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8.03SC **Fall 2012**

Notes for Lecture #11: Fourier Analysis

This lecture introduces Fourier analysis. It turns out that the previous discussion of representing an arbitrary vibration in terms of a superposition of normal modes is equivalent to Fourier analysis. This will be particularly useful in dealing with arbitrary initial conditions. In representing the motion as a superposition of normal modes we have $y(x,t) = \sum_{n=1}^{\infty} B_n \sin(k_n x)\cos(\omega_n t)$ with $k_n = n\pi/L$ and $\omega_n = vk_n$ (**2:10**). The initial condition corresponds to $t = 0$ and that function is $y(x,0) = B_1 \sin\left(\frac{\pi x}{L}\right) + B_2 \sin\left(\frac{2\pi x}{L}\right) + B_3 ....$ Fourier analysis can show us what values of B_n correspond to a specific initial condition, but first we need some general considerations about Fourier series. For any single-valued regular function, a series can be formed (**4:45**):

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos(mx) + \sum_{m=1}^{\infty} B_m \sin(mx).$$

We want to find the values for the A_m and B_m given the function defining the initial conditions at all points. The procedure will involve taking various integrals. We first just integrate the function $f(x)$ itself from $-\pi$ to π. Since $\sin(m\pi) = \sin(-m\pi) = 0$ and $\cos(m\pi) = cos(-m\pi)$, the integrals from $-\pi$ to $+\pi$ of the second and third terms in the representation of $f(x)$ will be zero and therefore: $\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{A_0}{2} dx = 2\pi \frac{A_0}{2} = \pi A_0$. We can thus write (**6:35**)

$$A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx \; .$$

To get the other A values, consider the integral $\displaystyle\int_{-\pi}^{\pi} f(x)\cos(nx)dx$, with n a positive integer. The first part gives $\displaystyle\int_{-\pi}^{\pi} \frac{A_0}{2}\cos(nx)dx = \frac{A_0}{2}\int_{-\pi}^{\pi}\cos(nx)dx$, which is, for exactly the reason just explained, 0. The third part gives $\displaystyle\int_{-\pi}^{\pi} \cos(nx)\sin(mx)dx$. This is 0 because $\cos nx \sin mx$ is an odd function for any value of n or m (recall both are positive integers). So integration over any symmetric range in x must give 0. That leaves us with the second part, which has $\displaystyle\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx$. It may not be obvious, but if $m \neq n$, this is also 0, as can be shown using integration of complex exponentials. However, if $m = n$, we have $\displaystyle\int_{-\pi}^{\pi} \cos^2(nx)dx$. Again, you do this integral with complex exponentials or trig identities and the answer is simply π. Thus $\displaystyle\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \int_{-\pi}^{\pi} A_m \cos(mx)\cos(nx)dx = \pi A_m$, or (**9:40**)

$$A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx \; .$$

By a very similar procedure, we find that:

$$B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx.$$

The equations $A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx$, and $B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx$ make up the recipe for Fourier analysis. In fact, if we let m start at 0, we can forget the first of these since $\cos(0) = 1$ (**12:30**). A very simple example of a plucked string is to lift it up to a value a over its whole length from 0 to L. Since this is not periodic, we imagine that, in fact, the function extends from 0 to $2L$ and is of value $-a$ between $x = L$ and $x = 2L$. The period is $2L$ and the average value (and thus A_0) is 0. In the equations above, the variable x was in radians and we want in now to be in meters. We do the conversion using the fact that the wavelength is $2L$ (and so $x = 2L$ is 2π radians) and the modified formulas are (**18:30**):

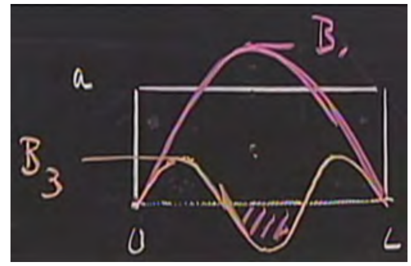
$$y(x) = f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos\left(\frac{mx\pi}{L}\right) + \sum_{m=1}^{\infty} B_m \sin\left(\frac{mx\pi}{L}\right)$$

which lead to the modified recipes:

$$A_m = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{mx\pi}{L}\right)dx \quad B_m = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{mx\pi}{L}\right)dx.$$

It is immediately clear that the cos terms cannot represent the square-wave function well, since cos is even, while the function is odd (**25:30**). All the A values must be zero! The sin terms *can* represent the function well, or at least the first one does so. However the second term (B_2) contributes in the wrong places, as do all even B_n (**28:45**). These must also be zero!

The integral can be split and done separately in the two intervals where the function changes sign, $B_m = \frac{a}{L}\int_0^L \sin\left(\frac{mx\pi}{L}\right)dx - \frac{a}{L}\int_L^{2L}\sin\left(\frac{mx\pi}{L}\right)dx$. Using $\int \sin(\alpha x)dx = -(\frac{1}{\alpha})\cos\alpha x$ with $\alpha = \frac{m\pi}{L}$, $\int_a^b \sin(mx\pi/L)dx = -\frac{L}{m\pi}\cos\frac{mx\pi}{L}\Big|_a^b$. So $B_m = -\frac{a}{L}\frac{L}{m\pi}\cos\left(\frac{mx\pi}{L}\right)\Big|_0^L + \frac{a}{L}\frac{L}{m\pi}\cos\left(\frac{mx\pi}{L}\right)\Big|_L^{2L}$ (**31:55**). If m is odd, then $\cos(mx\pi/L)\Big|_0^L = \cos(m\pi L/L) - \cos 0 = -1 - 1 = -2$. Similarly $\cos(mx\pi/L)\Big|_L^{2L} = \cos(m\pi 2L/L) - \cos(m\pi L/L) = 1-(-1) = 2$. For odd values of m, the final result is $B_m = -\frac{a}{L}\frac{L}{m\pi}(-2) + \frac{a}{L}\frac{L}{m\pi}2 = \frac{4a}{m\pi}$. However, if m is even, then the cos terms, and therefore the B_ms, are zero. The final result is $y(x,0) = \frac{4a}{\pi}\left[\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{L}\right) + ...\right]$ (**36:50**).



A drawing of the first two terms makes a good start at reproducing the function, with $B_1 \approx 1.27a$, $B_3 \approx 0.42a$, etc. The terms can be added electronically and displayed on an oscilloscope making a convincing demo of generating a square wave with sine waves (**42:40**).

Having a full description of the initial shape of the string at $t = 0$ as a summation of normal modes, we can now investigate its time development by including the time dependence of each normal mode (**45:00**). This gives (with $\omega_m = vk_m$, $k_m = m\pi/L$, and $v = \sqrt{T/\mu}$)

$$y(x,t) = \frac{4a}{\pi}\left[\sin\left(\frac{\pi x}{L}\right)\cos(\omega_1 t) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right)\cos(\omega_3 t) + \frac{1}{5}\sin\left(\frac{5\pi x}{L}\right)\cos(\omega_5 t) + ...\right].$$

The behavior of the time development is not what one might at first expect. If a triangular pulse is released, pulses of half the initial amplitude travel out in *both* directions. There is a very non-trivial equivalence between thinking of independent normal modes, each oscillating at their individual frequencies, and these two oppositely-directed traveling waves. Demos (**50:00**) for a variety of initial shapes dramatically show this equivalence, including both the two traveling waves as well as all of the oscillating harmonics (both their amplitudes and detailed time dependences).

The discussion now moves to the energy content of a signal (**1:00:00**) We can consider the energy in a signal made up of propagating waves and if we take $A^2 + B^2$ for each mode, we would get a power spectrum showing how much energy is present at each frequency. The *Fast Fourier Transform* (FFT) can do this spectral analysis with a digitized real-time signal on a computer. Several examples are shown, including relatively pure tones, tones with one or two higher harmonics, and more complicated sounds.

If enough data is available, the FFT can extract power information even from a noisy signal such as that of X-rays from a pulsar (**1:11:00**). In this case one sees a huge spike in the power spectrum at 401 Hz due to the rotation of an object of about the mass of the Sun (a neutron star). The FFT is a powerful and universally used technique in signal analysis.

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These viewing notes were written by Prof. Martin Connors in collaboration with Dr. George S.F. Stephans.

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