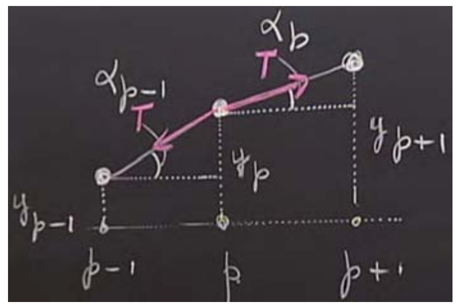
Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Notes for Lecture #7: Many Coupled Oscillators & Wave Equations

Carts on the air track, coupled with springs, supported *longitudinal* oscillations. Oscillation perpendicular to the alignment of the oscillators is called *transverse* oscillation. We are already familiar with these ideas, and the idea of polarization in transverse oscillation has already been discussed. Transverse motion of a series of beads with mass m separated by a distance ℓ along a string is examined. The motion is considered purely transverse and this is called the y direction. The "boundary conditions" are that the two ends are fixed and do not move.

For modest amplitudes, the tension T is constant, and motion along the string can be neglected (**3:00**). Although the tensions are the same in all segments of the string, the direction changes (slightly) at each bead. Therefore, the components of tension acting in the y direction on each bead are slightly different, resulting in a net force. Newton's Second Law for the p^{th} bead (one for each p from 1 to N) is:



$$m\ddot{y}_p = -T\sin\alpha_{p-1} + T\sin\alpha_p = -T\frac{(y_p - y_{p-1})}{\ell} + T\frac{(y_{p+1} - y_p)}{\ell} \quad (\mathbf{6{:}35})$$

Dividing by m, we get $\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) = 0$ where $\omega_0^2 = \dfrac{T}{m\ell}$. Note that the definition of ω_0 not only makes sense dimensionally, but reflects reality in that tighter strings will have higher frequency, and strings with more mass or longer length (between particles) will have lower frequency. The boundary conditions are $y_0 = 0$ and $y_{N+1} = 0$ at the fixed ends. (**9:20**)

For simplicity in visualization, we now consider a two-particle system. Based on what was done before, we can easily guess that the normal modes will be a lower frequency one with both particles oscillating up and down together, and a higher one with them out of phase, oscillating on opposite sides of equilibrium. We get differential equations for the two points: $\ddot{y}_1 + 2\omega_0^2 y_1 - \omega_0^2(y_2 + y_0) = 0$ and $\ddot{y}_2 + 2\omega_0^2 y_2 - \omega_0^2(y_3 + y_1) = 0$. Since $y_0 = 0$ and $y_3 = 0$, we have: $\ddot{y}_1 + 2\omega_0^2 y_1 - \omega_0^2 y_2 = 0$ and $\ddot{y}_2 + 2\omega_0^2 y_2 - \omega_0^2 y_1 = 0$. As a trial function for normal modes, we use $y_p = A_p \cos\omega t$, where for generality p goes from 0 to $N+1$ (**12:30**), and can immediately differentiate and substitute giving the two equations: $-\omega^2 A_1 + 2\omega_0^2 A_1 - \omega_0^2(A_2 + A_0) = 0$ and $-\omega^2 A_2 + 2\omega_0^2 A_2 - \omega_0^2(A_3 + A_1) = 0$. As before, the absence of first derivatives means that all terms have a common $\cos\omega t$ which can be divided out. Keeping A_0 and A_3 (even though they are 0 in this particular case), we can write these equations in a more generic form: (**14:40**):

$$-\omega^2 A_p + 2\omega_0^2 A_p - \omega_0^2(A_{p+1} + A_{p-1}) = 0.$$

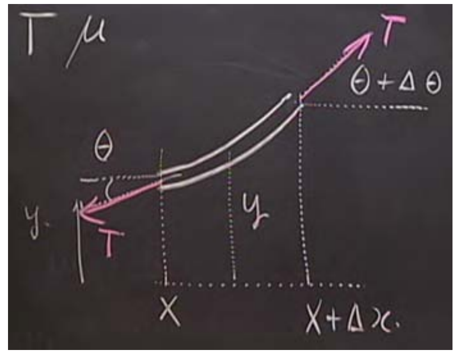
There are N linear equations to solve, which is not easy if N is large. Based on intuition, we guess that normal modes should look like sin curves with increasing number of bumps. So we can try, for mode n, $A_{p,n} = C_n \sin\left(\frac{pn\pi}{N+1}\right)$ (**17:00**). Notice that for $p = 0$ and $p = N+1$, this is always 0, thus automatically satisfying the boundary conditions. For $n = 1$, $A_{p,1} = C_1 \sin\left(\frac{p\pi}{N+1}\right)$, there are zeroes only at the ends: $p = 0$ and $p = N+1$. The next mode, $A_{p,2} = C_2 \sin\left(\frac{2p\pi}{N+1}\right)$, has an additional zero in the middle at $p = \frac{1}{2}(N+1)$, and so on.

Now, we need to determine the frequencies of the normal modes (**21:30**). From the general form above, we get $\frac{A_{p+1} + A_{p-1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$. Trigonometry[1] can be used to show that the left side of the equation (the ratio of As) is given by $2\cos\left(\frac{n\pi}{N+1}\right)$. From that, the mode frequencies are $\omega_n = 2\omega_0 \sin\frac{n\pi}{2(N+1)}$ (**24:20**). Folding in the amplitude already found, the displacement of particle p in mode n is $y_{p,n} = A_{p,n}\cos\omega_n t$. In the case $N = 5$ ($N+1 = 6$), $\omega_1 = 2\omega_0 \sin\frac{\pi}{2(6)} \approx 0.51\omega_0$ (**27:50**). It is a useful exercise to calculate the other four mode frequencies and compare them to what is in the lecture. Note that the numbers are not in a simple ratio. A general solution will be a linear superposition of all of these normal modes. Because the ratios of the frequencies are not ratios of integers, superpositions of these modes will never repeat exactly the shape at any given time! (**32:30**) A sketch shows the particles lying on sinusoids, but it is pointed out that they are really connected with straight lines, not arcs, and particles often do not reach the nominal amplitude of their mode (if they are not in the right place). A computer demo follows (**36:00**). Individual modes are shown, and then superpositions ("cocktails"). The simple normal modes, when all added together, lead to a very complex or even chaotic motion (**42:50**). Longitudinal motion can be treated in a mathematically identical way.

Now consider a continuous system, a good approximation for a solid material (for example a string) which has many atoms (**45:00**). A disturbance moving down a string, upon reflection at a fixed endpoint, comes back inverted. The analysis for a small length of string is very similar to that for a bead on a string (**49:45**). Again, we assume tension is constant, due to low amplitude, and we do not consider any motion in the x direction, only y.

[1] Pages in "French", mentioned from time to time in the lectures, refer to pages in the textbook used at that time for 8.03: A.P. French *Vibrations and Waves (1971) ISBN: 9780393099362.*

The force acting on a short length of string with an angle from the horizontal of θ at its left end and $\theta + \Delta\theta$ at its right end is (**52:20**):



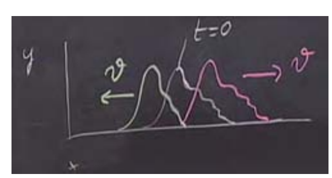
$$F_y = -T\sin\theta + T\sin(\theta + \Delta\theta) \approx -T\theta + T(\theta + \Delta\theta) = T\Delta\theta$$

We take the mass of the short length of string to be dm and Newton's Second Law for this is written as $(dm)\ddot{y} = T\Delta\theta$, and if we have a mass per unit length of μ, then the mass is $\mu\Delta x$. This gives $(\mu\Delta x)\ddot{y} = T\Delta\theta$. From the geometry, for small Δx, the tangent of the angle can be related to the derivative of the function describing the shape of the string $\tan\theta = \dfrac{\partial y}{\partial x}$. The symbol "$\partial$" (a "partial derivative") is used because y is a function both of position x, and of time t. When taking a partial derivative with respect to one of several variables, the others are held fixed.

The forces acting at any instant of time depend on the tension in the string and the angle it is away from equilibrium. So in doing Newton's Second Law, we wish to take the derivative of y only with respect to x, not taking into account that the string is also changing its displacement with time. The discussion near (**54:20**) involving $\tan\theta$ is more complicated than it needs to be. Like $\sin\theta$, $\tan\theta$ also reduces to θ (in radians) in the small angle approximation. Thus, we can write $\theta = \dfrac{\partial y}{\partial x}$ and then take $\dfrac{d\theta}{dx} = \dfrac{\partial^2 y}{\partial x^2}$. The force equation $(\mu\Delta x)\ddot{y} = T\Delta\theta$ can be rewritten as $\mu\ddot{y} = T\dfrac{\Delta\theta}{\Delta x} \approx T\dfrac{d\theta}{dx} = T\dfrac{\partial^2 y}{\partial x^2}$.

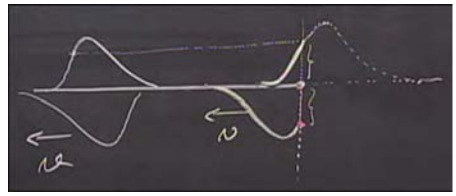
To stress that $\ddot{y}$ reflects time change only, we write it $\dfrac{\partial^2 y}{\partial t^2}$, so we get $\dfrac{\mu}{T}\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2}$ (**56:50**). (The reference to the "18.0 whatever people" is to the MIT Math department.)

The solution of this equation is actually very easy. *Any* function of the form $y = f(x \pm Ct)$ will solve this equation. Using the chain rule, the second derivative of y with respect to time is C^2 times $f''(x \pm Ct)$, the second derivative of f with respect to its argument, whereas the second derivative with respect to x is just $f''(x \pm Ct)$. (Note, the argument itself is $x \pm Ct$.) This will satisfy the wave equation if $C = \sqrt{T/\mu}$. Since T is a force with units kg·m/s^2, and μ is kg/m, it is clear the units of C are m/s, so it is a speed (**58:45**). We can use v as a symbol instead of C. Then we can write the wave equation as $\dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2}$.



The function $f(x - vt)$ will have the same value whenever $x - vt$ has the same value. Since t always increases, this means we get the same value of f whenever x has also increased to offset the greater value of vt. For the opposite case, $f(x+vt)$, $x+vt$ must have the same value. Since t

always increases, this means we get the same value of f whenever x has decreased to offset the lesser value of vt. So the $-$ sign case corresponds to a function moving toward $+x$ and the $+$ sign case, to one moving toward $-x$ (**1:02:00**).



An important discussion starts near (**1:03:20**). How can one generate an exact negative pulse when reflecting one coming in? Simply holding the end fixed automatically generates the forces needed to make this negative pulse! This explains why a "mountain comes back as a valley" (**1:07:30**). If this force was *not* applied, meaning that the end could move up and down freely, then no inversion would occur (**1:09:00**). This is a very different boundary condition. In this case, the *slope* of the string at the end must be zero, because otherwise the massless end of the string would undergo infinite acceleration. The fixed and moving alternatives are referred to as "closed" and "open" ends, respectively (**1:11:50**). For an open end, reflection still occurs but without the inversion of the pulse. Note that the moving end case gives twice the amplitude at the time of reflection, in a sense the "opposite" of the amplitude being zero always for a fixed end. This is demonstrated with a set of coupled torsional oscillators (**1:14:10**).

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These viewing notes were written by Prof. Martin Connors in collaboration with Dr. George S.F. Stephans.