Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Final Exam Solutions

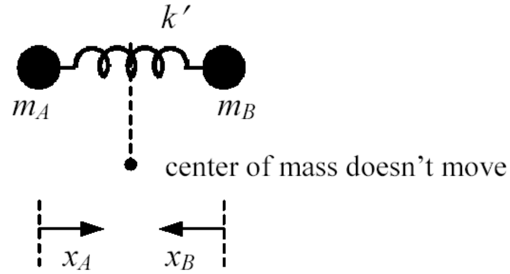
Problem 1 – Coupled oscillators

a) If $m_A = \infty$, then $\omega_A = 0$ and $\omega_B = \sqrt{(k' + k)/m_B}$.

b) If $k = 0$, then m_A, m_B and k' form an isolated system and the center of mass of this system doesn't move. Therefore, $m_A x_A = -m_B x_B$ The force applied on m_A is given by $m_A \ddot{x}_A = -k'(x_A - x_B)$ Solving them gives

$$\ddot{x}_A + x_A k'\Big(\frac{m_A + m_B}{m_A m_B}\Big) = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{k'}{M}}$$

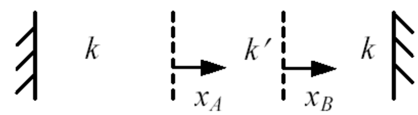
where $M = \dfrac{m_A m_B}{m_A + m_B}$ is called the "reduced mass".



c) If $k' = 0$, then $\omega_A = \sqrt{k/m_A}$ and $\omega_B = \sqrt{k/m_B}$.

d) For the general situation, the coupled equations of motion are

$$m_A \ddot{x}_A = -kx_A + k'(x_B - x_A) \qquad m_B \ddot{x}_B = -kx_B - k'(x_B - x_A)$$



e) Assume that $x_A = A\cos\omega t$ and $x_B = B\cos\omega t$. The equations of motion become

$$-\omega^2 A + \frac{k + k'}{m_A}A - \frac{k'}{m_A}B = 0 \qquad -\omega^2 B + \frac{k + k'}{m_B}B - \frac{k'}{m_B}A = 0$$

Rewrite them with A and B as the variables

$$A\Big(\omega^2 - \frac{k + k'}{m_A}\Big) + \frac{k'}{m_A}B = 0 \qquad A\frac{k'}{m_B} + \Big(\omega^2 - \frac{k + k'}{m_B}\Big)B = 0$$

For A and B to have non-zero roots, the determinant should be equal to zero, that is

$$\Big(\omega^2 - \frac{k + k'}{m_A}\Big)\Big(\omega^2 - \frac{k + k'}{m_A}\Big) - \frac{k'^2}{m_A m_B} = 0$$

$$\omega^4 - \omega^2(k + k')\Big(\frac{1}{m_A} + \frac{1}{m_B}\Big) + \frac{(k + k')^2 - k'^2}{m_A m_B} = 0$$

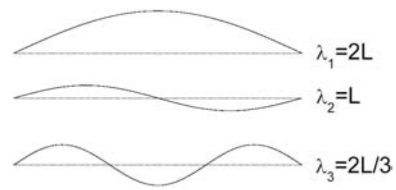
The solutions for ω are $\omega_{1,2}^2 = \dfrac{p}{2} \pm \dfrac{1}{2}\sqrt{p^2 - 4q} = \dfrac{p}{2} \pm \sqrt{\dfrac{p^2}{4} - q}$ where $p = (k + k')\left(\dfrac{1}{m_A} + \dfrac{1}{m_B}\right)$ and $q = \dfrac{(k + k')^2 - k'^2}{m_A m_B}$.

Problem 2 – Dispersive string

a) The phase velocity is $\quad v_p = \dfrac{\omega}{k} = \sqrt{\dfrac{T}{\mu} + \alpha k^2}$

b) The group velocity is $v_g = \frac{d\omega}{dk} = \sqrt{\frac{T}{\mu} + \alpha k^2} + \frac{\alpha k^2}{\sqrt{(T/\mu) + \alpha k^2}} = \frac{(T/\mu) + 2\alpha k^2}{\sqrt{(T/\mu) + \alpha k^2}}$

c) There are three normal oscillation modes of the string with wave numbers $k_n = n\pi/L$ and frequencies $\omega_n = v_{p_n} k_n = 2\pi v_{p_n}/\lambda_n$. Therefore:



$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{T}{\mu} + \alpha\left(\frac{\pi}{L}\right)^2} \quad \omega_2 = \frac{2\pi}{L}\sqrt{\frac{T}{\mu} + \alpha\left(\frac{2\pi}{L}\right)^2}$$

$$\omega_3 = \frac{3\pi}{L}\sqrt{\frac{T}{\mu} + \alpha\left(\frac{3\pi}{L}\right)^2}$$

$$y(x,t) = \sin\left(\frac{\pi x}{L}\right)\cos(\omega_1 t) + 4\sin\left(\frac{2\pi x}{L}\right)\cos(\omega_2 t) + 9\sin\left(\frac{3\pi x}{L}\right)\cos(\omega_3 t)$$

d) The three frequencies are not integer multiples of each other. It could be a long wait for the string to return to the shape $y(x,0)$, and it may NEVER happen!

Problem 3 – Transmission line

a) The voltage wave is $V_i(z<0) = V_0\cos(\omega t - kz)$.

b) For $Z_L = 0$, the circuit is shorted at $z = 0$, thus $V_t = 0$ and $\frac{V_r}{V_i} = \frac{Z_L - Z_0}{Z_L + Z_0} = -1 \quad \Rightarrow \quad V_r = -V_i$.

c) At $z = 0$, we have 2 resistors in parallel, each is 100 Ω. Thus the net impedance is 50 Ω.

d) Since the effective (net) impedance is the same as Z_0 (both are 50 Ω), we have an impedance-matched situation, i.e., $V_r = 0$, thus $V_t = V_i$. $\frac{V_r}{V_i} = \frac{50-50}{100} = 0$

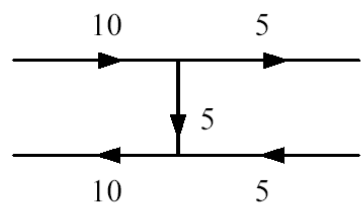
e) The maximum incident current $|I_{i_{max}}| = \frac{|V_{i_{max}}|}{Z_0} = 10$ A. $I_r = 0$ (see part (d)).

i) 10 A will be the maximum current in the lower wire for $z < 0$.

ii) The current splits equally between the load and the second transmission wire. Thus I_{max} through the load is 5 A.

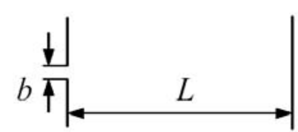
iii) Upper wire for $z > 0$: $I_{max} = 5$ A.

iv) Lower wire for $z > 0$: $I_{max} = 5$ A.



Problem 4 – Design your own pinhole camera

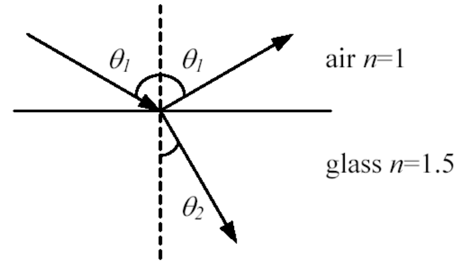
See the solution to PSet Problem 9.7. The best resolution is achieved when $1.2L\lambda \simeq b^2$, where b is the diameter of the circular hole. Given $L = 0.7$ m, and $\lambda = 5\times10^{-7}$ m, thus $b \simeq 0.65$ mm.



Problem 5 – Reflection of light

a) Snell's law $\sin\theta_1 = 1.5\sin\theta_2$, $\theta_1 = 40° \quad \Rightarrow \quad \theta_2 \simeq 25.4°$. Since the incident light is unpolarized, the intensity of the parallel and the perpendicular components are each 50% of the incident light intensity. The reflectivity for parallel and perpendicular components are

$$r_{\parallel} = \frac{-\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)} \simeq \frac{-0.26}{2.18} \simeq -0.12 \qquad r_{\perp} = \frac{-\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)} \simeq \frac{-0.25}{0.91} \simeq -0.28$$



thus $r_{\parallel}^2 \simeq 0.014$, $r_{\perp}^2 \simeq 0.077$. The fraction of the incoming 10k W that is reflected is 0.5(0.014+0.077)=4.55%.

b) Degree of linear polarization: $\dfrac{0.077 - 0.014}{0.077 + 0.014} \simeq 0.69 \simeq 70\%$ linearly polarized in favor of the $\perp$ direction. Notice that for $\theta_1 = 56°$ (the Brewster angle), $r_{\parallel} = 0$ and the reflected light would have been 100% linearly polarized in the $\perp$ direction.

Problem 6 – Oscillator in a viscous medium

a) The equation of motion for m is $m\ddot{x} = -kx - b\dot{x}$ that is, $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ where $k/m = \omega_0^2$ and $b/m = \gamma$.

b) Now, the equation of motion of m is $m\ddot{x} = -kx - b\dot{x} + b\dot{d}_1$ where $d_1 = D_1\cos(\omega t)$. In terms of ω_0 and γ, $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = -\gamma\omega D_1 \sin(\omega t)$

c) The steady state amplitude is $|A| = \dfrac{\gamma\omega D_1}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$

d) For this case, the equation of motion of m is
$m\ddot{x} = -k(x - d_2) - b\dot{x} + b\dot{d}_1$ where $d_2 = D_2\cos(\omega t + \phi)$.
In terms of ω_0 and γ, $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = -\gamma\omega D_1\sin(\omega t) + \omega_0^2 D_2 \cos(\omega t + \phi)$.

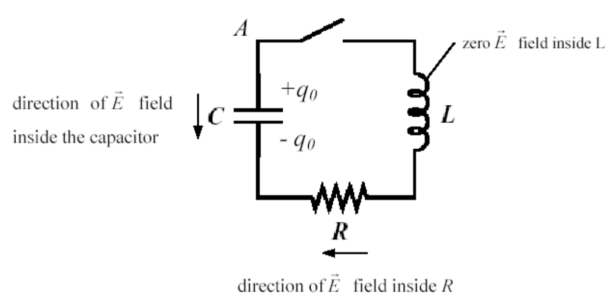
e) In the absence of an external driver, there is no motion in the steady state. This will be the case when $\gamma\omega D_1 \sin(\omega t) = \omega_0^2 D_2\cos(\omega t + \phi)$ Thus, $D_2 = \dfrac{\gamma\omega D_1}{\omega_0^2}$ and $\phi = -\dfrac{\pi}{2}$.

Problem 7 – Discharging a capacitor

a) I close the switch. A current will start flowing in the clockwise direction. Faraday's law is

$$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial \phi_B}{\partial t}$$

I start at A, go clockwise, and return to A.
$\oint_{A\to A} \neq 0$, Kirchhoff's voltage rule does NOT apply! Instead $0 + IR - V_c = -L\dfrac{dI}{dt}$
Here $V_c = \dfrac{q}{C}$ and $I = -\dfrac{dq}{dt}$.



Notice I is clockwise, thus q decreases in time. Therefore $I = -\dfrac{dq}{dt}$.

It now follows that $-L\ddot{q} - \dot{q}R - \dfrac{q}{C} = 0 \quad \Rightarrow \quad \ddot{q} + \dfrac{R}{L}\dot{q} + \dfrac{q}{LC} = 0$

b) $q(t=0) = q_0$, $\dot{q}(t=0) = 0$.

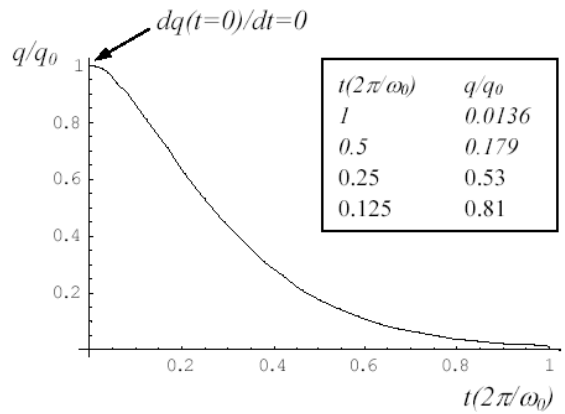
c) For critical damping $\omega_0 = \dfrac{\gamma}{2} \quad \omega_0 = \dfrac{1}{\sqrt{LC}}$, $\gamma = \dfrac{R}{L}$, thus: $\dfrac{1}{\sqrt{LC}} = \dfrac{R}{2L} \quad \Rightarrow \quad R = 2\sqrt{\dfrac{L}{C}}$.

d) The general formula for $q(t)$ is $q(t) = (A_1 + A_2 t)e^{-\gamma t/2}$ where A_1 has the dimension of charge and A_2 has the dimension of current.

$q(0) = q_0 \Rightarrow A_1 = q_0 \qquad \dot{q}(t) = -\dfrac{\gamma}{2}e^{-\gamma t/2}(A_1 + A_2 t) + A_2 e^{-\gamma t/2}$

At $t = 0$, $\dot{q} = 0$ so $-\dfrac{\gamma}{2}A_1 + A_2 = 0 \Rightarrow A_2 = \dfrac{\gamma}{2}q_0$ Thus, $q(t) = q_0(1 + \dfrac{\gamma}{2}t)e^{-\dfrac{R}{2L}t}$

e) For the case of critical damping, the charge versus time looks like this:

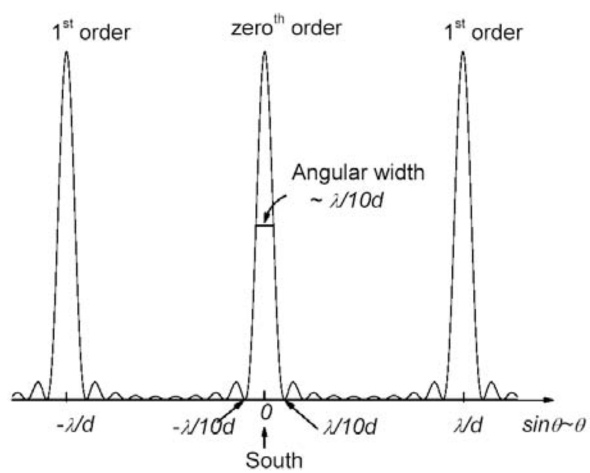


Problem 8 – Interferometric Radio Telescope

a) The intensity pattern of this array of telescopes looks like this figure where $\lambda = 6$ cm, $d = 800$ m, and so $\lambda/d \simeq 7.5 \times 10^{-5}$ radians.

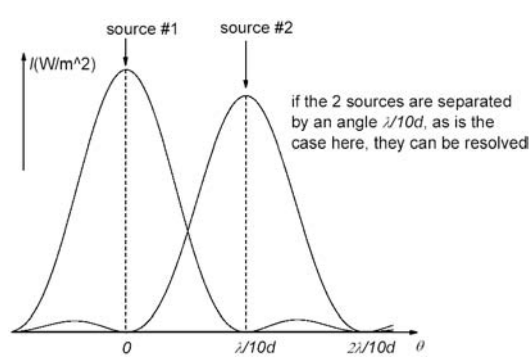
b) The angular distance from zero to first order is $\lambda/d \simeq 7.5 \times 10^{-5}$ radians$\simeq$ 15 arcsec.

c) The angular width of all orders is about $\lambda/(10d) \simeq 7.5\mu$ rad$\simeq$ 1.5 arcsec.



d) The size of the radio dishes does not enter into the angular resolution. A larger dish, however, is more sensitive, just like a larger optical telescope (ground based). Larger dishes cost $$$! It's a matter of economy.

If 2 radio sources, of approximatedly equal strength, are separated in the sky in the E−W direction by an angle of about $\lambda/(10d)$ ($\simeq$ 1.5 arcsec), the interferometer will be able to resolve them. The sketch shows the response to 2 such nearby sources. I made source #2 somewhat weaker than #1.



You can see that the diameter of the dishes does not enter into this. Also notice that d/D is 32 for a dish size of $D = 25$ m and $d/D = 8$ for $D = 100$ m. The influence of D (diffraction) shows up in the term $(\sin\beta/\beta)^2$. This alters the heights of the maxima (although not at zero order). For $D = 25$ m, at first order (where $\sin\theta = \lambda/d$), $(\sin\beta/\beta)^2 \simeq 0.997$. At second order, it is about 0.987. If D were 100 m, there would be no change in height at zero order, but there would be a $\simeq 5\%$ reduction at first order as $(\sin\beta/\beta)^2 \simeq 0.95$. But none of that would affect the angular resolution of the array. Notice, however, that the field of view of the array is ONLY dictated by D. It is about λ/D radians ($\simeq 8$ arcmin for $D = 25$ m).

Due to the rotation of the Earth, the sources move in the sky. Each source will produce its own pattern of maxima (see the figure for part (a)) as they move through the field of view of the array. The two patterns can be resolved if the angular separation of the sources (E−W) is larger than about 1.5 arcsec. 1 arcsec is the angle at which you see a dime at a distance of about 2.3 km, this is about 1/1800 of the diameter of the sun and the moon.

8.03SC Physics III: Vibrations and Waves
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