Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Problem Set #3 Solutions

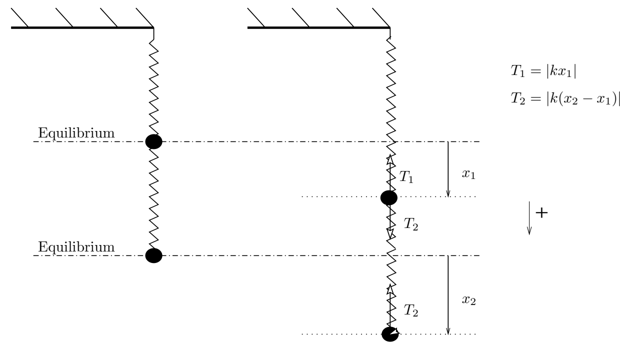
Problem 3.1: (French 5-10)[1] Coupled Oscillators using two springs

Let the displacement from the equilibrium positions for masses m_1 and m_2 be x_1 and x_2 respectively. Then the tensions in the two strings are $T_1 = kx_1$ and $T_2 = k(x_2 - x_1)$, respectively. Now

$$\begin{aligned} m_1\ddot{x}_1 &= +k(x_2 - x_1) - kx_1 \\ m_2\ddot{x}_2 &= -k(x_2 - x_1) \end{aligned}$$

Substituting $m_1 = m_2 = m$ and $\omega_s^2 = k/m$:

$$\begin{aligned} \ddot{x}_1 &= \omega_s^2(x_2 - 2x_1) \\ \ddot{x}_2 &= \omega_s^2(x_1 - x_2) \end{aligned} \tag{1}$$



Let $x_1 = C_1\cos(\omega t)$, and $x_2 = C_2\cos(\omega t)$. Now using these in Eq. 1

$$\begin{aligned} -\omega^2 C_1 &+ 2\omega_s^2 C_1 = \omega_s^2 C_2 \\ -\omega^2 C_2 &+ \omega_s^2 C_2 = \omega_s^2 C_1 \end{aligned} \tag{2}$$

Method I: Without using Cramer's Rule

From Eq. 2 we get

$$\frac{C_1}{C_2} = \frac{\omega_s^2}{2\omega_s^2 - \omega^2} = \frac{\omega_s^2 - \omega^2}{\omega_s^2} \qquad \omega_s^4 = 2\omega_s^4 - 3\omega_s^2\omega^2 + \omega^4 \qquad \omega^4 - 3\omega_s^2\omega^2 + \omega_s^4 = 0$$

$$\omega^2 = \frac{3\omega_s^2 \pm \sqrt{9\omega_s^4 - 4\omega_s^4}}{2} = \left(3 \pm \sqrt{5}\right)\frac{\omega_s^2}{2} = \left(3 \pm \sqrt{5}\right)\frac{k}{2m}$$

$$\omega_+ = \sqrt{(3+\sqrt{5})\frac{k}{2m}} \qquad \omega_- = \sqrt{(3-\sqrt{5})\frac{k}{2m}} \qquad \frac{\omega_+}{\omega_-} = \sqrt{\frac{3+\sqrt{5}}{3-\sqrt{5}}} = \frac{\sqrt{5}+1}{\sqrt{5}-1}$$

For $\omega_+ = \sqrt{(3+\sqrt{5})k/2m}$ $\qquad \dfrac{C_1}{C_2} = \dfrac{\omega_s^2}{2\omega_s^2 - \omega_+^2} = \dfrac{2\omega_s^2}{4\omega_s^2 - (3+\sqrt{5})\omega_s^2} = \dfrac{2}{1-\sqrt{5}}$

For $\omega_- = \sqrt{(3-\sqrt{5})k/2m}$ $\qquad \dfrac{C_1}{C_2} = \dfrac{\omega_s^2}{2\omega_s^2 - \omega_-^2} = \dfrac{2\omega_s^2}{4\omega_s^2 - (3-\sqrt{5})\omega_s^2} = \dfrac{2}{1+\sqrt{5}}$

Method II: Using Cramer's Rule

On collecting coefficients of C_1 and C_2 in Eq. 2 we get $(2\omega_s^2 - \omega^2)C_1 - \omega_s^2 C_2 = 0$ and $-\omega_s^2 C_1 + (\omega_s^2 - \omega^2)C_2 = 0$.

[1] The notation "French" indicates where this problem is located in one of the textbooks used for 8.03 in 2004: French, A. P. *Vibrations and Waves. The M.I.T. Introductory Physics Series.* Cambridge, MA: Massachusetts Institute of Technology, 1971. ISBN-10: 0393099369; ISBN-13: 9780393099362.

$$C_1 = \frac{\begin{vmatrix} 0 & -\omega_s^2 \\ 0 & \omega_s^2 - \omega^2 \end{vmatrix}}{\begin{vmatrix} 2\omega_s^2 - \omega^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_s^2 - \omega^2 \end{vmatrix}} \qquad C_2 = \frac{\begin{vmatrix} 2\omega_s^2 - \omega^2 & 0 \\ -\omega_s^2 & 0 \end{vmatrix}}{\begin{vmatrix} 2\omega_s^2 - \omega^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_s^2 - \omega^2 \end{vmatrix}}$$

Non-zero solutions for C_1 and C_2 only possible if

$$\begin{vmatrix} 2\omega_s^2 - \omega^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_s^2 - \omega^2 \end{vmatrix} = 0 \quad \Rightarrow \quad \omega^4 - 3\omega_s^2\omega^2 + \omega_s^4 = 0$$

This is the same as the last equation in line 1 of Method I. From here on, the solution is identical.

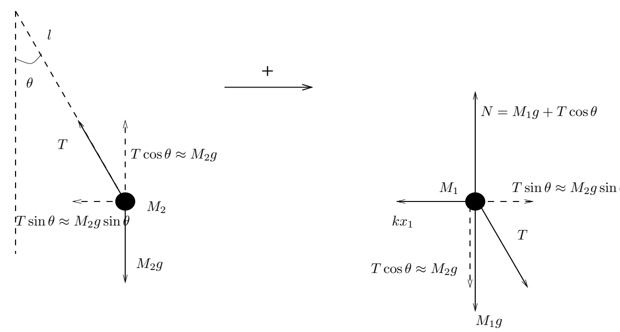
Problem 3.2: (French 5-11) Coupled spring and pendulum

a) The tension in the string is $T \approx M_2 g$. The equation of motion for mass M_2 in the x-direction is as follows

$$\begin{aligned} M_2\ddot{x}_2 &= -M_2 g\sin(\theta) \\ M_2\ddot{x}_2 &= -M_2\frac{g}{l}(x_2 - x_1) \end{aligned}$$

and for mass M_1 is

$$M_1\ddot{x}_1 = -kx_1 + M_2\frac{g}{l}(x_2 - x_1)$$



b) & c) Substituting $\omega_s^2 = k/M_2$, $\omega_p^2 = g/l$ and $M_1 = M_2 = M$ we get

$$\begin{aligned} \ddot{x}_2 + \omega_p^2 x_2 - \omega_p^2 x_1 &= 0 \\ \ddot{x}_1 + (\omega_s^2 + \omega_p^2)x_1 - \omega_p^2 x_2 &= 0 \end{aligned}$$

Let $x_1 = C_1\cos(\omega t)$, $x_2 = C_2\cos(\omega t)$. Substituting gives:

$$\begin{aligned} -\omega^2 C_2 + \omega_p^2 C_2 &= \omega_p^2 C_1 \\ -\omega^2 C_1 + (\omega_s^2 + \omega_s^2)C_1 &= \omega_p^2 C_2 \end{aligned} \tag{3}$$

Method I: Without using Cramer's Rule

$$\frac{C_1}{C_2} = \frac{-\omega^2 + \omega_p^2}{\omega_p^2} = \frac{\omega_p^2}{-\omega^2 + \omega_p^2 + \omega_s^2} \qquad \omega_p^4 = \omega^4 - \omega^2\omega_s^2 - 2\omega^2\omega_p^2 + \omega_p^2\omega_s^2 + \omega_p^4$$

$$\omega^4 - (2\omega_p^2 + \omega_s^2)\omega^2 + \omega_p^2\omega_s^2 = 0 \qquad \omega^2 = \frac{2\omega_p^2 + \omega_s^2}{2} \pm \frac{1}{2}\sqrt{(2\omega_p^2 + \omega_s^2)^2 - 4\omega_p^2\omega_s^2}$$

$$\omega^2 = \frac{2\omega_p^2 + \omega_s^2}{2} \pm \frac{1}{2}\sqrt{4\omega_p^4 + \omega_s^4} \qquad \omega_\pm = \left[\frac{2\omega_p^2 + \omega_s^2}{2} \pm \frac{1}{2}\left(4\omega_p^4 + \omega_s^4\right)^{1/2}\right]^{1/2}$$

For ω_+ $\quad \dfrac{C_1}{C_2} = \dfrac{-\omega_+^2 + \omega_p^2}{\omega_p^2} = \dfrac{-\omega_s^2 - \sqrt{4\omega_p^4 + \omega_s^4}}{2\omega_p^2}$

For ω_- $\quad \dfrac{C_1}{C_2} = \dfrac{-\omega_-^2 + \omega_p^2}{\omega_p^2} = \dfrac{-\omega_s^2 + \sqrt{4\omega_p^4 + \omega_s^4)}}{2\omega_p^2}$

Method II: Using Cramer's Rule

Collecting coefficients in Eq. 3: $\omega_p^2 C_1 + (\omega^2 - \omega_p^2)C_2 = 0$ and $(-\omega^2 + \omega_p^2 + \omega_s^2)C_1 - \omega_p^2 C_2 = 0$.

$$C_1 = \frac{\begin{vmatrix} 0 & \omega^2 - \omega_p^2 \\ 0 & -\omega_p^2 \end{vmatrix}}{\begin{vmatrix} \omega_p^2 & \omega^2 - \omega_p^2 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_p^2 \end{vmatrix}}$$

$$C_2 = \frac{\begin{vmatrix} \omega_p^2 & 0 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & 0 \end{vmatrix}}{\begin{vmatrix} \omega_p^2 & \omega^2 - \omega_p^2 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_p^2 \end{vmatrix}}$$

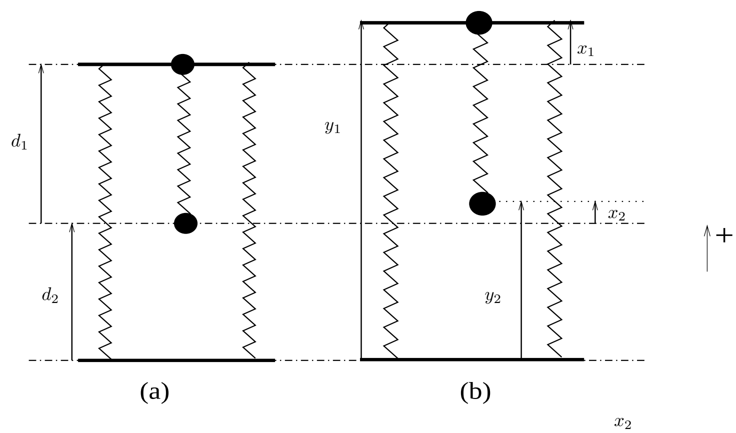
Non-zero values of C_1 and C_2 only possible if

$$\begin{aligned} \begin{vmatrix} \omega_p^2 & \omega^2 - \omega_p^2 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_p^2 \end{vmatrix} &= 0 \qquad \Rightarrow \quad -\omega_p^4 - (\omega^2 - \omega_p^2)(-\omega^2 + \omega_p^2 + \omega_s^2) = 0 \\ \omega^4 - (\omega_s^2 + 2\omega_p^2)\omega^2 + \omega_p^2\omega_s^2 &= 0 \end{aligned}$$

This is the same as found for Method I (see second line). From here on, the solution is identical.

Problem 3.3: (Bekefi & Barrett 1.16)[2] Coupled oscillators using three springs

Side (a) of the figure shows the system at rest and side (b) shows it at some random time t. Displacements from Equilibrium are x_1 and x_2. Now $y_1 = d_1 + d_2 + x_1$ and $y_2 = d_2 + x_2$



a) The equations of motion are:

$$\begin{aligned} m\ddot{x}_1 &= -2kx_1 - k(x_1 - x_2) \\ &\Rightarrow \quad \ddot{x}_1 + 3\omega_0^2 x_1 - \omega_0^2 x_2 = 0 \\ m\ddot{x}_2 &= +k(x_1 - x_2) \\ &\Rightarrow \quad \ddot{x}_2 + \omega_0^2 x_2 - \omega_0^2 x_1 = 0 \end{aligned}$$

where $\omega_0^2 = k/m$

b) Substituting $x_1 = A\cos(\omega t)$ and $x_2 = B\cos(\omega t)$ in the equations of motion gives $A(3\omega_0^2 - \omega^2) = B\omega_0^2$ and $A\omega_0^2 = B(\omega_0^2 - \omega^2)$

[2] The notation "Bekefi & Barrett" indicates where this problem is located in one of the textbooks used in 8.03 in 2004: Bekefi, George, and Alan H. Barrett *Electromagnetic Vibrations, Waves, and Radiation.* Cambridge, MA: MIT Press, 1977. ISBN: 9780262520478.

$$\frac{A}{B} = \frac{\omega_0^2}{3\omega_0^2 - \omega^2} = \frac{\omega_0^2 - \omega^2}{\omega_0^2} \qquad \omega_0^4 = 3\omega_0^4 - 4\omega^2\omega_0^2 + \omega^4$$

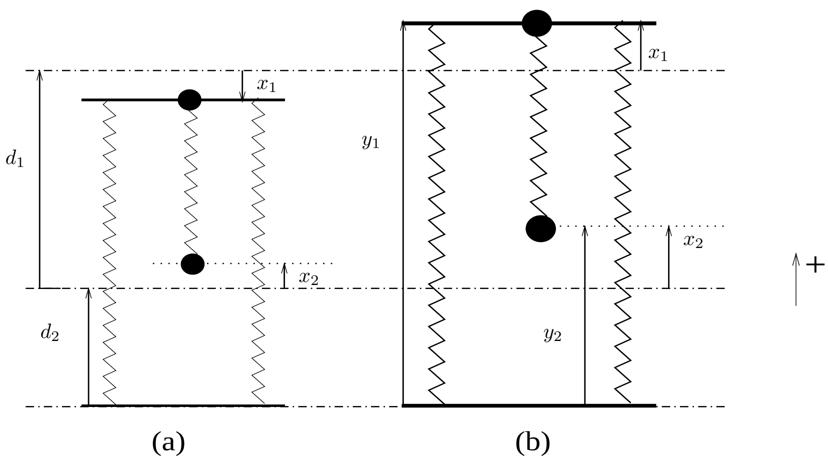
$$\omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0 \qquad \omega_\pm^2 = \omega_0^2(2 \pm \sqrt{2})$$

For $\omega_1 = \omega_0(2-\sqrt{2})^{1/2}$ $\quad \dfrac{B}{A} = 1+\sqrt{2}$ $\qquad$ For $\omega_2 = \omega_0(2+\sqrt{2})^{1/2}$ $\quad \dfrac{B}{A} = 1-\sqrt{2}$

Hence the general solutions are:

$$\begin{aligned} y_1(t) &= d_1 + d_2 + x_1(t) = d_1 + d_2 + A\cos(\omega_1 t + \alpha) + B\cos(\omega_2 t + \beta) \\ y_2(t) &= d_2 + x_2(t) = d_1 + d_2 + (1+\sqrt{2})A\cos(\omega_1 t + \alpha) + (1-\sqrt{2})B\cos(\omega_2 t + \beta) \end{aligned} \tag{4}$$

c) Side (a) of the figure shows the normal mode with higher frequency ω_2 such that $x_2(t) = (1-\sqrt{2})x_1(t)$. Side (b) shows the normal mode with lower frequency ω_1 such that $x_2(t) = (1+\sqrt{2})x_1(t)$.

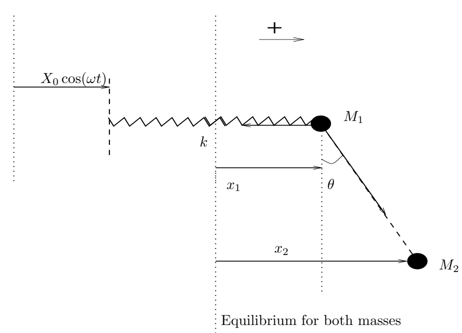


x_2

Problem 3.4: Driven coupled oscillator

a) The equation of motion for mass M_2 is unchanged $M_2\ddot{x}_2 = -M_2 g\sin(\theta)$ and for mass M_1 is

$$\begin{aligned} M_1\ddot{x}_1 &= -k[x_1 - X(t)] + M_2\frac{g}{l}(x_2 - x_1) \\ M_1\ddot{x}_1 &+ kx_1 + M_2\frac{g}{l}(x_1 - x_2) = kX_0\cos(\omega t) \end{aligned}$$



b) Substituting $\omega_s^2 = k/M_2$, $\omega_p^2 = g/l$ and $M_1 = M_2 = M$,

$$\begin{aligned} \ddot{x}_2 + \omega_p^2 x_2 - \omega_p^2 x_1 &= 0 \\ \ddot{x}_1 + (\omega_s^2 + \omega_p^2)x_1 - \omega_p^2 x_2 &= \omega_s^2 X_0 \end{aligned} \tag{5}$$

Let $x_1 = C_1\cos(\omega t)$ $\quad x_2 = C_2\cos(\omega t)$. Now using these in Eq. 5

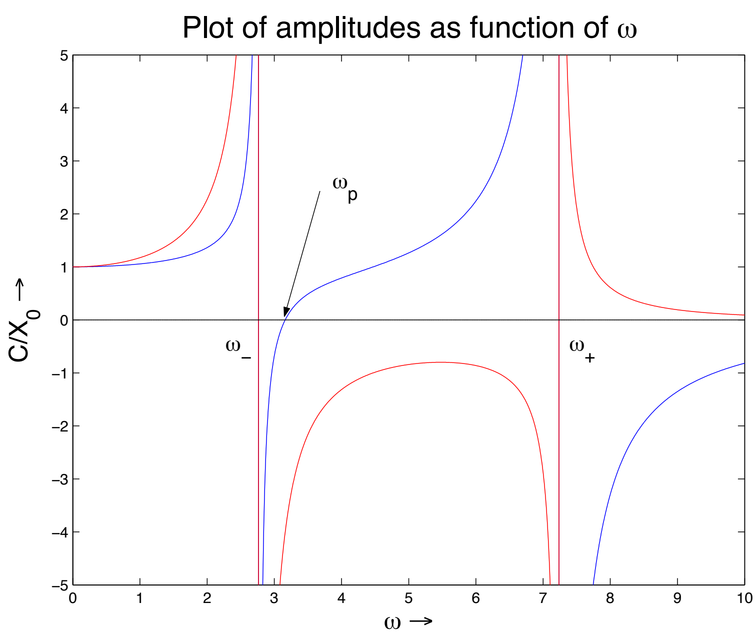
$$\omega_p^2 C_1 + (\omega^2 - \omega_p^2)C_2 = 0 \qquad (-\omega^2 + \omega_s^2 + \omega_s^2)C_1 - \omega_p^2 C_2 = \omega_s^2 X_0 \tag{6}$$

$$C_1 = \frac{\begin{vmatrix} 0 & \omega^2 - \omega_p^2 \\ \omega_s^2 X_0 & -\omega_p^2 \end{vmatrix}}{\begin{vmatrix} \omega_p^2 & \omega^2 - \omega_p^2 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_p^2 \end{vmatrix}} = \frac{kX_0(g - l\omega^2)}{Ml\omega^4 - (2Mg + kl)\omega^2 + kg}$$

$$C_2 = \frac{\begin{vmatrix} \omega_p^2 & 0 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & \omega_s^2 X_0 \end{vmatrix}}{\begin{vmatrix} \omega_p^2 & \omega^2 - \omega_p^2 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_p^2 \end{vmatrix}} = \frac{kgX_0}{Ml\omega^4 - (2Mg + kl)\omega^2 + kg}$$

These are the steady state solutions. The general problem is a linear combination between the transient problem and the steady state solutions. Notice that the transient problem has four adjustable parameters which follow from the initial conditions.

c) The figure shows the plot of amplitudes C_1 (blue line) and C_2 (red line) as a function of the frequency. Note: At $\omega = 0$, the amplitudes are $C_1 = C_2 = X_0$. This figure is unrealistic. It was derived (i) under the small angle approximation and (ii) for zero damping. Thus, the very large amplitude for C_1 and C_2 as shown are meaningless. If you add sufficient damping, and if you cannot make the small angle approximations, because the angles are large, the problem becomes substantially more complicated. But it can be solved numerically. You will then find meaningful values for the amplitudes. A more insightful way to express (and plot) the amplitude of the pendulum would be to do this in terms of the angle θ, rather than C_2.



d) We can note from the functional form of C_2 that it cannot have the value zero (except for $\omega \to \infty$). However, C_1 will be zero when $g - l\omega^2 = 0 \quad \omega = \omega_p = \sqrt{\frac{g}{l}} \quad C_1 \to 0$. This is the resonance frequency of the pendulum. Thus, at this frequency, the two horizontal forces on the upper mass, $kX_o \cos(\omega t)$ and $T \sin(\theta)$, cancel. Since $\sin(\theta) < 1$, kX_o must always be smaller than T. At first sight, this inequality seems a bit bizarre, as, according to our derivation, the frequency at which the upper mass stands still is independent of the spring constant k. Also, keep in mind that we never had to make any assumption regarding k in our derivation (the inequality must have been met automatically without our realizing it).

You SHOULD also ask yourself the question: **How on Earth can the pendulum swing if the mass attached to the spring does not move at all; what is driving the pendulum?** The answer is simple: **it is not possible!** It is only possible in our dream-world of zero damping. In the presence of damping, no matter how little, the peculiar state is unstable. This can easily be seen as follows.

Assume that the system is in that state. That means that at any moment in time the net horizontal force on the pendulum mass is zero. Thus, the vectorial sum of the spring force and $T\sin(\theta)$ must be ZERO. However, if the mass on the spring is not moving, the pendulum is no longer driven, and thus its amplitude will decay, and the net force on the mass on the spring is no longer zero, and thus that mass will start to move. Thus, the peculiar state is unstable. You will be able to go through that "special" state by varying ω, but you cannot "stop" there. However, I demonstrated in lectures (9/28) using 3 different driven systems, that you can get very close to those "special" states, and that is already amazing (and very non-intuitive).