance of closest approach b to the lens, and then be deflected by an angle δ so that it subsequently passes through O. The apparent angular distance between the lens and the image of the source I, is θ (also indicated on the sketch).

There are two convenient approximations that one can make in doing this problem. (1) All angles are taken to very small such that $\tan x \simeq \sin x \simeq x$. (2) The light path from the source to the observer may be considered as two straight-line segments with a deflection of angle δ taking place at the point of closest approach to the lens.

Utilize the following steps to derive the relation between the angles θ and β:

$$\delta = \frac{4GM}{c^2 b} \ ; \ \alpha = \frac{h}{D_{\rm S}} \ ; \ \theta = \frac{b}{D_{\rm L}} \ ; \ \delta = \frac{h}{D_{\rm LS}} \ .$$

(a) In particular, show that:

$$\beta = \theta - \frac{\theta_{\rm E}^2}{\theta} \ ,$$

where

$$\theta_E^2 = \frac{4GM}{c^2} \frac{D_{\rm LS}}{D_{\rm L} D_{\rm S}} \ ,$$

where θ_E is defined as the angle of the "Einstein Ring", i.e., the value of θ when $\beta \to 0$ (the lens and the source are along a line).

(b) Show that there are two solutions for θ, the apparent position of the source (i.e., the image) for each value of β, and find these two solutions. Make a sketch, analogous to the one above, to indicate what the geometry of these two solutions looks like.

Optional problem F

"Gravitational Acceleration on the Spherical Shell"

Exercise 9, Chapter 3, page 3–31 of Taylor & Wheeler.

Optional problem G

"Energy Measured by a Shell Observer"

Start with the general expression for $d\tau/dt$ in the Schwarzschild metric:

$$\frac{d\tau}{dt} = \left[\left(1-\frac{2M}{r}\right) - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2 - r^2\left(\frac{d\phi}{dt}\right)^2\right]^{1/2} ,$$

and show that

$$\frac{d\tau}{dt} = \left\{\left(1-\frac{2M}{r}\right)\left[1-\left(\frac{dr_s}{dt_s}\right)^2 - r^2\left(\frac{d\phi}{dt_s}\right)^2\right]\right\}^{1/2} .$$

Carefully examine all the terms and argue that the quantity in square brackets ('[ ]') is, in fact, $\gamma_{\rm shell}^{-2}$, and therefore:

$$\frac{d\tau}{dt} = \left(1-\frac{2M}{r}\right)^{1/2}\frac{1}{\gamma_{\rm shell}} .$$

Finally, combine this relation with our expression for the conserved quantity, E:

$$\frac{E}{m_0} = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} ,$$

to relate E and $E_{\rm shell}$ as follows:

$$E = \left(1-\frac{2M}{r}\right)^{1/2} E_{\rm shell} ,$$

where $E_{\rm shell} \equiv m_0\gamma_s$, the relativistic energy measured by the shell observer.

Optional problem H

"Speed of Light in a Schwarzschild Metric"

Follow the derivation of equations (14) and (15) in the Boxed Exercise "Motion of Light in Schwarzschild Geometry", page 5–8 of the Taylor & Wheeler book.

(a) From these two equations derive equation 5–16 on page 5–7.

(b) For purely radial motion of the photon, show that the speed of light as "reckoned" by the bookkeeper is:

$$\frac{dr}{dt} = \pm\left(1-\frac{2GM}{c^2 r}\right)c .$$

(c) In the weak field limit, find the bookeeper's time for a photon to move radially (with respect to a central 1 $M_\odot$ star) from a distance of $r = 10^8$ m to $r = 10^{10}$ m. How much longer does this take than for a photon making the same trip in the absence of the star?