Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Problem Set #1 Solutions

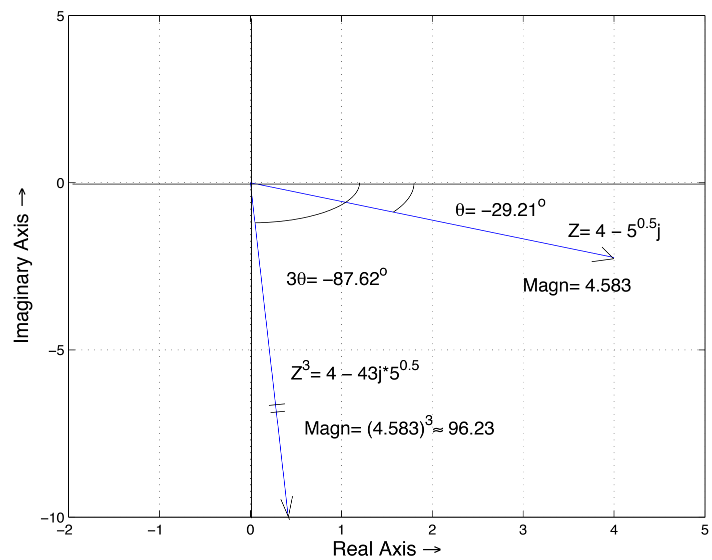
Problem 1.1: Manipulation of complex vectors

a)
$$\begin{aligned}(4-\sqrt{5}j)^3 &= 4^3 - 3\cdot 4^2\cdot\sqrt{5}j + 3\cdot 4\cdot(\sqrt{5}j)^2 - (\sqrt{5}j)^3 = 64 + 5\sqrt{5}j - 48\sqrt{5}j - 60\\ &= 4 - 43\sqrt{5}j\end{aligned}$$

Magnitude:
$|(4 - \sqrt{5})^3| = \sqrt{4^2 + (43\sqrt{5})^2} = \sqrt{16+9245} = \sqrt{9261} = 96.23$

Direction: $\arctan\left(\dfrac{-43\sqrt{5}}{4}\right) = -87.62^\circ$

We show a graphical representation. Raising the complex vector Z to the power 3 means that the new angle is 3 times larger than that of Z, and the length of the new vector is the length $|Z|^3$. The length of the vector Z^3 is not to scale ($|Z|^3 \approx 96$).



b)
$$\begin{aligned}\frac{Ae^{j(\omega t+\pi/2)}}{4+5j} &= \frac{A(\cos(\omega t+\pi/2) + j\sin(\omega t+\pi/2))}{4+5j}\\ &= \frac{A(\cos(\omega t+\pi/2) + j\sin(\omega t+\pi/2))}{4+5j}\times\frac{4-5j}{4-5j}\\ &= \frac{A[4\cos(\omega t+\pi/2) + 5\sin(\omega t+\pi/2) + j[4\sin(\omega t+\pi/2) - 5\cos(\omega t+\pi/2)]]}{4^2+5^2}\end{aligned}$$

Real Part $\quad \dfrac{A}{41}[4\cos(\omega t+\pi/2) + 5\sin(\omega t+\pi/2)]$

Imaginary Part $\quad j\dfrac{A}{41}[(4\sin(\omega t+\pi/2) - 5\cos(\omega t+\pi/2)]$

c) Remember $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$$\begin{aligned}Z_1 &= j^j = [e^{j(\pi/2\pm 2n\pi)}]^j = e^{j(\pi/2\pm 2n\pi)\times j} = e^{j^2(\pi/2\pm 2n\pi)} = e^{-(\pi/2\pm 2n\pi)}\\ &\simeq 0.208\ ,\ 3.88\times 10^{-4}\ ,\ 1.11\times 10^{2}\ \ldots\quad (n = 0, 1\ldots)\end{aligned}$$

Note: All values are real!

$$\begin{aligned}Z_2 &= j^{8.03} = [e^{j(\pi/2\pm 2n\pi)}]^{8.03} = e^{j([8.03\times(\pi/2\pm 2n\pi)]}\\ &= \cos\left[8.03\times(\frac{\pi}{2}\pm 2n\pi)\right] + j\sin\left[8.03\times(\frac{\pi}{2}\pm 2n\pi)\right]\\ &= 0.999 + 0.047j\ ,\ 0.9724 + 0.233j\ ,\ 0.990 - 0.141j\ \ldots\quad (n = 0, 1\ldots)\end{aligned}$$

Problem 1.2: (French 1-10)[1] SHM of y as a function of x

$$\begin{aligned}
y &= A\cos(kx) + B\sin(kx) \Rightarrow \quad \frac{dy}{dx} = -Ak\sin(kx) + Bk\cos(kx) \\
\Rightarrow \quad \frac{d^2y}{dx^2} &= -Ak^2\cos(kx) - Bk^2\sin(kx) = -k^2[A\cos(kx) + B\sin(kx)] \quad \frac{d^2y}{dx^2} = -k^2y
\end{aligned}$$

Hence the given differential equation has $y = A\cos(kx) + B\sin(kx)$ as its solution.

Now to express the equation in the desired form, we divide and multiply it by $\sqrt{A^2+B^2}$. When we substitute $\cos(\alpha) = A/\sqrt{A^2+B^2}$ and $\sin(\alpha) = -B/\sqrt{A^2+B^2}$, the equation takes the form:

$$\begin{aligned}
y &= \frac{A\cos(kx) + B\sin(kx)}{\sqrt{A^2+B^2}} \times \sqrt{A^2+B^2} = \sqrt{A^2+B^2}[\cos(\alpha)\cos(kx) - \sin(\alpha)\sin(kx)] \\
&= \sqrt{A^2+B^2}\cos(kx+\alpha) \\
y &= \sqrt{A^2+B^2}\cos(kx+\alpha) = \sqrt{A^2+B^2}Re[e^{j(kx+\alpha)}] = Re[(\sqrt{A^2+B^2}e^{j\alpha})e^{jkx}]
\end{aligned}$$

where $C = \sqrt{A^2+B^2} \quad \alpha = \tan^{-1} -(\frac{B}{A})$

Problem 1.3: (French 1-11) Oscillating springs

a) The mass at the end of the spring oscillates with an amplitude of 5 cm and at a frequency of 1 Hz, hence the values of A and ω are: $A = 5$ cm $\quad \omega = 2\pi f = 2\pi \times 1 = 2\pi$ rad/s.

We are given that at time $t = 0$ the mass is at the position $x = 0$. Using this and substituting the values from above in the equation $x = A\cos(\omega t + \alpha)$ we get $0 = 5\cos(\alpha) \;\Rightarrow\; \cos(\alpha) = 0 \quad \alpha = \pm\frac{\pi}{2}$. Hence the possible equations of motion for the mass as a function of time are $x = 5\cos(2\pi t + \frac{\pi}{2})$ and $x = 5\cos(2\pi t - \frac{\pi}{2})$ cm where the values are $A = 5$ cm, $\omega = 2\pi$ rad/s, and $\alpha = \pm\frac{\pi}{2}$.

b) $x = A\cos(\omega t + \alpha) \;\Rightarrow\; dx/dt = -A\omega\sin(\omega t + \alpha) \;\Rightarrow\; d^2x/dt^2 = -A\omega^2\cos(\omega t + \alpha) = -\omega^2 x$
Substituting values for A, ω, and α from part (a); and putting $t = \frac{8}{3}$ sec, we get

$$\begin{aligned}
x &= 5\cos\left[\left(2\pi \times \frac{8}{3}\right) \pm \frac{\pi}{2}\right] = 5\cos\left(\frac{16\pi}{3} \pm \frac{\pi}{2}\right) \\
&= 5\cos\left(\frac{35\pi}{6}\right), 5\cos\left(\frac{29\pi}{6}\right) = 5\cos\left(\frac{11\pi}{6}\right), 5\cos\left(\frac{5\pi}{6}\right) = \pm\frac{5\sqrt{3}}{2}\text{ cm} = \pm 4.330\text{ cm} \\
\frac{dx}{dt} &= -5 \times 2\pi\sin\left[\left(2\pi \times \frac{8}{3}\right) \pm \frac{\pi}{2}\right] = -10\pi\sin\left(\frac{16\pi}{3} \pm \frac{\pi}{2}\right) = \pm 5\pi\text{ cm/s} = \pm 15.708\text{ cm/s} \\
\frac{d^2x}{dt^2} &= 5 \times (2\pi)^2\cos\left[\left(2\pi \times \frac{8}{3}\right) \pm \frac{\pi}{2}\right] = 20\pi^2\cos\left(\frac{16\pi}{3} \pm \frac{\pi}{2}\right) \\
&= \pm 10\sqrt{3}\pi^2\text{ cm/s}^2 = \pm 170.95\text{ cm/s}^2
\end{aligned}$$

[1] The notation "French" indicates where this problem is located in one of the textbooks used for 8.03 in 2004: French, A. P. *Vibrations and Waves. The M.I.T. Introductory Physics Series.* Cambridge, MA: Massachusetts Institute of Technology, 1971. ISBN-10: 0393099369; ISBN-13: 9780393099362.

Problem 1.4: (French 3-4) Floating Cylinder

a) The diameter of the floating cylinder is d and it has l of its length submerged in water. The volume of water displaced by the submerged part of the cylinder in equilibrium condition is $\pi d^2 l/4$. Let the density of water be ρ_w and that of the cylinder be ρ_{cyl}. Hence the mass of the cylinder is:

$$M_{cyl} = \rho_w V_{displaced} = \rho_w \pi \frac{d^2 l}{4} = \rho_{cyl} \pi \frac{d^2 L}{4}$$

When the cylinder is submerged by an additional length x from its equilibrium position, the restoring force acting on it is $F_{restoring} = -\frac{\rho_w g \pi d^2}{4} x \Rightarrow M_{cyl}\ddot{x} = -\frac{\rho_w g \pi d^2}{4} x \Rightarrow 0 = \ddot{x} + \frac{\rho_w g \pi d^2 x}{4M_{cyl}} \Rightarrow$ $\omega^2 = \rho_w \frac{g\pi d^2}{4M_{cyl}} = \frac{g}{l} \Rightarrow\ x(t) = A\cos(\omega t + \alpha) = A\cos(\sqrt{\frac{g}{l}}t + \alpha)$ Hence the angular frequency of the oscillations is $\omega = \sqrt{g/l}$ rad/s and the frequency in cycles per second is $f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ Hz

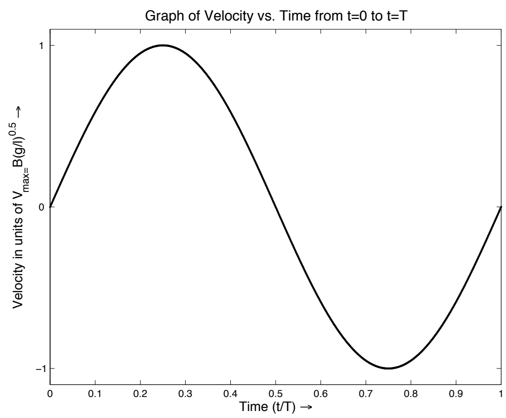
b) The equation of motion is of the form:
$x(t) = B\cos(\omega t + \alpha)$. We assume *up* to be the positive and *down* to be the negative direction. At $t = 0$, $x = -B$

$$\begin{aligned} x(0) = -B &= B\cos(\sqrt{g/l} \times 0 + \alpha) \\ \alpha &= \cos^{-1}(-1) = \pi \end{aligned}$$

The velocity of the mass is

$$\dot{x}(t) = -B\sqrt{g/l}\sin(\sqrt{g/l}t + \pi) = B\sqrt{g/l}\sin(\sqrt{g/l}t)$$

The amplitude of the velocity is $V_{max} = B\sqrt{g/l}$.



Problem 1.5: (French 3-14) A damped oscillating spring

The mass of the object is $m = 0.2$ kg and the spring constant is $k = 80$ N/m. The resistive force providing the damping force has the value of $-bv$, where v is velocity in m/s.

a) Let the oscillations of the spring be along the x axis. The spring force and damping force acting on the mass are: $F_{restoring} = -kx \quad F_{damping} = -bv = -b\dot{x}$

Newton's 2^{nd} law: $m\ddot{x} = F_{net} = F_{restoring} + F_{damping} = -kx - b\dot{x}\ \ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x}$

Hence the differential equation describing the motion of the mass is:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \quad \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0 \quad \text{where} \quad \gamma = \frac{b}{m} \quad \omega_0^2 = \frac{k}{m}$$

b) We are given that the damped frequency is $\omega = 0.995\omega_0$. The value of the damped frequency in terms of the undamped frequency and damping parameter is:

$$\begin{aligned} \omega^2 &= \omega_0^2 - \frac{\gamma^2}{4} \qquad (0.995\omega_0)^2 = 0.99{\omega_0}^2 = \omega_0^2 - \frac{\gamma^2}{4} \\ \frac{\gamma^2}{4} = \frac{b^2}{4m^2} &= 0.01\omega_0^2 = 0.01\frac{k}{m} \qquad b = \sqrt{0.04km} = 0.2\sqrt{km} \end{aligned}$$

substituting the values given in the problem, we find $b = 0.8$ Ns/m =0.8 kg/s.

c) $\omega_0 = \sqrt{\frac{k}{m}} = 20$ rad/s $\quad \gamma = \frac{b}{m} = 4$ rad/s $\quad Q = \frac{\omega_0}{\gamma} = 5$. Four complete cycles imply that the time $t = 8\pi/\omega$. The envelope for the damped oscillatory motion as a function of time $A(t) = A_0 \exp\left(-\frac{\gamma t}{2}\right) = A_0 \exp\left(-\frac{4 \cdot 4\pi}{0.995 \cdot 20}\right) = A_0 \exp(-0.804\pi) \quad \frac{A(t)}{A_0} = \exp(-0.804\pi) = 0.08$

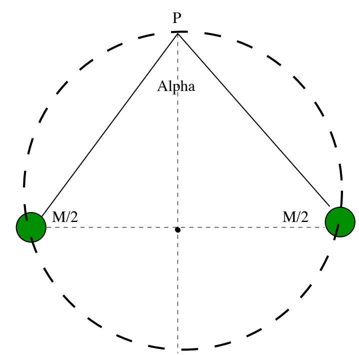
The factor by which the amplitude is reduced after four complete cycles is 0.08.

d) The equation defining the decay of energy of the system is: $E(t) = E_0 e^{-\gamma t}$. Substituting values from above, we get $\frac{E(t)}{E_0} = \exp(-\gamma t) = \exp(-1.608\pi) = 0.0064$. The factor by which the energy is reduced after four complete cycles is 6.4×10^{-3}; this is the square of the ratio of the amplitudes.

Problem 1.6: A physical pendulum

a) To solve this problem, we first consider the simpler case of a two mass rigid pendulum, both of whose masses are equidistant from the pivot point at P. All three points lie on a circle of diamater D and subtend an angle α at the pivot, as shown. In this system, let the distance of each mass from the pivot point be l.

The moment of inertia of the two masses together is $I_p = Ml^2/2 + Ml^2/2 = Ml^2$. At equilibrium the position of each mass is $l\cos(\alpha/2) = l^2/D$ below P. The gravitational potential energy of the system, after being displaced over a small angle θ is $U \approx Mg\frac{l^2}{D}\frac{\theta^2}{2}$



$$E \approx \frac{1}{2}Ml^2\dot{\theta}^2 + \frac{1}{2}g\frac{Ml^2}{D}\theta^2 \quad \frac{dE}{dt} = Ml^2\dot{\theta}\ddot{\theta} + Mg\frac{l^2}{D}\theta\dot{\theta} = 0$$

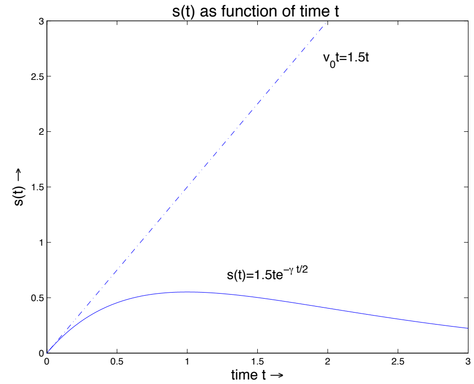
$$0 = \ddot{\theta} + \frac{g}{D}\theta \quad T = 2\pi\sqrt{\frac{D}{g}}$$

Hence the period is independent of the mass M and angle α. It only depends on the diameter D of the circle. So now considering the circular arc system whose period we have to calculate, we now realize that we can see it as a collection of many such two-mass pendulums. Since the period of all those pendulums is the same $T = 2\pi\sqrt{D/g}$, the period of the arc is also $T = 2\pi\sqrt{D/g} = 2\pi\sqrt{2R/g}$.

b) The period of the oscillations is independent of the length of the arc and the 120° angle. Hence when we complete the arc to form the hoop, the period of the hoop is same as the period of the small angle oscillations of the arc.

Problem 1.7: Damped oscillator and initial conditions

a) The solution for the case of critical damping ($\gamma/2 = \omega_0$) is of the form $s = (A + Bt)e^{-\gamma t/2}$. We know that $s(t = 0) = 0$ and $\dot{s}(t = 0) = v_0$. So
$s(0) = Ae^0 = 0 \Rightarrow \ A = 0$
$\dot{s}(0) = Be^{-\gamma t/2} - \frac{\gamma}{2}(A + Bt)e^{-\gamma t/2} = -\frac{\gamma}{2}A + B = v_0$
$\Rightarrow \quad B = v_0$. Hence the time evolution of the displacement of the pen is $s(t) = v_0 te^{-\gamma t/2} = v_0 te^{-\omega_0 t}$ and so $s(t)$ does *not* change sign before it settles to its equilibrium position as $s = 0$.

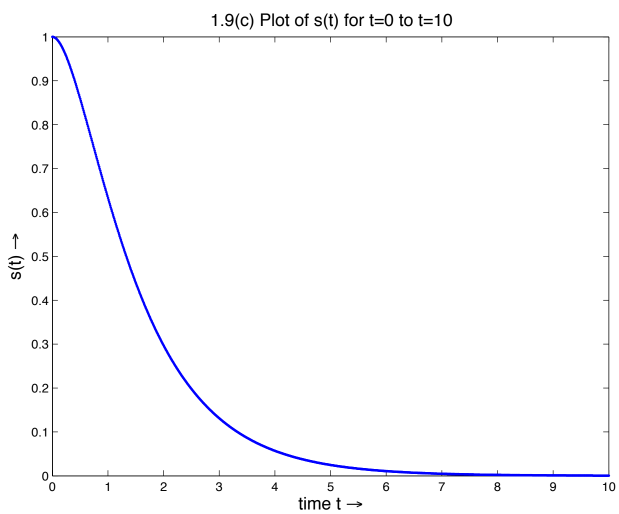


b) The solution describing the evolution of an overdamped system is $s = A_1 e^{-(\gamma/2+\beta)t} + A_2 e^{-(\gamma/2-\beta)t}$.
Now $s\,(0) = A_1 + A_2 = s_0 \quad \dot{s}\,(0) = -A_1\left(\frac{\gamma}{2}+\beta\right)e^{-(\gamma/2+\beta)t} - A_2\left(\frac{\gamma}{2}-\beta\right)e^{-(\gamma/2-\beta)t}$

$$\begin{aligned} 0 &= -A_1\left(\frac{\gamma}{2}+\beta\right) - A_2\left(\frac{\gamma}{2}-\beta\right) \quad 0 = (A_2 - s_0)\left(\frac{\gamma}{2}+\beta\right) - A_2\left(\frac{\gamma}{2}-\beta\right) \\ 2A_2\beta &= s_0\left(\frac{\gamma}{2}+\beta\right) \Rightarrow \quad A_2 = s_0\frac{1}{2\beta}\left(\frac{\gamma}{2}+\beta\right) \Rightarrow \quad A_1 = s_0\left[1 - \frac{1}{2\beta}\left(\frac{\gamma}{2}+\beta\right)\right] \end{aligned}$$

$$s = s_0\left[1 - \frac{1}{2\beta}\left(\frac{\gamma}{2}+\beta\right)\right]e^{-(\gamma/2+\beta)t} + s_0\frac{1}{2\beta}\left(\frac{\gamma}{2}+\beta\right)e^{-(\gamma/2-\beta)t} \text{ where } \beta = \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

c) Plot of s(t) for the given values



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