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PROFESSOR: OK, so I want to start out by reviewing a few things and putting some machinery together. Unfortunately, this thing is sort of stuck. We're going to need a later, so I don't know. I'll put it up for now. So first just a bit of notation. This symbol, you should think of it like the dot product, or the inner product. It's just saying that bracket f g is the integral. It's a number that you get.

So this is a number that you get from the function f and the function g by taking f, taking its complex conjugate, multiplying it by g, and then integrating overall positions. All right? So it's a way to get a number. And you should think about it as the analog for functions of the dot product for vectors. It's a way to get a number out of two vectors.

And so, for example, with vectors we could do v dot w, and this is some number. And it has a nice property that v dot v, we can think it as v squared, it's something like a length. It's strictly positive, and it's something like the length of a vector. Similarly, if I take f and take its bracket with f, this is equal to the integral dx of f squared, and in particular, f could be complex, so f norm squared.

This is strictly non-negative. It could vanish, but it's not negative at a point, hence the norm squared. So this will be zero if and only if what? f is 0, f is the 0 function, right. So the same way that if you take a vector, and you take its dot product with itself, take it the norm, it's 0 if an only if the vector is 0. So this beast satisfies a lot of the properties of a dot product. You should think about it as morally equivalent. We'll talk about that in more detail later.

Second, basic postulate of quantum mechanics, to every observable is associated an operator, and it's an operator acting on the space of functions or on the space of

wave functions. And to every operator corresponding to an observable in quantum mechanics are associated a special set of functions called the eigenfunctions, such that when the operator acts on that function, it gives you the same function back times a constant. What these functions mean, physically, is they are the wave functions describing configurations with a definite value of the corresponding observable. If I'm in an eigenfunction of position with eigenvalue x naught, awesome. Thank you, AV person, thank you.

So if your system is described by a wave function which is an eigenfunction of the position operator with eigenvalue x naught, that means you can be confident that the system is in the configuration corresponding to having a definite position x naught. Right? It's not a superposition of different positions. It is at x naught. Similarly, momentum, momentum has eigenfunctions, and we know what these guys are. These are the exponentials, e to the iKX's. They're the eigenfunctions, and those are the wave functions describing states with definite value of the momentum, of the associated observable.

Energy as an operator, energy is described by an operator, which has eigenfunctions which I'll call phi sub n, with energy as E sub n, those are the eigenvalues. And if I tell you that your wave function is the state phi sub 2, what that tells you is that the system has a definite energy, E sub 2, corresponding to that eigenvalue. Cool? And this is true for any physical observable. But these are sort of the basic ones that we'll keep focusing on, position, momentum, and energy, for the next while.

Now a nice property about these eigenfunctions is that for different eigenvalues, the associated wave functions are different functions. And what I mean by saying they're different functions is that they're actually orthogonal functions in the sense of this dot product. If I have a state corresponding to be at x 0, definite position x 0, that means they're in eigenfunction of position with eigenvalue x 0, and I have another that corresponds to being at x1, an eigenfunction of the position operator or the eigenvalue x1, then these wave functions are orthogonal to each other. And we get 0 if x 0 is not equal to x1. Everyone cool with that?

Now, meanwhile not only are they orthogonal but they're normalized in a particular way. The inner product gives me a delta function, which goes beep once, so that if I integrate against it I get a 1. Same thing with momentum. And you do this, this you're checking on the problem set. I don't remember if it was last one or this one. And for the energies, energy 1, if I know the system is in state energy 1, and let's say e sub n and e sub m, those are different states if n and m are not equal to each other. And this inner product is 0 if n and m are not equal to each other and 1 if they are. Their properly normalized. Everyone cool with that? Yeah.

AUDIENCE: Is it possible that two eigenfunctions have the same eigenvalue?

PROFESSOR: Absolutely. It is absolutely possible for two eigenfunctions to have the same eigenvalue. That is certainly possible.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, good. Thank you, this is a good technicality that I didn't want to get into, but I'll go and get into it. It's a very good question. So the question is, is it possible for two different eigenfunctions to have the same eigenvalue. Could there be two states with the same energy , different states, same energy? Yeah, that's absolutely possible. And we'll run into that. And there's nice physics encoded in it. But let's think about what that means. The subsequent question is well, if that's the case, are they really still orthogonal?

And here's the crucial thing. The crucial thing is, let's say I take one function, I'll call the function phi 1, consider the function phi 1. And let it have energy E1, so that E acting on phi 1 is equal to E1 phi 1. And let there be another function, phi 2, such that the energy operator acting on phi 2 is also equal to E1 phi 2. These are said to be degenerate. Degenerate doesn't mean you go out and trash your car, degenerate that the energies are the same. So what does this tell me?

This tells me a cool fact. If I take a wave function phi, and I will call this phi star, in honor of Shri Kulkarni, so I've got this phi star, which is a linear combination alpha phi 1 plus beta phi 2, a linear combination of them, a superposition of those two

states. Is this also an energy eigenfunction? Yeah, because if I act on phi star with E, then it's linear, so E acting on phi star is E acting on alpha phi 1, alpha's a constant, doesn't care. Phi 1 gives me an E1. Similarly, E acting on phi 2 gives me an E1.

So if I act with E on this guy, this is equal to, from both of these I get an overall factor of E1. So notice that we get the same vector back, times a constant, a common constant. So when we have degenerate eigenfunctions, we can take arbitrary linear combinations to them, get another degenerate eigenfunction. Cool? So this is like, imagine I have a vector, and I have another vector. And they share the property that they're both eigenfunctions of some operator.

That means any linear combination of them is also, right? So there's a whole vector space, there's a whole space of possible functions that all have the same eigenvalue. So now you say, well, look, are these two orthogonal to each other? No. These two? No. But here's the thing. If you have a vector space, if you have a the space, you can always find orthogonal guys and a basis for that space, yes?

So while it's not true that the eigenfunctions are always orthogonal, it is true-- we will not prove this, but we will discuss the proof of it later by pulling the mathematician out of the closet-- the proof will say that it is possible to find a set of eigenfunctions which are orthogonal in precisely this fashion, even if there are degeneracies. OK? That theorem is called the spectral theorem. And we'll discuss it later.

So it is always possible to do so. But you must be alert that there may be degeneracies. There aren't always degeneracies. In fact, degeneracies are very special. Why should two numbers happen to be the same? Something has to be forcing them to be the same. That's going to be an important theme for us. But it certainly is possible. Good question. Other questions? Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, so using the triangular brackets-- so there's another notation for the same thing, which is f g, but this carries some slightly different weight. It mean something

slightly-- you'll see this in books, and this means something very similar to this. But I'm not going to use this notation. It's called Dirac notation. We'll talk about it later in the semester, but we're not going to talk about it just yet. But when you see this, effectively it means the same thing as this. This is sort of like dialect. You know, it's like French and Quebecois. Other questions? My wife's Canadian. Other questions? OK.

So given this fact, given the fact that we can associate observables to operators, operators come with special functions, the eigenfunctions, those eigenfunctions corresponding to have a definite value of the observable, and they're orthonormal. This tells us, and this is really the statement of the spectral theorem, that any function can be expanded in a basis of states with definite values of some observable.

So for example, consider position. I claim that any wave function can be expanded as a superposition of states with definite position. So here's an arbitrary function, here's this set of states with definite position, the delta functions. And I can write any function as a superposition with some coefficients of states with definite position, integrating over all possible positions, x0.

And this is also sort of trivially true, because what's this integral? Well, it's an integral, dx0 over all possible positions of this delta function. But we're evaluating at x, so this is 0 unless x is equal to x0. So I can just put in x instead of x0, and that gives me psi of x. Sort of tautological We can do the same thing for momentum eigenfunctions. I claim that any function can be expanded in a superposition of momentum eigenfunctions, where I sum over all possible values in the momentum with some weight. This psi tilde of K is just telling me how much amplitude there is at that wave number. Cool?

But this is the Fourier theorem, it's a Fourier expansion. So purely mathematically, we know that this is true. But there's also the physical statement. Any state can be expressed as a superposition of states with definite momentum. There's a math in here, but there's also physics in it. Finally, this is less obvious from a mathematical

point of view, because I haven't even told you what energy is, any wave function can be expanded in states with definite energy.

So this is a state, my state En, with definite energy, with some coefficient summed over all possible values of the energy. Given any physical observable, any physical observable, momentum, position, angular momentum, whatever, given any physical observable, a given wave function can be expanded as some superposition of having definite values of that. Will it in general have definite values of the observable? Well a general state be an energy eigenfunction? No. But any state is a superposition of energy eigenfunctions. Will a random state have definite position? Certainly not. You could have this wave function. Superposition. Yeah.

AUDIENCE: Why is the energy special such that you can make an arbitrary state with a countable number of energy eigenfunctions rather than having to do a continuous spectrum?

PROFESSOR: Excellent question. So I'm going to phrase that slightly differently. It's an excellent question, and we'll come to that at the end of today's lecture. So the question is, those are integrals, that is a sum over discrete things. Why? Why is the possible values of the position continuous, possible values of momentum continuous, and possible values of energy discrete? The answer to this will become apparent over the course of your next few problem sets. You have to do some problems to get your fingers dirty to really understand this. But here's the statement, and we'll see the first version of this at the end of today's lecture.

Sometimes the allowed energies of a system, the energy eigenvalues, are discrete. Sometimes they are continuous. They will be discrete when you have bound states, states that are trapped in some region and aren't allowed to get arbitrarily far away. They'll be continuous when you have states that can get arbitrarily far away. Sometimes the momentum will be allowed to be discrete values, sometimes it will be allowed to be continuous values. And we'll see exactly why subsequently.

But the thing I want to emphasize is that I'm writing this to emphasize that it's possible that each of these can be discrete or continuous. The important thing is

that once you pick your physical system, you ask what are the allowed values of position, what are the allowed values of momentum, and what are the allowed values of energy. And then you sum over all possible values.

Now, in the examples we looked at yesterday, or last lecture, the energy could have been discrete, as in the case of the infinite well, or continuous, as in the case of the free particle. In the case of a continuous particle this would have been an integral. In the case of the system such as a free particle, where the energy could take any of a continuous number of possible values, this would be a continuous integral. To deal with that, I'm often going to use the notation, just shorthand, integral sum. Which I know is a horrible bastardization of all that's good and just, but on the other hand, emphasizes the fact that in some systems you will get continuous, in some systems discrete, and sometimes you'll have both continuous and discrete.

For example, in hydrogen, in hydrogen we'll find that there are bound states where the electron is stuck to the hydrogen nucleus, to the proton. And there are discrete allowed energy levels for that configuration. However, once you ionize the hydrogen, the electron can add any energy you want. It's no longer bound. It can just get arbitrarily far away. And there are an uncountable infinity, a continuous set of possible states. So in that situation, we'll find that we have both the discrete and continuous series of possible states. Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, sure, if you work on a lattice. So for example, consider the following quantum system. I have an undergraduate. And that undergraduate has been placed in 1 of 12 boxes. OK? Now, what's the state of the undergraduate? I don't know. Is it a definite position state? It might be. But probably it's a superposition, an arbitrary superposition, right? Very impressive undergraduates at MIT. OK, other questions. Yeah.

AUDIENCE: Do these three [INAUDIBLE] hold even if the probability changes over time?

PROFESSOR: Excellent question. We'll come back to that. Very good question, leading question.

OK, so we have this. The next thing is that energy eigenfunctions satisfy some very special properties. And in particular, energy eigenfunctions have the property from the Schrodinger equation $i\hbar\, d_t$ on psi of x and t is equal to the energy operator acting on psi of x and t.

This tells us that if we have psi x 0 time t 0 is equal to phi n of x, as we saw last time, then the wave function, psi at x at time t is equal to phi n of x. And it only changes by an overall phase, e to the minus $i E_n t$ over $\hbar$. And this ratio E_n upon $\hbar$ will often be written omega n is equal to E_n over $\hbar$. This is just the Dupre relations. Everyone cool with that? So are energy eigenfunctions-- how to say. No wave function is more morally good than another. But some are particularly convenient.

Energy eigenfunctions have the nice property that while they're not in a definite position and they don't necessarily have a definite momentum, they do evolve over time in a particularly simple way. And that and the superposition principle allow me to write the following. If I know that this is my wave function at psi at x at time 0, so let's say in all these cases, this is psi of x at time 0, how does this state evolve forward in time? It's kind of complicated.

How does this description, how does psi tilde of k evolve forward in time? Again, kind of complicated. But when expressed in terms of the energy eigenstates, the answer to how it evolves forward in time is very simple, because I know that this is a superposition, a linear combination of states with definite energy. States with definite energy evolve with a phase. And the Schrodinger equation is linear, so solutions of the Schrodinger equation evolve to become solutions of the Schrodinger equation. So how does this state evolve forward in time? It evolves forward with a phase, e to the minus i omega n t. One for every different terms in this sum. Cool?

So we are going to harp on energy functions, not because they're more moral, or more just, or more good, but because they're more convenient for solving the time evolution problem in quantum mechanics. So most of today is going to be about this

expansion and qualitative features of energy eigenfunctions. Cool? OK. And just to close that out, I just want to remind you of a couple of examples that we did last time, just get them on board.

So the first is a free particle. So for free particle, we have that our wave functions-- well, actually let me not write that down. Actually, let me skip over the free particle, because it's so trivial. Let me just talk about the infinite well. So the potential is infinite out here, and it's 0 inside the well, and it goes from 0 to L. This is just my choice of notation. And the energy operator, as usual, is p squared upon 2m plus u of x. You might say, where did I derive this, and the answer is I didn't derive this. I just wrote it down. It's like force in Newton's equations. You just declare some force and you ask, what system does is model.

So here's my system. It has what looks like a classical kind of energy, except these are all operators. And the potential here is this guy, it's 0 between 0 and L, and it's infinite elsewhere. And as we saw last time, the solutions to the energy eigenvalue equation are particularly simple. Phi sub n of x is equal to root properly normalized 2 upon L sine of Kn x, where kn is equal to n plus 1 pi, where n is an integer upon L.

And these were chosen to satisfy our boundary conditions, that the wave function must vanish here, hence the sine, and K was chosen so that it turned over and just hit 0 as we got to L. And that gave us that the allowed energies were discrete, because the En, which you can get by just plugging into the energy eigenvalue equation, was equal to h bar squared Kn squared upon 2m. So this tells us a nice thing.

First off, in this system, if I take a particle and I throw it in here in some arbitrary state so that at time t equals zero the wave function x 0 is equal to sum over n phi n of x Cn. OK? Can I do this? Can I just pick some arbitrary function which is a superposition of energy eigenstates? Sure, because any function is. Any function can be described as a superposition of energy eigenfunctions. And if I use the energy eigenfunctions, it will automatically satisfy the boundary conditions. All good things will happen. So this is perfectly fine initial condition.

What is the system at time t? Yeah, we just pick up the phases. And what phase is this guy? It's this, e to the minus i omega n t. And when I write omega n, let me be more explicit about that, that's En over h bar. So that's h bar Kn squared upon 2m t. Cool? So there is our solution for arbitrary initial conditions to the infinite square well problem in quantum mechanics. And you're going to study this in some detail on your problem set.

But just to start with a little bit of intuition, let's look at the wave functions and the probability distributions for the lowest lying states. So for example, let's look at the wave function for the ground state, what I will call psi sub 0. And this is from 0 to L. And I put these bars here not because we're looking at the potential. I'm going to be plotting the real part of the wave function. But I put these walls here just to emphasize that that's where the walls are, at x equals 0 and x equals L.

So what does it look like? Well, the first one is going to sine of Kn x. n is 0. Kn is going to be pi upon L. So that's again just this guy. Now, what's the probability distribution associated with psi 0? Where do you find the particle? So we know that it's just the norm squared of this wave function and the norm squared is here at 0, it's 0 and it rises linearly, because sine is linear for small values. That makes this quadratic, and a maximum, and then quadratic again. So there's our probability distribution.

Now, here's a funny thing. Imagine I take a particle, classical particle, and I put it in a box. And you put it in a box, and you tell it, OK, it's got some energy. So classically it's got some momentum. So it's sort of bouncing back and forth and just bounces off the arbitrarily hard walls and moves around. Where are you most likely to find that particle? Where does it spend most of its time? It spends the same amount of time at any point. It's moving at constant velocity. It goes boo, boo, boo, boo, right? So what's the probability distribution for finding it at any point inside, classically? Constant. Classically, the probability distribution is constant. You're just as likely to find it near the wall as not near the wall.

However, quantum mechanically, for the lowest lying state that is clearly not true.

You're really likely to find it near the wall. What's up with that? So that's a question that I want to put in your head and have you think about. You're going to see a similar effect arising over and over. And we're going to see at the very end that that is directly related, the fact that this goes to 0, is directly related, and I'm not kidding, to the transparency of diamond. OK, I think it was pretty cool. They're expensive. It's also related to the transparency of cubic zirconium, which I guess is less impressive.

So the first state, again, let's look at the real part of psi 1, the first excited state. Well, this is now a sine with one extra-- with a 2 here, 2 pi, so it goes through 0. So the probability distribution associated with psi 1, and I should say write this as a function of x, looks like, well, again, it's quadratic. But it has a 0 again in the middle. So it's going to look like-- oops, my bad art defeats me. OK, there we go. So now it's even worse. Not only is unlikely to be out here, it's also very unlikely to be found in the middle. In fact, there is 0 probability you'll find it in the middle. That's sort of surprising.

But you can quickly guess what happens as you go to very high energies. The real part of psi n let's say 10,000, 10 to the 4, what is that going to look like? Well, this had no 0s, this had one 0, and every time you increase n by 1, you're just going to add one more 0 to the sign. That's an interesting and suggestive fact. So if it's size of 10,000, how many nodes are there going to be in the middle of the domain? 10,000. And the amplitude is going to be the same. I'm not to be able to do this, but you get the idea.

And now if I construct the probability distribution, what's the probability distribution going to be? Probability of the 10,000th psi sub 10 to the 4 of x. Well, it's again going to be strictly positive. And if you are not able to make measurements on the scale of L upon 10,000, but just say like L over 3, because you have a thumb and you don't have an infinitely accurate meter, what do you see?

You see effectively a constant probability distribution. And actually, I shouldn't draw it there. I should draw it through the half, because sine squared over 2 averages to one half, or, sorry, sine squared averages to one half over many periods. So what

we see is that the classical probability distribution constant does arise when we look at very high energy states. Cool?

But it is manifestly not a good description. The classical description is not a good description. Your intuition is crappy at low energies, near the ground state, where quantum effects are dominating, because indeed, classically there was no minimum energy. Quantum effects have to be dominating there. And here we see that even the probability distribution's radically different than our intuition. Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Keep working on it. So I want you all to think about what-- you're not, I promise you, unless you've already seen some quantum mechanics, you're not going to be able to answer this question now. But I want you to have it as an uncomfortable little piece of sand in the back of your oyster mind-- no offense-- what is causing that 0? Why are we getting 0? And I'll give you a hint. In quantum mechanics, anytime something interesting happens it's because of superposition and interference. All right.

So with all that said, so any questions now over this story about energy eigenfunctions expanding in a basis, et cetera, before we get moving? No, OK. In that case, get out your clickers. We're going to test your knowledge. Channel 41, for those of you who have to adjust it.

[CHATTER]

Wow. That's kind of worrying. Aha. OK, ready? OK, channel 41, and here we go. So go ahead and start now. Sorry, there was a little technical glitch there.

So psi 1 and psi 2 are eigenstates. They're non-degenerate, meaning the energies are different. Is a superposition psi 1 plus psi 2 also an eigenstate? All right, four more seconds. All right. I want everyone to turn to the person next to you and discuss this. You've got about 30 seconds to discuss, or a minute.

[CHATTER]

All right. I want everyone, now that you've got an answer, click again, put in your current best guess. Oh, wait, sorry. For some reason I have to start over again. OK, now click. This is the best. I'm such a convert to clickers, this is just fantastic. So you guys went from, so roughly you all went from about 30, 60, 10, to what are we now? 8, 82, and 10. So it sounds like you guys are predicting answer b. And the answer is-- I like the suspense. There we go. B, good.

So here's a quick question. So why? And the reason why is that if we have E on psi 1 plus psi 2, this is equal to E on psi 1 plus E on psi 2, operator, operator, operator, but this is equal to E 1 psi 1 E 2 psi 2, which if E1 and E2 are not equal, which is not equal to E times psi 1 plus psi 2. Right? Not equal to E anything times psi 1 plus psi 2. And it needs to be, in order to be an eigenfunction, an eigenfunction of the energy operator. Yeah.

AUDIENCE: So I was thinking about this, if this was kind of a silly random case where one of the energies is 0. Does this only happen if you have something that's infinite?

PROFESSOR: Yeah, that's a really good question. So first off, how do you measure an energy? Do you ever measure an energy? Do you ever measure a voltage, the actual value of the scalar potential, the electromagnetic scalar potential? No. You measure a difference. Do you ever measure the energy? No, you measure a difference in energy. So the absolute value of energy is sort of a silly thing. But we always talk about it as if it's not. We say, that's got energy 14. It's a little bit suspicious.

So to answer your question, there's nothing hallowed about the number 0, although we will often refer to zero energy with a very specific meaning. What we really mean in that case is the value of the potential energy at infinity. So when I say energy, usually what I mean is relative to the value at infinity.

So then let me ask your question again. Your question is it possible to have energy 0? Absolutely, and we'll see that. And it's actually going to be really interesting what's true of states with energy 0 in that sense. Second part of your question, though, is how does energy being 0 fit into this?

Well, does that save us? Suppose one of the energies is 0. Then that says E on psi 1 plus psi 2 is equal to, let's say E2 is 0. Well, that term is gone. So there's just the one E1. Are we in energy eigenstate? No, because it's still not of the form E times psi 1 plus psi 2. So it doesn't save us, but it's an interesting question for the future.

All right. Next question, four parts. So the question says x and p commute to i h bar. We've shown this. Is p x equal to i h bar, and is ip plus cx the same as cx plus ip? If you're really unsure you can ask the person next to you, but you don't have to. OK, so this is looking good. Everyone have an answer in? No? Five, four, three, two, one, OK, good. So the answer is C, which most of you got, but not everyone. A bunch of you put D. So let's talk through it.

So remember what the definition of the commutator is. x with p by definition is equal to xp minus px. If we change the order here, px is equal to minus this, px minus xp. It's just the definition of the commutator. So on the other hand, if you add things, does 7 plus 6 equal 6 plus 7? Yeah. Well, of course 6 times 7 is 7 times 6. So that's not a terribly good analogy. Does the order of addition of operators matter? No. Yeah. Yeah, exactly. Exactly. So it's slightly sneaky.

OK, next question. OK, this one has five. f and g are both wave functions. c is a constant. Then if we take the inner product c times f with g, this is equal to what? Three, two, one, OK. So the answer is-- so this one definitely discuss. Discuss with the person next to you.

[CHATTER]

All right. OK, go ahead and enter your guess again, or your answer again, let it not be a guess. OK, 10 seconds. Wow. OK, fantastic. That works like a champ. So what's the answer? Yes, complex conjugation. Don't screw that one up. It's very easy to forget, but it matters a lot. Cursor keeps disappearing.

OK, next one. A wave function has been expressed as a sum of energy eigenfunctions. Here I'm calling them mu rather than phi, but same thing. Compared to the original wave function, the set of coefficients, given that we're using the

energy basis, the set of coefficients contains more or less the same information, or it can't be determined. OK, five seconds. All right. And the answer is C, great.

OK, next one. So right now we're normalizing. OK. All stationary states, or all energy eigenstates, have the form that spatial and time dependence is the spatial dependence, the energy eigenfunction, times a phase, so that the norm squared is time independent. Consider the sum of two non-degenerate energy eigenstates psi 1 and psi 2. Non-degenerate means they have different energy. Is the wave function stationary? Is the probability distribution time independent or is it time dependent?

This one's not trivial. Oh, shoot. I forgot to get it started. Sorry. It's particularly non-trivial if you can't enter your answer. Right. So go ahead and enter your answer. Whoo, yeah. This one always kills people. No chatting just yet. Test yourself, not your neighbor. It's fine to look deep into your soul, but don't look deep into the soul of the person sitting next to you. All right. So at this point, chat with your neighbor. Let me just give you some presage. The parallel strategy's probably not so good, because about half of you got it right, and about half of you got it wrong.

[CHATTER]

All right. Let's vote again. And hold on, starting now. OK, vote again. You've got 10 seconds to enter a vote. Wow. OK, two seconds. Good. So the distribution on this one went from 30, 50, 20 initially, to now it is 10, 80, and 10. Amazingly, you guys got worse. The answer is C. And I want you to discuss with each other why it's C.

[CHATTER]

All right. OK. So let me talk you through it. So the wave function, we've said psi of x and t is equal to phi 1 at x, e to the minus i omega 1 t plus phi 2 of x e to the minus i omega 2 t. So great, we take the norm squared. What's the probability to find it at x at time t. The probability density is the norm squared of this guy, psi squared, which is equal to phi 1 complex conjugate e to the plus i omega 1 t plus phi 2 complex conjugate e to the plus i omega 2t times the thing itself phi 1 of x e to the minus i

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omega 1 t plus phi 2 of x e to the minus i omega 2t, right?

So this has four terms. The first term is psi 1 norm squared. The phases cancel, right? You're going to see this happen a billion times in 804. The first term is going to be phi 1 norm squared. There's another term, which is phi 2 norm squared. Again the phases exactly cancel, even the minus i omega 2 t to the plus i omega 2 t. Plus phi 2 squared. But then there are two cross terms, the interference terms. Plus phi 1 complex conjugate phi 2 e to the i omega 1 t e to the plus i omega 1 t, i omega 1 t, and e to the minus i omega 2t, minus omega 2.

So we have a cross-term which depends on the difference in frequencies. Frequencies are like energies [INAUDIBLE] in h bar, so it's a difference in energies. And then there's another term, which is the complex conjugate of this guy, phi 2 star times phi 1 phi 2 complex conjugate phi 1 and the phases are also the complex conjugate e to the minus i omega 1 minus omega 2 t of x of x of x of x.

So is there time dependence in this, in principle? Absolutely, from the interference terms. Were we not in the superposition, we would not have interference terms. Time dependence comes from interference, when we expand in energy eigenfunctions. Cool?

However, can these vanish? When? Sorry, say again? Great, so when omega 1 equals omega 2, what happens? Time dependence goes away. But omega 1 is e 1 over h bar, omega 2 is e 2 over h bar, and we started out by saying these are non-degenerate. So if they're non-degenerate, the energies are different, the frequencies are different, so that doesn't help us.

How do we kill this time dependence? Yes. If the two functions aren't just orthogonal in a functional sense, but if we have the following. Suppose phi 1 is like this. It's 0 everywhere except for in some lump that's phi 1, and phi 2 is 0 everywhere except here. Then anywhere that phi 1 is non-zero, phi 2 is zero. And anywhere where phi 2 is non-zero, phi 1 is zero.

So this can point-wise vanish. Do you expect this to happen generically? Does it

happen for the energy eigenfunctions in the infinite square well? Sine waves? No. They have zero at isolated points, but they're non-zero generically. Yeah, so it doesn't work there. What about for the free particle? Well, those are just plain waves. Does that ever happen? No. OK, so this is an incredibly special case. We'll actually see it in one problem on a problem set later on. It's a very special case.

So technically, the answer is C. And I want you guys to keep your minds open on these sorts of questions, when does a spatial dependence matter and when are there interference terms. Those are two different questions, and I want you to tease them apart. OK? Cool? Yeah?

AUDIENCE: Is a valid way to think about this to think that you're fixing the initial [INAUDIBLE]

PROFESSOR: That's a very good way to think about it. That's exactly right. That's a very, very good question. Let me say that subtly differently, and tell me if this agrees with what you were just saying. So I can look at this wave function, and I already know that the overall phase of a wave function doesn't matter. That's what it is to say a stationary state is stationary. It's got an overall phase that's the only thing, norm squared it goes away. So I can write this as e to the minus i omega 1 t times phi 1 of x plus phi 2 of x e to the minus i omega 2 minus omega 1 t. Is that what you mean?

So that's one way to do this. We could also do something else. We could do e to the minus i omega 1 plus omega 2 upon 2 t. And this is more, I think, what you were thinking of, a sort of average frequency and then a relative frequency, and then the change in the frequencies on these two terms. Absolutely. So you can organize this in many, many ways.

But your question gets at a very important point, which is that the overall phase doesn't matter. But relative phases in a superposition do matter. So when does a phase matter in a wave function? It does not matter if it's an overall phase. But it does matter if it's a relative phase between terms in a superposition. Cool? Very good question. Other questions? If not, then I have some.

So, consider a system which is in the state-- so I should give you five-- system is in

a state which is a linear combination of n equals 1 and n equals 2 eigenstates. What's the probability that measurement will give us energy E1? And it's in this superposition. OK, five seconds. OK, fantastic. What's the answer? Yes, C, great. OK, everyone got that one.

So one's a slightly more interesting question. Suppose I have an infinite well with width L. How does the energy, the ground state energy, compare to that of a system with a wider well? So L versus a larger L. OK, four seconds. OK, quickly discuss amongst yourselves, like 10 seconds.

[CHATTER]

All right. Now click again. Yeah. All right. Five seconds. One, two, three, four, five, great. OK, the answer is A. OK, great, because the energy of the infinite well goes like K squared. K goes like 1 over L. So the energy is, if we make it wider, the energy if we make it wider is going to be lower.

And last couple of questions. OK, so t equals 0. Could the wave function for an electron in an infinite square well of width a, rather than L, be A sine squared of pi x upon a, where A is suitably chosen to be normalized? All right, you've got about five seconds left. And OK, we are at chance. We are at even odds, and the answer is not a superposition of A and B, so I encourage you to discuss with the people around you.

[CHATTER]

Great. What properties had it better satisfy in order to be a viable wave function? What properties should the wave function have so that it's reasonable? Yeah. Is it zero at the ends? Yeah. Good. Is it smooth? Yeah. Exactly. And so you can write it as a superposition. Excellent. So the answer is? Yeah.

All right. Vote again. OK, I might have missed a few people. So go ahead and start. OK, five more seconds. All right. So we went from 50-50 to 77-23. That's pretty good. What's the answer? A. Why? Is this an energy eigenstate? No. Does that

matter? No.

What properties had this wave function better satisfy to be a reasonable wave function in this potential? Say again? It's got to vanish at the walls. It's got to satisfy the boundary conditions. What else must be true of this wave function? Normalizable. Is it normalizable? Yeah. What else? Continuous. It better not have any discontinuities. Is it continuous? Great. OK. Is there any reason that this is a stupid wave function? No. It's perfectly reasonable. It's not an energy eigenfunction, but-- Yeah, cool? Yeah.

AUDIENCE: This is sort of like a math question. So to write that at a superposition, you have to write it like basically a Fourier sign series? Isn't the [INAUDIBLE] function even, though?

PROFESSOR: On this domain, that and the sines are even. So this is actually odd, but we're only looking at it from 0 to L. So, I mean that half of it. The sines are odd, but we're only looking at the first peak. So you could just as well have written that as cosine of the midpoint plus the distance from the midpoint. Actually, let me say that again, because it's a much better question I just give it shrift for.

So here's the question. The question is, look, so sine is an odd function, but sine squared is an even function. So how can you expand sine squared, an even function, in terms of sines, an odd function? But think about this physically. Here's sine squared in our domain, and here's sine.

Now what do you mean by even? Usually by even we mean reflection around zero. But I could just as well have said reflection around the origin. This potential is symmetric. And the energy eigenfunctions are symmetric about the origin. They're not symmetric about reflection around this point. But they are symmetric about reflection around this point. That's a particularly natural place to call it 0.

So I was calling them sine because I was calling this 0, but I could have called it cosine if I called this 0, for the same Kx. And indeed, can we expand this sine squared function in terms of a basis of these sines on the domain 0 to L?

Absolutely. Very good question.

And lastly, last clicker question. Oops. Whatever. OK. At t equals 0, a particle is described by the wave function we just saw. Which of the following is true about the wave function at subsequent times? 5 seconds. Whew. Oh, OK. In the last few seconds we had an explosive burst for A, B, and C. So our current distribution is 8, 16, 10, and 67, sounds like 67 is popular. Discuss quickly, very quickly, with the person next to you.

[CHATTER]

OK, and vote again. OK, five seconds. Get your last vote in. All right. And the answer is D. Yay. So let's think about the logic. Let's go through the logic here. So as was pointed out by a student up here earlier, the wave function sine squared of pi x can be expanded in terms of the energy eigenfunction. Any reasonable function can be expanded in terms of a superposition of definite energy states of energy eigenfunctions. So that means we can write psi at some time as a superposition Cn sine of n pi x upon a e to the minus i e n t upon h bar, since those are, in fact, the eigenfunctions. So we can do that.

Now, when we look at the time evolution, we know that each term in that superposition evolves with a phase. The overall wave function does not evolve with a phase. It is not an energy eigenstate. There are going to be interference terms due to the fact that it's a superposition. So its probability distribution is not time-independent. It is a superposition. And so the wave function doesn't rotate by an overall phase.

However, we can solve the Schrodinger equation, as we did before. The wave function is expanded at time 0 as the energy eigenfunctions times some set of coefficients. And the time evolution corresponds to adding two each independent term in the superposition the appropriate phase for that energy eigenstate. Cool? All right. So the answer is D. And that's it for the clicker questions. OK, so any questions on the clicker questions so far? OK, those are going to be posted on the

web site so you can go over them.

And now back to energy eigenfunctions. So what I want to talk about now is the qualitative behavior of energy eigenfunctions. Suppose I know I have an energy eigenfunction. What can I say generally about its structure? So let me ask the question, qualitative behavior. So suppose someone hands you a potential U of x. Someone hands you some potential, U of x, and says, look, I've got this potential. Maybe I'll draw it for you. It's got some wiggles, and then a big wiggle, and then it's got a big wiggle, and then-- do I want to do that? Yeah, let's do that. Then a big wiggle, and something like this.

And someone shows you this potential. And they say, look, what are the energy eigenfunctions? Well, OK, free particle was easy. The infinite square well was easy. We could solve that analytically. The next involved solving a differential equation. So what differential equation is this going to lead us to?

Well, we know that the energy eigenvalue equation is minus h bar squared upon 2 m phi prime prime of x plus U of x phi x, so that's the energy operator acting on phi, is equal to, saying that it's an energy eigenfunction, phi sub E, says that it's equal to the energy operator acting on this eigenfunction is just a constant E phi sub E of x. And I'm going to work at moment in time, so we're going to drop all the t dependence for the moment.

So this is the differential equation we need to solve where U of x is this god-awful function. Do you think it's very likely that you're going to be able to solve this analytically? Probably not. However, some basic ideas will help you get an intuition for what the wave function should look like. And I cannot overstate the importance of being able to eyeball a system and guess the qualitative features of its wave functions, because that intuition, that ability to estimate, is going to contain an awful lot of physics. So let's try to extract it.

So in order to do so, I want to start by massaging this equation into a form which is particularly convenient. So in particular, I'm going to write this equation as phi sub E prime prime. So what I'm going to do is I'm going to take this term, I'm going to

notice this has two derivatives, this has no derivatives, this has no derivatives. And I'm going to move this term over here and combine these terms into E minus U of x, and I'm going to divide each side by 2m upon h bar squared with a minus sign, giving me that phi prime prime of E of x upon phi E of x dividing through by this phi E is equal to minus 2m over h bar squared. And let's just get our signs right. We've got the minus from here, so this is going to be E minus U of x.

So you might look at that and think, well, why is that any better than what I've just written down. But what is the second derivative of function? It's telling you not its slope, but it's telling you how the slope changes. It's telling about the curvature of the function. And what this is telling me is something very, very useful. So for example, let's look at the function. Let's assume that the function is real, although we know in general it's not. Let's assume that the function is real for simplicity.

So we're going to plot the real part of phi in the vertical axis. And this is x. Suppose the real part of phi is positive at some point. Phi prime prime, if it's positive, tells us that not only is the slope positive, but it's increasing. Or it doesn't tell us anything about the slope, but it tells us that whatever the slope, it's increasing. If it's negative, the slope is increasing as we increase x. If it's positive, it's increasing as we increase x.

So it's telling us that the wave function looks like this, locally, something like that. If phi is negative, if phi is negative, then if this quantity is positive, then phi prime prime has to be negative. But negative is curving down. So if this quantity, which I will call the curvature, if this quantity is positive, it curves away from the axis. So this is phi prime prime over phi greater than 0. If this quantity is positive, the function curves away from the axis. Cool?

If this quantity is negative, phi prime prime upon phi less than 0, exactly the opposite. This has to be negative. If phi is positive, then phi prime prime has to be negative. It has to be curving down. And similarly, if phi is negative, then phi prime prime has to be positive, and it has to curve up. So if this quantity is positive, if the curvature is positive, it curves away from the axis. If the curvature is negative, if this

quantity is negative, it curves towards the axis.

So what does that tell you about solutions when the curvature is positive or negative? It tells you the following. It tells you that, imagine we have a function where phi prime prime over phi is constant. And in particular, let's let phi prime prime over phi be a constant, which is positive. And I'll call that positive constant kappa squared. And to emphasize that it's positive, I'm going to call it kappa squared. It's a positive thing. It's a real number squared. What does the solution look like?

Well, this quantity is positive. It's always going to be curving away. So we have solutions that look like this or solutions that look like this. Can it ever be 0? Yeah, sure, it could be an inflection point. So for example, here the curvature is positive, but at this point the curvature has to switch to be like this. What functions are of this form? Let me give you another hint. Here's one. Is this curvature positive? Yes. What about this one? Yup. Those are all positive curvature. And these are exponentials.

And the solution to this differential equation is e to the plus kappa x or e to the minus kappa x. And an arbitrary solution of this equation is a superposition A e to the kappa x plus B e to the minus kappa x. Everyone cool with that? When this quantity is positive, we get growing and collapsing exponentials. Yeah?

On the other hand, if phi prime prime over phi is a negative number, i.e. minus what I'll call k squared, then the curvature has to be negative. And what functions have everywhere negative curvature? Sinusoidals. Cool? And the general solution is A e to the i K x plus B e to the minus i K x. So that differential equation, also known as sine and cosine. Cool?

So putting that together with our original function, let's bring this up. So we want to think about the wave functions here. But in order to think about the energy eigenstates, we need to decide on an energy. We need to pick an energy, because you can't find the solution without figuring the energy. But notice something nice here. So suppose the energy is e. And let me just draw E. This is a constant. The

energy is this. So this is the value of E. Here we're drawing potential. But this is the value of the energy, which is a constant. It's just a number.

If you had a classical particle moving in this potential, what would happen? It would roll around. So for example, let's say you gave it this energy by putting it here. And think of this as a gravitational potential. You put it here, you let go, and it falls down. And it'll keep rolling until it gets up here to the classical turning point. And at that point, its kinetic energy must be 0, because its potential energy is its total energy, at which point it will turn around and fall back. Yes?

If you take your ball, and you put it here, and you let it roll, does it ever get here, to this position? No, because it doesn't have enough energy. Classically, this is a forbidden position. So given an energy and given a potential, we can break the system up into classically allowed zones and classically forbidden zones. Cool?

Now, in a classically allowed zone, the energy is greater than the potential. And in a classically forbidden zone, the energy is less than the potential. Everyone cool with that? But this tells us something really nice. If the energy is greater than the potential, what do you know about the curvature? Yeah. If we're in a classically allowed zone, so the energy is greater than the potential, then this quantity is positive, there's a minus sign here, so this is negative. So the curvature is negative. Remember, curvature is negative means that we curve towards the axis.

So in a classically allowed region, the wave function should be sinusoidal. What about in the classically forbidden regions? In the classically forbidden regions, the energy is less than the potential. That means in magnitude this is less than this, this is a negative number, minus sign, the curvature is going to be minus times a minus is a positive, so the curvature's positive. So the solutions are either growing exponentials or shrinking exponentials or superpositions of them. Everyone cool with that?

So let's think about a simple example. Let's work through this in a simple example. And let me give you a little bit more board space. Simple example would be a potential that looks like this. And let's just suppose that we want to find an energy

eigenfunction with energy that's E. Well, this is a classically allowed zone, and these are the classically forbidden regions.

Now I want to ask, what does the wave function look like? And I don't want to draw it on top of the energy diagram, because wave function is not an energy. Wave function is a different quantity, because it's got different axes and I want it drawn on a different plot. So but as a function of x-- so just to get the positions straight, these are the bounds of the classically allowed and forbidden regions. What do we expect? Well, we expect that it's going to be sinusoidal in here. We expect that it's going to be exponential growing or converging out here, exp.

But one last important thing is that not only is the curvature negative in here in these classically allowed regions, but the magnitude of the curvature, how rapidly it's turning over, how big that second derivative is, depends on the difference between the energy and the potential. The greater the difference, the more rapid the curvature, the more rapid the turning over and fluctuation. If the differences between the potential and the true energy, the total energy, is small, then the curvature is very small. So the derivative changes very gradually.

What does that tell us? That tells us that in here the wave function is oscillating rapidly, because the curvature, the difference between the energy and the potential is large, and so the wave function is oscillating rapidly. As we get out towards the classical turning points, the wave function will be oscillating less rapidly. The slope will be changing more gradually.

And as a consequence, two things happen. Let me actually draw this slightly differently. So as a consequence two things happen. One is the wavelength gets longer, because the curvature is smaller. And the second is the amplitude gets larger, because it keeps on having a positive slope for longer and longer, and it takes longer to curve back down. So here we have rapid oscillations. And then the oscillations get longer and longer wavelength, until we get out to the classical turning point.

And at this point, what happens? Yeah, it's got to be [INAUDIBLE]. Now, here we

have some sine, and some superpositions of sine and cosines, exponentials. And in particular, it arrives here with some slope and with some value. We know this side we've got to get exponentials. And so this sum of sines and cosines at this point must match the sum of exponentials. How must it do so?

What must be true of the wave function at this point? Can it be discontinuous? Can its derivative be discontinuous? No. So the value and the derivative must be continuous. So that tells us precisely which linear combination of positive growing and shrinking exponentials we get. So we'll get some linear combination, which may do this for awhile. But since it's got some contribution of positive exponential, it'll just grow exponentially off to infinity.

And as the energy gets further and further away from the potential, now in their negative sine, what happens to the rate of growth? It gets more and more rapid. So this just diverges more and more rapidly. Similarly, out here we have to match the slope. And we know that the curvature has to be now positive, so it has to do this. So two questions. First off, is this sketch of the wave function a reasonable sketch, given what we know about curvature and this potential of a wave function with that energy? Are there ways in which it's a bad estimate?

AUDIENCE: [INAUDIBLE]

PROFESSOR: OK, excellent.

AUDIENCE: On the right side, could it have crossed zero?

PROFESSOR: Absolutely, it could have crossed zero. So I may have drawn this badly. It turned out it was a little subtle. It's not obvious. Maybe it actually punched all the way through zero, and then it diverged down negative. That's absolutely positive. So that was one of the quibbles you could have.

Another quibble you could have is that it looks like I have constant wavelength in here. But the potential's actually changing. And what you should chalk this up to, if you'll pardon the pun, is my artistic skills are limited. So this is always going to be sort of inescapable when you qualitatively draw something. On a test, I'm not going

to bag you points on things like that. That's what I want to emphasize.

But the second thing, is there something bad about this wave function? Yes, you've already named it. What's bad about this wave function? It's badly non-normalizable. It diverges off to infinity out here and out here. What does that tell you? It's not physical. Good. What else does it tell you about this system? Sorry? Excellent. Is this an allowable energy? No.

If the wave function has this energy, it is impossible to make it continuous, assuming that I drew it correctly, and have it converge. Is this wave function allowable? No, because it does not satisfy our boundary conditions. Our boundary conditions are that the wave function must vanish out here and it must vanish out here at infinity in order to be normalizable. Here we failed.

Now, you can imagine that-- so let's decrease the energy a little bit. If we decrease the energy, our trial energy just a little tiny bit, what happens? Well, that's going to decrease the curvature in here. We decrease, we bring the energy in just a little tiny bit. That means this is a little bit smaller. The potential stays the same. So the curvature in the allowed region is just a little tiny bit smaller. And meanwhile, the allowed region has got just a little bit thinner.

And what that will do is the curvature's a little less, the region's a little less, so now we have-- Sorry, I get excited. And if we tweak the energy, what's going to happen? Well, it's going to arrive here a little bit sooner. And let's imagine something like this. And if we chose the energy just right, we would get it to match to a linear combination of collapsing and growing exponentials, where the contribution from the growing exponential in this direction vanishes.

There's precisely one value of the energy that lets me do that with this number of wiggles. And so then it goes through and does its thing. And we need it to happen on both sides. Now if I take that solution, so that it achieves convergence out here, and it achieves convergence out here, and I take that energy and I increase it by epsilon, by just the tiniest little bit, what will happen to this wave function? It'll diverge. It will no longer be normalizable.

When you have classically forbidden regions, are the allowed energies continuous or discrete? And that answers a question from earlier in the class. And it also is going to be the beginning of the answer to the question, why is the spectrum of hydrogen discrete. See you next time.