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**PROFESSOR:** OK, so, we're going to pick up on our study of periodic potentials and our search for the explanation of the physics of solids. And I want to quickly remind you of the logic we went through last time. This is one of the more subtle bits of logic in the semester, so I'm gonna do it again, a little faster. I'm mostly gonna do it in a slightly different form. I want to think for a moment about just a free particle, but I want to use all the language and formalism we used for the periodic potential last time-- translation operators, the block wave functions-- I want to use all of that for just a free particle.

So, first let's just remind ourselves how the free particle story goes. So, for a free particle, by which I mean the energy operator is  $p^2/2m$ , and no potential, plus 0. We have a very important consequence of being free-- that the energy can be written as just  $p^2$ -- which is that the energy commutes with the momentum operator. And what this tells us is that we can find a basis of eigenfunctions which are common eigenfunctions of  $E$  and  $p$ , right? We can expand every wave function at any arbitrary-- fantastic. Thank you, sir. Oh! Brilliant!

**GUEST:** Plan A, Plan B.

**PROFESSOR:** I don't know. That's an awful huge straw. Pay no attention to the device beneath the table. So, 'cause you know, it's quantum mechanics, right? Bowl of water--

So, the energy and momentum commute. And as a consequence, we can find a basis of eigenfunctions exists which are common eigenfunction of  $E$ ,  $E$  on  $\phi$ , and I'll label  $k$ , by the eigenvalue of  $k$ , is equal to  $E \phi = E_k \phi$ . And  $p$  on  $\phi = \hbar k \phi$ . You'll notice, of course, that I don't label it by the eigenvalue, but I label it by something that's determined by the label, the eigenvalue's determined by the

labels,  $\hbar k$ .

Now, here's the thing, can we allow  $E$  and  $k$  to be any values we want independently? Can I make  $E$ , 3, and  $k$ , 47? No. There's a consistency relation, which involves satisfying the energy eigenvalue equation. We actually have to find solutions of these equations. And once we find the solutions are those equations, we'll find relations between  $k$  and  $E$ . And in particular, for a free particle, we find that the relation that we need is  $E$  is equal to-- and I will write  $E_{k--}$  is equal to  $\hbar^2 k^2 / 2m$ . This is also known as  $\omega$  is equal to  $\hbar k^2 / 2m$ . Just to remind you, the [INAUDIBLE]  $E$  is  $\hbar \omega$ ,  $\omega$  sub  $k$ . That's not the thing I wanted to look at. This is.

And when we work this way, we find that the energy eigenvalues,  $E$ , as a function of  $k$ , take a determined form. And that determined form is quite simple, it's a parabola. Everyone cool with that? So, if you know  $k$ , you know  $E$ . But if you know  $E$ , it turns out it's doubly degenerate. So, if you know  $E$ , you're one of two  $k$ 's. You could be either the  $ikx$ -- oh, sorry, and I needed to say-- that the function, so, from solving the equation, gave us that the energy eigenfunctions,  $\phi_{E_k}$  are equal to some normalization  $1/\sqrt{2\pi}$  as conventional,  $e^{ikx}$ .

OK. So, if I tell you the energy, you don't know which momentum state I'm talking about. But if I tell you the momentum, you know the energy. So, there's a little degeneracy. That degeneracy is coming from the parity symmetry, the system, right? And note that what we've done here, did we have to write the wave functions in terms of these, the  $ikx$ 's. Is there another basis we could've used for the energy eigenstates? Yeah. Because these are degenerate, we could've taken any linear combinations of  $E$  to the  $ikx$  and  $E$  to the minus  $ikx$ , and we could have got, for example, sine and cosine. That would have had the advantage that the energy eigenfunctions would have been pure real, which is one we proved was possible. You can always choose pure real energy eigenfunctions. However, it would have had the disadvantage that these states would not-- sine and cosine-- are not eigenstates of momentum, right?

It's convenient to work with common eigenstates of momentum and energy, so we give up on having a real wave function and that's the cost of finding shared eigenfunctions of energy and momentum. Everyone cool? Any questions? OK. So, this is the usual story for the free particle. But we learn a couple of things about it immediately. Just by eyeballing the wave function we see some things to note. Note, all states-- all eigenfunctions--  $\psi_E$ , which are of the form  $e^{i(kx - \omega t)}$ -- and I'm gonna add the time dependence in now--  $e^{i(kx - \omega t)}$ , are extended. They have equal probability density everywhere, and the probability distribution does not fall off to 0 as we go far away.

And so, what this tells us we have to do is we have to build-- if we want to study real localized states that are normalizable-- we have to build wave packets of the form  $\psi(x,t) = \int dk \sum_{\text{overall momentum modes}} f(k) e^{i(kx - \omega t)}$ . Where this  $f$  we will generally take-- although we don't have to take, to be a real function-- but, is peaked at some  $k_0$ . So, this is what we did when we studied the evolution of a wave packet of this form. But in particular, such wave packets have a couple of nice properties. The first property we've studied. And if you're not totally comfortable with the idea of group velocity, you really need to go over it again. It's be an excellent thing to show up in recitation, for example, stationary phase. So, the group velocity of such a wave packet-- as long as it's of this form and peaked around a single momentum value,  $k_0$ -- the group velocity is  $d\omega/dk$  evaluated at  $k_0$ , at the peak value of the distribution.

And note, I wanna just quickly note that this is equal for the free particle.  $\omega = \hbar k^2 / 2m$ , take a derivative with respect to  $k$ , the 2 cancels. We get  $\hbar k$  upon  $m$ . Which notably, is equal to the expectation value of the momentum over the mass. Everyone cool with that? So, this gives us a nice way to, by the way, if you didn't know the mass, but you could measure the system, for example, you can compute on average the momentum, and you could compute the group velocity by watching wave packets move, this is a nifty way to compute the mass. The mass can be given by, well, put the mass up here, and divide by the group velocity. Measure the momentum and divide by the group velocity. OK, so, I can take this, and notice, that this just exactly gives me the mass. So, this is a way to, given a

system with a momentum and a group velocity, I can compute an effective mass. Which for the free particle, is just the mass. Cool? Just a side observation.

But this is all a consequence of the fact that we have wave packets of this form. Their plane waves, or at least things that have a phase, times what could be a real-- though it could but an overall phase on the whole thing-- a real function, which is peaked at a particular value of  $k_0$ . Usual wave packet story. Any questions? Good. So, this should all be pretty familiar.

Now, here's the thing, when we talk about a periodic potential, what I want to do next is, I want to talk about a periodic potential. So, now, we have a system that has maybe steps, and it's periodic, with period  $l$ , OK? And then, ditto. It's gonna be periodic. And I want to run through the same analysis. I want to find the energy eigenfunctions. I want to organize them in a nice way. And in particular, I want to know the following, if this is equal to  $v$  of  $x$ , and  $v$  of  $x$ , and I call the energy operator  $E$  sub  $g$  is equal to  $p$  squared upon  $2m$  plus a coefficient,  $g$ , times-- let me call it  $g$  sub  $0$ , just to make it unambiguous-- times  $v$  of  $x$ . Notice that when  $g_0$  is  $0$ , this is just the free particle. I added a periodic potential times  $0$ . That is stupid, but doable.

But as I slowly turn on  $g_0$ , what's going to happen? The energy eigenfunctions are going to slowly change. And the energy eigenvalues are going to slowly change. And the actual system we care about has some actual value of, you know, like  $7$ , or  $e$ , or whatever. But this is a way to connect the free particle to the particle in a periodic potential. You know, and if you just turn on a little weak potential, you don't expect the system to radically change. So, one way to organize the question of periodic potentials is, what happens as you slowly turn on this periodic potential?

But here's the first thing I want to notice, the first thing I want to notice is that it's no longer true that the energy commutes with the momentum. So, it would be silly to try to organize our wave functions in terms of energy eigenfunctions, which are also momentum eigenfunctions. Because that will only be a sensible thing to do when this interaction term-- when this potential-- is  $0$ , right? So, instead, I want to organize them in terms of energy eigenvalues and any other operator that

commutes with the energy eigenfunction. In particular, this is not invariant under arbitrarily translations, which it would have to be to be invariant to conserve momentum.

But it is invariant under translations by  $L$ . And as a consequence, we have that the energy operator commutes with translations by  $L$ . Where translations by  $L$ . Takes a function and shifts it over by  $L$ . And so, because translations by  $L$  as we showed in a problem set long ago commutes with  $p$  squared, because this is just an exponential of  $Lp$ , with coefficients, with an  $i$  and an  $\hbar$ . And because the potential is invariant under shifting by  $L$ , and so it commutes with  $T_L$ , this is 0. So, what this tells us is we can find common eigenfunctions. In fact, a common eigenbasis of  $E$  and  $T_L$ .

And here's the thing that's nice about this, I could've made this argument whether the potential was 0 or not. For the free particle, it's also true that the system commutes with translations by  $L$ . It's free, there is no potential. It sort of silly, because I could've just used the momentum. But let's see what happens if we use the translate by  $L$  operator. Everyone cool with that?

So, we want to find common eigenfunctions. So, as you guys showed on a problem set, the eigenfunctions of  $T_L$ -- so, in order to do this, what are the eigenfunctions of  $T_L$ ?-- you showed that the eigenfunctions of  $T_L$  must take the form-- sorry--  $\phi$  sub  $q$  of  $x$  is equal to  $E$  to the  $iqx$  times  $u$  of  $x$ , for some  $q$  where  $q$  is-- where  $u$ -- is a real periodic-- I don't really need the real, but it's going to simplify my life, so I'll take it to be real-- periodic function. Well, let's just drop real altogether, because we don't need that. We'll use a periodic function, and  $\phi$  of  $q$  is this phase. It times the periodic function. So in particular,  $\phi$  is not periodic. And as a consequence,  $T_L$  on  $\phi$   $q$  is equal to  $E$  to the  $iqL$   $\phi$   $q$ , because this guy shifts, and this is invariant periodic.

So, this is the form-- general form-- of the eigenfunctions of translate by  $L$ . And periodic means that  $u$  of  $x$  plus  $L$  is equal to  $u$  of  $x$ , just be to explicit. OK. So, what this tells us is that the energy eigenfunctions can be put in a form-- can be organized-- as  $\phi$  sub  $E_q$ , such that  $E$  acting on  $\phi$  sub  $E_q$  is equal to  $E$   $\phi$  sub  $E_q$ .

Just like this line,  $\psi_{Eq}$ . And  $T_L \psi_{Eq}$  is equal to  $E \psi_{Eq}$ . And it's just saying then we have common eigenfunctions, yeah, with energy  $E$  and  $T_L$  eigenvalue,  $q$ .

**AUDIENCE:** [INAUDIBLE]?

**PROFESSOR:** No, because under translation by  $l$ , this  $x$  goes to  $x + l$ . Oh, I've slightly changed notation from earlier in the course. You're right. I've slightly switched. So, ah! Good. So, yes. So, thank you. So, let's fix notation.  $T_L$  takes  $f$  of  $x$  to  $f$  of  $x - l$ , is what we've used previously, right? But I'm going to just switch notation, and call this  $x + l$ . So, for all the notes in here, this is  $x + l$ . Yeah, it doesn't really make a difference, you just have to be consistent. And you're right, I switched between lecture-- what was it?-- eight, and this one. Sorry about that. Yeah, good catch. Thank you for pointing that out. So, in which case, this shifts by  $x$  to  $x + l$ . And we get  $E \psi_{Eq}$ . Good catch. Good.

So, again, we have this property that we can find common eigenfunctions. However, can  $E$  and  $q$  be specified independently? We know on general grounds that it's possible to find eigenfunctions which are common eigenfunctions, because these guys commute--  $E$  and  $T_L$  commute-- so we can find common eigenfunctions. And here they are. And, meanwhile, we know that the form of these eigenfunctions is of this form. Because any eigenfunction of  $T_L$  is of this form, a phase times a periodic functions. Does that, by itself, tell us a relationship between the eigenvalues  $E$  and  $q$ ? No, right? Just like  $E$  and  $k$  were independent, until we solved the actual eigenvalue equations, right? And it required things like regularity, and no singularities.

So, what we have to do at this point, is solve the differential equations for continuity, periodicity, and solving the energy eigenvalue equations. And when we solve the equations, we'll get a relationship between  $E$  and  $q$ . And in particular, if we do that, here-- do I want to do it this way? Yes. So, in particular for-- write this in the following way-- for  $g$  equals 0--  $g_0$  equals 0, which is a free particle-- we find  $E$  is equal to  $\hbar^2 q^2$ .  $E_{q}$  is equal to  $\hbar^2 q^2$  upon  $2m$ . And yeah, and the wave functions are of the form  $\psi_{qE}$  are equal to some

normalization,  $1$  over  $2\pi$ , times  $E$  to the  $iqx$ . And the coefficient function,  $u$ , is just constant, which is nice. So, times constant.

And so, this is reassuring because it's exactly the same form of the wave functions, with exactly the same form of the energy. But where here, the label-- the value--  $q$  is representing the possible value of-- it's controlling the eigenvalue of  $T_L$ . So, there was a question?

**AUDIENCE:** Yes.

**PROFESSOR:** Yes?

**AUDIENCE:** So, for the translation operator--

**PROFESSOR:** Yes?

**AUDIENCE:** --that  $Eiqx$  equal to  $x$  will work for any complex  $q$  in any  $u$  with period [INAUDIBLE]? Or any periodic  $u$  with [INAUDIBLE]?

**PROFESSOR:** Yeah, although, if  $q$  is complex, then this is badly non-normalizable. Because if  $q$  has an imaginary part, then that function is gonna diverge in one direction or the other. So,  $q$  had better be real. But, that's correct. This works for-- this expression-- A function of this form, a periodic function-- periodic, with period  $L$ -- times an  $E$  to the  $iqx$ , for any real value  $q$ , and any periodic  $u$ , will be an eigenfunction of  $T_L$ , translation.

**AUDIENCE:** [INAUDIBLE]?

**PROFESSOR:** No. It's not just non-hermitian. It's, in fact, unitary. I mean, what exactly is your question? Sorry, I'm misunderstanding.

**AUDIENCE:** So, I mean, just the translation operator has a really, really large number of eigenfunctions?

**PROFESSOR:** Indeed, it does. And that number is the same as the number of momentum eigenfunctions. Because they can be both used as a basis. They're both operators

whose eigenfunctions form a basis. That's exactly right. So, I think the difficult thing is to think that-- we have a tendency to think that there aren't very many momentum functions, for example. But there's enough momentum eigenfunctions to be complete basis. So, if this seems like a vastly larger set, then yeah. Indeed. OK.

So, when we do our analysis for the free particle of this form, we get exactly the same story. And I want to point out a couple of consequences for this. Working with this TL operation-- or working with TL eigenfunctions--- we have exactly the same properties. Our eigenfunctions are, again, extended. We have to use wave packets. And those wave packets move with a group velocity, which is given by  $d\omega/dk$ . On the other hand, here, when we write super positions, our general wave function-- our general wave pack, which I'm gonna write here--  $\psi$ , is gonna be of the form  $\int dq \delta(E - \hbar^2 q^2/2m) u(q) e^{iqx - i\omega(q)t}$  or sorry,  $f(q) u(q)$ .

So, as long as this is a periodic function, then the-- oh, and I should say minus  $\omega(q)t$ , to get the time dependence in there-- as long as this is a nice periodic function, especially for this case was, when it's real, and this is a sharply peaked function, which is sharply peaked around some  $q_0$ , then we will find that the group velocity is, again, the  $d\omega/dq$ , evaluated at the peak value,  $q_0$ . And that's just a general consequence of this form of our wave packet. As I encourage the recitation searchers to review by stationary phase in recitation.

And secondly, again, we can find this nice notion of a mass, which is the expectation value of the momentum divided by the group velocity. And, again, this, for our simple free particle, this is just a nice way of measuring the mass, if you happen to want to do so for your free particle. No one told you the mass. Everyone cool with that? OK. Questions? Yeah?

**AUDIENCE:** Is that still the momentum when we have  $g_0$  is not equal to 0?

**PROFESSOR:** Excellent question! So, now, let's study the case with  $g_0$  not equal to 0. OK? So, so far we looked at for  $g_0$  no equal to-- or  $g$  not equal to 0-- that was just this line. But everything else, was for general case of studying common eigenfunctions of  $E$  and  $TL$ . So, now, let's study the case for an actual periodic potential. So, in particular,



repeat for  $g_0$  not equal to 0. And this is the analysis we did last time. We did exactly this analysis. And we looked for the common eigenfunctions of  $E$  and translate by  $L$ , right? And we found that the common eigenfunctions took the form  $E_q$  is equal to  $E$  to the  $iq$  of  $x$  times some  $u$  of  $x$ .

And now, we need to find what's the relationship between  $E$  and  $q$ ? Are any  $E$  and any  $q$  allowed? And do those give you eigenfunctions? From the free particle, we expect that that's certainly not the case. On the other hand, we have a guess as to what it should look like. The energy eigenfunction for-- let me, actually, write this here-- the energy eigenfunctions and energy eigenvalues, more importantly, of the free particle organized in terms of  $q$ , we just immediately see the energy is just  $\hbar^2 q^2$  upon  $2m$ . So, the potential-- or sorry, the energy as a function of  $q$ -- is, again, a parabola. OK, so, here's  $q$ , and here's the energy. And this is for a free particle, right? Free particle.  $E$  equals  $\hbar^2 q^2$  upon  $2m$ .

Now, I want to ask what happens as we slowly turn on the potential? As we take  $g_0$  not equal 0. So, we did this last time. We turned on the potential. We turned on the a potential, which is, in fact, delta functions. So, now, in order to solve the differential equations, we have to actually know what the potential is. There's no getting around it. And so, we took a bunch of delta functions with dimensionless strength  $g_0$ , and spaced by  $L$ . And what we found was an equation relating  $E$  and  $L$ , which took the following form-- and for simplicity, I'm gonna write  $E$  is equal to, this has nothing to do with momentum, although it looks like momentum, I'm just going to write  $E$  formally as  $\hbar^2 k^2$  upon  $2m$ -- so,  $k$ , here, should just be understood as  $\sqrt{2mE}$  over  $\hbar$ , which I hate writing out over, and over again. So, let me, actually, write that the other way around.

$k$  is equal to the square root of  $2mE$  upon  $\hbar^2$ . So, when you see  $k$  think  $E$ . When you see  $E$ , think  $k$ . OK. So, we found that the condition-- in order that we solve the differential equations-- the condition, in order that  $E$  and  $q$  label energy eigenstates which actually are energy eigenfunctions-- common energy eigenfunctions-- of the energy and the translation by operator, we found that

there was a relation, which was cosine of  $qL$ --

**AUDIENCE:** [SNEEZE]

**PROFESSOR:** --is equal to-- Bless you-- cosine of  $kL$  plus  $g_0$  upon  $2kL$  sine  $kL$ , right?

So, first off, let's make sure that this agrees with what we found when  $g_0$  was equal to 0. When  $g_0$  was equal to 0 we found that  $E$  was equal to  $\hbar^2 q^2$  for any  $q$ . And, well, when  $g_0$  is equal to 0, this term goes away. And we get cosine of  $qL$  is equal to cosine of  $kL$ . And an example of a solution for that is  $qL$  is equal to  $kL$ . Or  $q$  is equal to  $k$ . In which case,  $E$  is equal to  $\hbar^2 k^2$  upon  $2m$ , by plugging in that expression. Cool? So, this reproduces the free particle result when we need it to. But in general, it's not the same. So, what does it look like?

So, let's take a look at that. So, here, first off, what I'm plotting here is the plot that we draw on the board last time, only here I'm plotting it here horizontally is the energy,  $E$ , and vertically is cosine of  $qL$ . Now, for  $q$  a real number, cosine of  $qL$  goes between 1 and negative 1. And so, any value in here corresponds to a valid value of cosine-- or of  $q$ -- sorry. So, any value, any point, between 1 and minus 1 is an allowable value of  $q$ . So, for example, this point, right here, corresponds to this value is  $q$ , OK? Just horizontal lines. Meanwhile, this curve is the right hand side of that expression. It's cosine of  $kL$  plus-- or sorry-- in this case, it's cosine of  $\sqrt{2mE}$  upon  $\hbar^2 L$ . This right-hand side plugged in here, and I'm plotting it as a function of energy. Just to be explicit, so this is energy-- or energy-- and the  $q$ . And here's that curve.

So, for  $g$  equals 0-- which is what I've set it to here--  $g$  is equal to 0-- that didn't work, there we go-- when  $g$  is equal to 0, we see that it goes between 0, and this is just cosine of root  $E$ . And there's cosine of root  $E$ . It gets spread out because it's square root of  $E$ , rather than  $E$ . Everyone cool with that? And so, here's my question, what pairs of  $E$  and  $q$  correspond to allowed energy eigenfunctions? How do you look at this diagram and decide which values of  $E$  and  $q$  are good energy eigenfunctions?

**AUDIENCE:** Well, the fact that the boundary conditions of the  $q$  is such that cosine of  $qL$  is between negative 1 and 1?

**PROFESSOR:** Good

**AUDIENCE:** Or all values are allowed?

**PROFESSOR:** Awesome. So, let me be explicit. So, how would I find? That's exactly right. Let me rephrase that and be a little more explicit. So, what we need is we need a solution of cosine  $qL$  is equal to the right hand side, in this case, cosine of root energy,  $L$ . So, that means we need to find a horizontal line corresponding to a particular value of  $q$ -- or a particular value of cosine  $qL$ -- and a point on the curve, cosine of root  $EL$ , which is this guy. So, at any point where we are inside these two extreme values, and this  $E$ , will give us a value. So, for example, for this point-- ah, I wish I could draw-- for this point, that corresponds to a particular value of  $q$ , right? Where the cosine of  $q$  is equal to that vertical value. And it corresponds to a particular value of  $E$ . So, each point, here, corresponds to a value of  $E$ -- an allowed value of  $E$ -- on the horizontal, and an allowed value of  $q$ , such that cosine  $qL$  is the vertical value. Everyone cool with that? So, what val-- Yeah?

**AUDIENCE:** [INAUDIBLE]  $p$  degeneracy for values of  $q$ . Like for a specific  $q$ , we're going to find a lot of [INAUDIBLE]?

**PROFESSOR:** For specific values of  $q$ , why does it--

**AUDIENCE:** [INAUDIBLE] horizontal line. You'll have a lot of intersections, right?

**PROFESSOR:** Yeah, that's, excellent. OK. So, good. So, there are two ways of answering that. That's a very good question. So, the question is, look, this is slightly funny state of affairs. If we plot-- if we take a horizontal line-- corresponding to a particular value  $q$ , it hits at various different values of the energy. Excellent observation. So, let me make two points about that. The first point is that when we defined  $q$ , how did we define  $q$ ? We said the translate be  $L$  on  $\phi E q$  was equal to  $E$  to the  $i q L \phi e q$ . But now, suppose we take  $q$  to  $q$  plus  $2\pi$  upon  $L$ , OK? OK, so if we take  $q$  to  $q$  plus  $2\pi$

upon  $L$ , we don't do anything to the eigenvalue. So, that's not enough to specify which function we're talking about. Because we changed the value of  $q$ , and it doesn't change the eigenvalue. So, you can't just focus on the eigenvalue.

There are two ways of thinking about that. One is, look, since  $q$  and  $q + 2\pi$  upon  $L$  give you the same eigenvalue of  $TL$ , we should just think of them as equivalent. So, one, we should just-- look, any time you have  $q$ , and you add plus  $2\pi$  upon  $L$ , you just think of this as  $q$ . It's not different. There's just, you know,  $q$  now is valued periodic. If you shift  $q$  by  $2\pi$ , you're actually back at the same point. So,  $q$  is now a periodic variable. And but, that's not enough. Different values of these  $q$  correspond to different values-- the same eigenvalue. So, you're getting multiple solutions as a consequence.

But there's another way to think about this. Which is just in terms of this graph. In terms of this graph, if what we're specifying is cosine of  $qL$ , but we're not specifying the value for  $q$ , yeah? So, if we instead plotted this as a function of  $q$ , without assuming  $q$  is equivalent to  $2\pi q$  plus  $2\pi$  upon  $L$ -- which we can do, we're perfectly entitled to do that-- what do we get? What do you think you get? For the free particle, we know exactly what we get for the free particle. The parabola. And if you just take this, and I'm going to give you guys the Mathematica files for all these things. Yeah, if you just take this, and open it up, instead of plotting it in terms of cosine of  $qL$ , plot it in terms of  $q$ , that's exactly where you get. The satisfying thing. And I'm going to come back to that in just a minute, but it was a really good question and we'll come back to it. Other questions?

OK, now, let's ask what happens as we turn on  $g_0$ . As we turn on  $g_0$ , the solutions to this equation are going to be different, because it's going to be a difference equation. So, let's do it. So, here it is. We have  $g_0$ , we can tune it. And I'm going to slowly turn on  $g_0$ . And watch what happens to these curves. So, now we see that the right hand side of the yellowish curve is now exceeding 1 and minus 1 in various places, right? And why is it doing that? Let's get some reasonable down here. So, why is it doing that's? It's doing that because this can now be greater than 1. Cosine is bounded between 1 and minus 1, but this is not.

And so, we overshoot in some places. And what does this tell us? What does this really telling us? What its really telling us is that-- there we go-- how do we find allowed values of  $E$  and  $q$  such that we have a good energy eigenfunction which is simultaneously and eigenfunction of  $E$  and of  $TL$ ? We need to find points where a horizontal line between 1 and minus 1 intersects the right hand side curve. But for the energies between here-- this value of the energy, and this value of the energy--- there is no such point. The energies in here-- any energy in here-- in order for those lines to intersect, line must be greater than 1, or less than minus 1 down here, or here.

So, there's no allowed value of  $q$ , such that there's a solution with energy between these two points. Everyone cool with that? So, that's telling us that there are values of energy where there are no energy eigenvalues with that corresponding energy. Because there simply aren't solutions of this equation for any value of  $q$  with that energy. On the other hand, when the right hand side is between 1 and minus 1, we have allowed values of energy. We have a value of  $q$ , and a value of  $E$  that correspond to each other, and they correspond to a solution of the energy eigenvalue equation and the  $TL$  eigenvalue equation.

So, let's see what that looks like in this presentation. Oh, before I do, I want to say one other thing about this. Let's take this plot, and let's use this periodicity. So, this periodicity is going to say that-- So, here's  $\pi$  upon  $L$ , here's minus  $\pi$  upon  $L$ -- let's see, yeah, good-- there's minus  $\pi$  upon  $L$ , there's  $2\pi$  upon  $L$ , and  $2\pi$  upon  $L$ . If these guys are periodic, than I could've just said, look, at this point-- this is  $\pi$  upon  $L$ , and minus  $\pi$  upon  $L$ -- I could've said this point is the same as this point. So, I could've taken this whole bit of the curve, and I could've moved it over here. Everyone cool with that?

Because this value of  $q$ , is the same as this value,  $q$ . This value of  $q$ , is the same as this value of  $q$ , right? So, I could have written this, and if we just take this over, you can see it's, you know, symmetric. As a consequence, we get-- And so, we can write-- we can draw-- this entire parabola folded up into one region, using the periodicity with  $q$  as  $2\pi$  upon  $L$ . And when you fold it up into one region, just using  $q$

as periodic with period  $2\pi$  over  $L$ , and now noting that there are several solutions for every value  $q$ -- corresponding to how far out on the parabola you went-- now we have a much simpler way of presenting this, where we don't need to see the whole parabola. This is a real advantage.

When we plot things in this way, the structure of the energy bands and energy gaps becomes considerably more simple. So, let me do that. First let me set the  $g$  to 0. OK, so, here is our free particle. And let's look at that plot. So, here's the plot that I was just describing. What you see here, is the parabola. And I've use a periodicity to box it up into one fundamental period between  $\pi$  and minus  $\pi$  upon  $L$ , OK? And so, the blue line corresponds to the energy as a function of  $q$ -- the allowed energy as a function of  $q$ -- and what the green background represents is I'm going to put the horizontal section in green anytime there's an allowed energy with that. So, energy, here, is vertical, and  $q$  is on the horizontal, going between minus  $\pi$  over  $L$  and  $\pi$  over  $L$ . Yeah?

**AUDIENCE:** I might be getting this just out, but is there a reason that as the energy increases the bands get broader and broader? Is that just an artifact of the math, or is there a good physical reason for that?

**PROFESSOR:** Yeah, there're both. So, one way to think about it, so just purely mathematically, this is saying that cosine of  $qL$  is cosine of root  $EL$ . So, as we make the energy larger, and larger, some small variation in  $q$  is going to correspond to a quadratic variation in  $E$ . And that's what's making it stretched out. If we plotted this as a function of  $kL$ , instead of  $E$ , then it would have looked much more constant period, OK? So, that's the mathematical answer.

The physical answer is this, if we have some potential, if I put you in a finite well-- So, there you are. You're in a finite well. And you look at the ceiling, you're like, damn it! And so, that's frustrating, right? And it's deeply frustrating if it's a deep well, but if it's a shallow well, it's not all that big a deal, you climb out. Being in a deep potential-- being at low energy, compared to the potential-- means that you're tightly bound, right? Takes a huge amount of energy to get out. And as a consequence,

what you saw in your problem set is that the bands are quite thin. But as you get closer, and closer to the top, or as you go to higher energy, the thickness of that band grows as a measure of the fact that you're less tightly constrained by the potential. OK? Good. So, that's the physical intuition. Did that make sense? Good. OK, so, yeah?

**AUDIENCE:** [INAUDIBLE]  $g_0$  different from 0, do you ever get to the point where the solution is always inside the [INAUDIBLE]. So, like the bumps [INAUDIBLE].

**PROFESSOR:** Ah! Good. Let's look. Let's look. So, let's answer that by looking. So, the question is, basically, look, as you crank up  $g_0$ , do you ever lose the bands entirely, or do the bands just disappear? Is that the question?

**AUDIENCE:** As you get to higher energies.

**PROFESSOR:** Yeah, as you go to higher energies. Good. So, we can answer that. Let me do that in two ways. So, the first thing I want to do is I want to ask the first of that question, which is what happens as we turn on the potential? Here we have the free particle, let's turn on the potential. We know what happens down here, I want to ask what happens to the picture of the allowed values of  $q$  and the allowed value-- sorry, the allowed values of energy-- and their corresponding values of  $q$ . So, here, it's easy. This particular point on the blue line corresponds to a value of the energy, and a value of  $q$ . What happens to those allowed pairs as we increase the potential? And as we increase the potential, you can see on the left that the right hand side is exceeding 1 and minus 1. And correspondingly, on the right, here the red dot is what was the free particle-- parabola-- and the blue line is the actual solutions. What you find is when you turn on the potential, the actual solution moves away from the free particle line in a very particular way.

In particular, for this value of energy-- around 100-- there are absolutely no allowed energy eigenstates. There are no solutions. And so as correspondingly, that area is not shaded green. There are only solutions where you have this blue line. Everyone see that? So, we get these gaps. And we get bands of continuously allowed energies. And then gaps between those allowed bands. And I'm going to post this--

the guy-- to play with on Stellar.

So, now, let's go up to very, very large values of the interaction. OK, and let's see what happens when you go to large of interaction. Well, one thing that happens it that Mathematica sort of panics down here, so it lost an entire band, which shouldn't happen. Let's try this, there we go. That's better There's a wonderful book on spectral methods in solving differential equations by a guy named Boyd. And he's pithy, if somewhat degenerate, and he says at one point, the definition of an idiot is- - or he says, idiot definition, someone who doesn't reproduce their numerics twice with different parameter values. And so, we could have just lost this band and written a paper, oh, band's disappear! This is great! Never believe Mathematica until you've checked it.

So, anyway, here you see a band, and another band, and another. But what's happened is the bands become very thin. But that makes sense from the earlier intuition. We're making the potential much stronger. We're more tightly binding particles into wells. And as a consequence, the bands are very tightly restricted. And that fits very much with what we saw on your phet simulations. OK, but-- hold on for one sec-- coming back to the question over here-- yours-- coming back to that question, if you look at higher and higher energy, what you're going to find is that the bands get wider and wider, and the gaps get thinner and thinner, as you go to high-- well, it's a little more complicated than that, it depends on exactly what you're doing, but-- the bands get wider, the gaps get wider, but both of them don't get wider at the same rate. So, this is actually something you'll look at on your problem set. But the important thing you will find is that the bands never disappear. Like a band in 1D doesn't just close and disappear altogether, and the bands don't overlap.

And here's an easy way to understand why the bands don't overlap. Let's look at what these states are. So in particular, I want to look at the states at the bottom of each loud energy band. So, let's go back down to small value of  $g_0$ . So, we have these big thick bands with a weak little potential. We have big, thick bands, little, tiny gaps, and I want to look at the states at the bottom of the gap. Let's make it maybe



a little bit bigger. So, we'll look at the states at the bottom of the gap. And there's an optional problem in your problem set that works through the mechanics of this. But I want to look through it in some detail. So, what we're plotting here-- Did that work? Crap. Unfortunately, this is-- OK, good.

So, here what we have is the ground state of the lowest band at zero interaction. So, here, is  $g_0$ . And it's 0.001, it's basically 0. As I turn on the delta function potentials, what's going to happen to the lowest energy state, which is just the constant, has momentum 0. What's going to happen to this guy? Well, we saw from this guy, that the lowest energy state got a little bit of a gap there. There is no state at 0 energy anymore. There's now a state only at slightly positive energy. A wave from what would have been the zero energy, zero momentum state. Everyone see that?

There's this little gap between 0, and the bottom. And if we make the interaction bigger, we'll see that a little more obviously. Here's the red is the zero energy, and it goes up to finite energy. But has that lowest state developed any nodes? Does the lowest state in a 1D potential have nodes? No, right? There's nodes here. 0 nodes, then one, then two. And do they ever switch orders? Does it ever go zero, two, one? No. There's the node theory. So, what happens to these wave functions is all the wave functions continuously change as we turn on the potential, but they don't switch order. Their ordering is completely fixed by the number of nodes.

So, let's watch what happens to the actual wave functions. So, here's that lowest state, and I'm going to turn on the potential. What's going to happen? Well, we're going to see the effect of the delta function. And there, you see the effect of the delta function turning on. And as we crank up the potential, you see that the wave function generates a kink where there is a delta function. And that state has zero nodes. That's the ground state. Everyone cool with that? OK.

Similarly, let's go to 0. Whoops. Oh, shoot. Don't give me infinite expressions. You're not allowed to divide by 0. Oh, shoot! OK, here's the second excited state, this guy. And let's look at what happens as we turn up the potential now. Exactly the

same thing. We see a little kink. We don't get new nodes. But what we do get are kinks at the potential. This is for the special case of the delta function potential, but it's illustrative of everything else. So, going back to the bands, what's going on here is that these states are getting compressed-- they're getting pushed up together-- but they're not getting swapped in their order. OK? And this was in the service of a question that was asked. What was the last question?

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** OK, I don't remember exactly which questions I was answering, I'm sorry, but--

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** Sorry?

**AUDIENCE:** I think it was if you lose the bands.

**PROFESSOR:** Ah, yeah. If you lose the bands. So, you know by the node theorem that you never lose states. States with three nodes can't disappear. That same with three nodes always has to be between the state with two-- thank you-- between the state with two nodes, and the state with four nodes. And they can't just disappear. So, those gaps never disappear. And they never overlap, because they never cross each other, again, by the node theorem. All you get is things separating into bands and squishing together, or spreading back apart and becoming the free particle.

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** Yeah?

**AUDIENCE:** [INAUDIBLE] strange periodic potential and you'll get the [INAUDIBLE].

**PROFESSOR:** Indeed. Indeed. So, that's exactly right. So, the observation is, look, this was for the delta function potential, but we didn't have to use delta function barriers. We could have use little square barriers, I could have used little, you know, goat-shaped barriers. I could have used whatever. And there's going to be some answer. And how's it going to change? Well, the way it's going to change is where the bands are,

how wide they are, and how wide the gaps are. But the rest of the story goes over in exactly the same way. The energy eigenfunctions take the form  $E$  to the  $iqx$  times the periodic function. Cool?

OK, and on your problem set, where you're gonna show-- one of the things you're gonna show-- and I love this result, because it encapsulates so much physics, you're going to show that if you have a potential-- a single potential barrier-- and you know the reflection amplitude, and the transmission amplitude-- or really, you just need to know one-- if you know the reflection amplitude-- if it's parity symmetric-- then you can write down the band structure entirely in terms of-- you can write down an equation for the allowed energies as a function of  $q$ -- entirely in terms of the reflection amplitude and the phase shift. Which is an amazing fact. So, you know about reflection off of one barrier, you make an infinite lattice of those barriers, and you deduce the structure of the allowed energy bands and gaps, which is cool.

So, the scattering information-- as I kept promising-- contains a huge amount of the physics of your system. So, we saw that it contains bound energies, but it also contains these band gaps. OK, questions. Yeah?

**AUDIENCE:** [INAUDIBLE] If we make the  $g_0$  go to infinity, would that approach a single [INAUDIBLE]?

**PROFESSOR:** Yeah, exactly. Excellent. So, the question was, what happens as  $g_0$  goes to infinity, is it like getting single wells? So, let's look at that. It's a very good observation. So, here's a band between-- I really should put a line here-- but between here, and here, we have this band. And that corresponds to between here, and here. As we make  $g_0$  stronger, the derivation from simple cosine gets larger and larger. The amplitude gets larger and larger, and those bands get steeper, and steeper, and steeper. And the allowed energy bands get thinner, and thinner, and thinner. And, in fact, they get so thin, that Mathematica loses track of some of them, which is the annoyance. So, here we have very, very thin bands.

What happens is we take  $g_0$  very, very large. So, let's make it 400. And, you can see, Mathematica doesn't like that. It's totally lost the band. So, let's try something a

little less extreme. Oh, no, still too big for Mathematica. Eight, you can do it, little buddy. Ah, there we go! We can keep track of one. So, what's happening is these bands are getting extremely thin. And what's happening, you're erecting infinitely high barriers between each period in the lattice. And if you have infinitely high barriers, what are the allowed eigenvalues? It's the same as the energy eigenvalues of the infinite well, because it has to be 0 at those infinite barriers, right? So, all of those states-- all of the states in the band-- become degenerate, and have the same energy, the energy of the periodic well, or of the infinite well. Very good question. OK.

So, let's let Mathematica relax a little bit again. Well done. OK. And so, I want to quickly ask why do we have bands in the first place? Why do we have gaps? So, what's going on? Why do we have gaps in the first place?

**AUDIENCE:** Because our math told us we had to?

**PROFESSOR:** OK. Good. This is an excellent answer. Not the one I was looking for. The answer is a good answer because we did the calculation and that's what the calculation tells us. And that is a completely valid answer for a theorist, right? At this stage of writing down what your theory is and making a prediction. But before you can write that paper, publish it, and say, aha! I've discovered bands! You must have an explanation to the following question, why? How do you know your calculation was correct, before you've done the experiment? So, one answer is, of course, you do the experiment, and it fits like a champ. But that's not a very illuminating experiment as you know you're going to find departures from your theory, you want to improve the situation. You want some intuition about why this works. And you actually all know the physics behind this. And we did it on the first problem set.

Remember the experiment that we studied on a problem set, the Davis and Germer Experiment, where you send plane waves onto a crystal, and due to interference effects, the transmission amplitude changes-- the transition probability changes-- as a function of the angle. And the reason it changes as a function of the angle was you had a crystal plane, right? So, atom, atom, atom, atom, atom, atom, atom,

atom, and a plane wave sent in could scatter off of any of these scatterers, right? Or it could transmit and scatter off a further one down. And if you look at the scattering off different scatterers, if the effect of those scatterings-- so one down, and then one much deeper down-- if those interfere destructively, then that's going to suppress the reflection. And if they interfere constructively, it's gonna enhance the reflection.

So, usually we think about that Davis and Germer Experiment, we think about it as a function of the angle of incidence, either straight at, or as a function of angle. And we ask, how does the transmission depend on those angles? But there's another way to phrase that question, you can ask, look, on average, how deep does that wave go? Does the wave propagate through the crystal? Or if it doesn't propagate through the crystal, it's gonna exponentially decay with some wavelength, with some decay-length. And what you find is that whether the wave propagates in, or whether it decays exponentially, depends on the energy and the lattice spacing of that crystal.

And this is exactly the same thing, only we have a crystal in both directions. We don't have empty air and we're sending in a plane wave. Now we have a plane wave, can it propagate through? Well, for some energies, it'll propagate with continuous energy, or it'll propagate continuously. And for some energies, it will damp out exponentially. But if it damps out exponentially, and it damps out exponentially in the other direction too, then you don't get any solution at all. It just completely dies. So, what's really going on here is the fact that all of these scatterers in this lattice are scattering centers. And if you send a little wave packet in-- which you should think of as a superposition of energy eigenstates-- if you send a wave packet in, the probability to get-- the probability amplitude to get from here to here-- is the probability amplitude to go from here to here, plus to scatter through-- to transmit through and reflect back-- plus to transmit, transmit, reflect, reflect-- or sorry, transmit, transmit, reflect, transmit-- plus all the other crazy things you could do. And you have to sum up over all those terms.

You have to sum the effect of how do you get from here to here? Every possible way to get from here to there, a la the two slit experiment. Sum of over all possible

paths where you bounce around scattering off each of these scattering centers. And that's why the band structure's encoded in the reflection amplitudes from an individual scattering center. Because now, it's just a common [INAUDIBLE] problems of how many different paths, and summing up the phase that you get. OK? So, it's just about coherent and incoherent scattering. Yeah?

**AUDIENCE:** [INAUDIBLE] crystal and I have an electron.

**PROFESSOR:** Yeah.

**AUDIENCE:** If I sent the electron through the crystal in just the right way, with just the right energy, won't it just go through in some states, without seeing it?

**PROFESSOR:** Excellent. Excellent. So, the question is under what conditions can you send an electron and it'll just cruise right on through the potential? So, let's answer that question. That's exactly the right question. So, let's pick up from here. And I want to ask, what have we learned? So, what are the lessons for  $g_0$  equal to 0? So, the first lesson is that the energy eigenvalues,  $E$  eigenvalues, are restricted to lie within a band-- this bands-- and the bands are separated by gaps. Two, the energy eigenstates-- so this was eigenvalues-- the energy eigenstates are all extended. And we didn't even need to find the full solution of this equation to know that. All we needed to know is that it was a solution. That the energy eigenfunction are eigenfunctions of the translation operator as well. And that tells us the probability of that form. And that tells us what the probability distribution is, which is the norm squared of that guy, is independent, it's periodic. And if it's periodic, it can't fall off exponentially one direction or the other.

So, we've seen that the eigenfunctions are all extended, and so as a consequence, we have to build wave packets. So, therefore, we must build wave packets. And, again, the form of the wave packets,  $\psi$  is equal to the integral  $dq$   $f$  of  $q$ -- sorry, I'll say  $f_{q_0}$ -- sharply peaked around the point  $q_0$  of  $q$ . And then the wave function's  $E$  to the  $iqz$  minus  $\omega t$  times  $u$  of  $x$  sub  $q$ . OK. So, we have to build wave packets. So, three, this tells us that those wave packets move with group velocity, by the same logic as for the free particle,  $V_{\text{group}}$  is equal to  $d\omega/dk$ -- or  $dq$ --

evaluated at  $q_0$ . OK, where  $q_0$  is where our wave packet is sharply peaked.

So, that's already telling you something really cool. Look at this lowest band. Let's make this a little more reasonable. So, let's look at this lowest band. And in particular, I'm going to make this big. So, focus in on the lowest energy band, here. So, there's the lowest energy band. It goes from the top of the band to the bottom of the band. It's just a little bit away from the free particle, because we're looking at a relatively weakly, perturbed system. I chose a small value of  $q_0$ . Actually, let's make it a little more exaggerated. OK, so there's the free particle curve, the red one, and the blue one is our actual band. So, that's the energy as a function of  $q$ .

And let's make it even bigger. This one goes to 11. Let's see. OK. And so what you can see is the following, it looks, again, it looks sort of like a parabola at the bottom. And at the top, it curves over, and, actually, goes to zero derivative. And it has to, because it's got to be periodic. Because  $q$  is periodic with period  $2\pi$  over  $L$ . So, what does that tell you about the velocity? Well, let's look at the velocity.  $V$  group velocity, of wave packet with-- whoops, say, here's 0-- as a function of  $q$  between minus  $2\pi$  over  $L$ , and  $2\pi$  over  $L$ .

What is the group velocity? And here's what I mean. What I mean is let's build a wave packet which has a reasonably well-localized momentum, say, at this value of  $q$ . So, I built a little wave packet it has a peak here, and what does that wave function do? Well, it has a group velocity. And the group velocity is given by  $d\omega/dq$ . And now, we can't solve that equation analytically, because it's [INAUDIBLE]. But we can see, just by eyeball, what this does. When  $q$  is equal to 0, what's the group velocity? Yeah. It's the slope, right?  $d\omega/dq$ . So, what's  $d\omega/dq$  at the origin? 0, right? 'Cause it's just horizontal. So, it's 0. Good. What about a little bit to the right of 0? What about if I give it a small positive  $q$ -- wave packet localized with a small positive  $q$ ? It's got a little bit of a slope now, right? Little bit of a positive slope. So, it increases. But then, it reaches a maximum slope, halfway through the domain. So, here, it hits a maximum value, and then the slope comes back down.

And if we'd went the other way, if we'd made  $q$  a little bit negative, the slope becomes negative. And that's the velocity. So, that's a slightly strange thing. As we increase  $q$ , it increases the velocity. Increasing  $q$  increases the velocity for a while. That's just like  $k$  and momentum. Momentum is  $\hbar k$ . So, that would be increasing the group velocity for a free particle. But now, as we increase the  $q$  to a reasonably large value, to  $2\pi$  upon  $L$ -- oh, sorry, this is  $\pi$  upon  $L$ , and  $\pi$  upon  $L$ -- so, if we do this to  $\pi$  upon  $2L$ , or roughly, to the middle, we get to a maximum value. And if we increase  $q$  further, what happens? The velocity goes down, right? If we increase  $q$  further, the velocity goes. This is strange. For a free particle, if you had momentum  $\hbar k$ , and you increased  $k$ , what happens to the expectation--? What happens to the group velocity? If you increase the momentum, what happens to the group velocity for a free particle?

**AUDIENCE:** [MURMURS]

**PROFESSOR:** It's gonna increase, right? But not for a particle in the periodic potential. The wave packet, as you increase the crystal momentum,  $q$ , past this maximum, as you increase the crystal momentum, you actually decrease the velocity. And, in fact, if you keep increasing  $q$ , eventually the velocity goes to 0. And you increase  $q$  a little more, and it actually goes negative. So, this is a funny thing, it's not behaving like normally you'd think of as a momentum. And, indeed, is there momentum conservation in this system? No. We have a periodic potential. There are forces at work. Is momentum conserved? No. This is not a system that conserves a momentum. Momentum can be exchanged between the particle and the potential.

$Q$  is a good quantity. It's the eigenfunction of  $TL$ , and that does commute with the energy operator. So, it's a perfectly reasonable quantity. But the momentum itself,  $p$ , is not conserved. So, this crystal momentum,  $q$ , is not the momentum, for all these reasons. But it plays a lot of the same role, especially near the bottom of the band. Near the top of the band, it's a little bit funnier, because if you increase  $q$ , you get a negative velocity. If you decrease  $q$ , you get a positive velocity. It's exactly the opposite of what you'd normally expect. But let's hold on to that thought for a second.



So, there's another sense in which the crystal momentum is like a momentum. And you're gonna show this on your problem set. But it's very important in giving you some understanding-- some intuition-- about what the crystal momentum is. On your problem set, you'll show the following. Suppose I induce a constant force. For example, imagine this particle in my periodic potential had a charge, and I turn on a constant, uniform electric field. Then that particle experiences a constant, uniform driving force, which I can write as a linearly increasing potential, right? So, what you showed on the problem set is that the force-- the expectation value of the force-- is equal to-- for a wave packet-- sharply localized around some value,  $q_0$ , the force, it gives the time rate of change,  $d/dt$ , of the expectation value of  $q$ . So, if you turn on a force, what happens is  $q$  increases linearly. Thank you. It's so hard. It's just so obviously 1. So, if the force is constant, then  $\hbar q$  increases constantly. So,  $q$  just increases linearly in time, yeah? But that's funny.

Well, we'll come back to that, just how funny that is in just a-- yeah? Question? No, OK. Well, let's think about how funny that it. Let's tackle this, right now. So, imagine we start with a wave packet, which is well localized around zero crystal momentum, around  $q$  equals 0. Then what is that wave function doing in time? How does that wave packet move? What's its group velocity? 0. So, it's just sitting there. It's just little wave packets slowly dispersing, quantum mechanically, yeah? But if we don't make it too tightly constrained, and  $p$  and  $2$  too tightly constrained. And the dispersion could be made quite slow. So, it's just a little wave packet, localized around a particular position. And localized around zero momentum, and it's just sitting there. Cool?

Now, let's apply a force. What happens? As we apply a force, a constant, force with a positive sign.  $Q$  is gonna increase. So, the central value-- I should say, yeah-- so, the central value of our wave packet is gonna go from 0 to something slightly positive. Because  $q$  is increasing. What that means is that the velocity is increasing at first. So, the velocity is gonna increase, and in equal units of time it's gonna march linearly along this way. So, the particle is going faster, and faster, and faster, and faster. And that makes sense. You're driving the system by putting on a

constant force. Of course it accelerates. The velocity accelerates linearly, so the position accelerates quadratically.

So, let's plot the position-- expectation value of the position-- as a function of time. At zero time-- it's at the origin-- it's at some trivial position,  $x_0$ . As we increase time, velocity's increasing linearly for awhile, and the position is gonna increase quadratically. Everyone cool with that? But eventually, at some point in time, the velocity-- or the  $q$ -- is gonna get to the point where we're at the maximum allowed velocity. And at that point, what is the curve for  $x$  gonna look like? It's gonna have an inflection point. And as we increase-- as we wait longer--  $q$  is gonna continue linearly increasing, which means the velocity is going to slow down. So, the thing slows down, so its slope decreases, until finally it's got zero velocity again. There we go-- zero velocity--  $dx/dt$ .

And then what happens? We keep forcing it. It's a constant driving force. But this is this, by periodicity of  $q$ , and the thing is, we increase  $q$  further linearly in time. As we increased  $q$  further, the velocity becomes negative. So, instead of continuing to accelerate-- instead of continuing to move-- in a positive  $x$  direction, it starts going backwards! That's weird. You're putting a force in this direction, and it's accelerating in that direction. That's a strange effect. And as we continue, eventually we get to maximum negative velocity. Again, an inflection point. And, at which point, the velocity starts getting less, and less negative, or closer, and closer to 0. And we return.

And so, in some period, capital  $T$ , which is determined by how big this constant driving force is, the particle returns to its original position. Everyone see that? Yeah?

**AUDIENCE:** Is it starting at  $q$  equals 0?

**PROFESSOR:** Well, yeah. At  $q_0$  equals 0. Exactly. So, at first point, so this is  $q$  equals 0, this is-- sorry,  $q$  equals 0-- is the initial point. And  $x$  is equal to  $x_0$ . So, under this process, the particle goes through an oscillation, called a Bloch Oscillation. It's named after the guy who invented the wave functions, the Bloch Wave Functions. And this is a deeply weird thing. Normally, you think, well, look, if I take a charged particle, and I

just put it in an empty space, or in a box, or whatever, and I turn on an electric field, what happens to it? It accelerates. But if you put that charged particle in a periodic lattice, like copper-- idealized copper, here, perfect uniform lattice-- and you turn on an electric field-- or you put a capacitor plate across the thing-- that electron, at first, wants to behave like an electron in free space. It wants to accelerate. But then it finds out it's in a lattice. And it goes backwards. And it just oscillates back and forth, [INAUDIBLE]-like, you know? I really want to get there, oh crap. I really want to get there. Oh no, I'm in a periodic potential.

And it just oscillates back and forth. Is there any conduction in this system due to the electromagnetic field that we've induced? None. That is a strange little beast. That is a strange, strange property. And yet, copper conducts. Copper, which is a uniform lattice of ions-- of potentials-- to which electrons are stuck in bands. Copper conducts! The electrons don't do this Bloch Oscillation. How come? Yeah. It's not perfect. And it's not perfect for a whole bunch of reasons. So, one reason it's not perfect is that, so, first off, in a lattice of, say, copper, or whatever, the ions have a finite mass.

So, what that means is they're not stuck in place, they also wiggle. When the electron moves, one of the ions move. So, the whole thing you should think of as a little wiggling piece of jello. And as a consequence, they're not perfectly periodic. This does a bunch of things. One is these are no longer the exact energy eigenfunctions. You now have to deal with the oscillations and wiggling of the lattice. But more importantly, here's something that can happen-- and electron can move along, and it can kick one of these ions-- it can bounce off one of the ions-- and scatter some of its momentum into the ion. Not into the rigid lattice, but it could just make one of those ions wiggle. So, it's changed the structure of the lattice. OK?

And what you can do then, is you can have an electron that is accelerating, and it hits something, and it stops. And it accelerates. And it stops. Accelerates, and it hits something, then it stops. OK? So, when you have lots of disorder in your system-- or when you allow the electron to bounce off things in the lattice-- you get this effective conductivity. Things start their Bloch Oscillation, then they collide, and

scatter off their momentum, and go back to zero momentum, but they've moved over a little bit in their oscillation already. They move up, and they move. So, what that picture looks like is, you accelerate, and then, boom, you scatter, and you fall back down to zero momentum by scattering off your momentum into something else.

Same thing, you move up, and then, now, you have zero momentum again. You move up, you have zero momentum again. You move up, you have zero momentum. You move up, you have zero momentum, again. And so, you get this effective drifting of the electrons, pointed out by a guy named Drude. That gives you, effectively, a conductivity and a solid. Now, there are a bunch of other effects that are important for conductivity. Normally, we think of disorder as something that, you know, if you make things messy, they're probably gonna transmit less well, right? But disorder is absolutely essential for conductivity in real solids. Both for this reason, because you scatter off of fluctuations, but also, when you have a chunk of copper, it's not a chunk of copper. It's a chunk of copper, but it's got, you know, here and there, there's a little carbon atom that got stuck, and maybe there's a bit of nickel, and, you know, palladium, or, you know, Berkelium-- pretty unlikely, but could be there-- so, you have all sorts of schmutz sort of distributed around.

And, again, those are things that make the potential not periodic. And, so, will change the conservation of this  $q$ . OK? Yeah?

**AUDIENCE:** [INAUDIBLE] but if you actually managed to put a alternating current-- or an alternating voltage-- at the right frequency, you could actually drive the electrons from one side to the other in this fashion?

**PROFESSOR:** Yeah, so, I--

**AUDIENCE:** --even though the conventional way of thinking about alternating currents is that the electrons really don't move at all.

**PROFESSOR:** That's a very good observation. It's a little subtle, so come to my office hours and ask me about that. But there's an interesting-- For a very specific reason, because

it's easy and unambiguous, I chose look the DC. But you're right, looking at the AC, is really a very entertaining example. So, yeah, I'm not gonna get into it. There's a cool story with parametric resonance. And yeah, there's a very nice story there. Come to my office hours, because that is a particularly fun story. OK, but I want to do one last thing on this before we move on. So, in this process, something very strange happened.

When we got to the top of the potential, what we found was as we increase-- as we continue constant, positive force-- the velocity decreased. Yeah? That is weird. Normally, when I take a force, and I take some object with some mass, and I take a force and I apply it to the mass, and then the acceleration is the force divided by the mass. In particular, mass is always a positive thing, so if I take my force, and I divide it by the positive mass, I get an acceleration in the same direction as the force. But here, we have an applied force, and we get an acceleration in the opposite direction. I thought our particle had a positive mass. Didn't it have a positive mass? We started off with a positive mass. We should still have a positive mass. So, what's going on here? Well, we have a rule for calculating what the mass is.

We take the expectation value of the momentum and divide by the group velocity. So, let's find out what the mass is for this guy. And I'll do that here. So, this is a slightly surreal moment with a nice little codon. So, I'm going to write that expression for the mass slightly differently. And you're gonna show in your problem set-- PSet-- you're gonna show that that definition of the mass--  $M$  star is  $p$  upon the group velocity-- leads to the following expression, you can write that as-- you can use that to derive-- that the mass can be written as  $1$  upon  $\hbar$  squared  $d$  squared  $E$  dq squared.

So, the group velocity is the first derivative of the energy with respect to  $q$ . The mass-- or really  $q$  over the mass-- is proportional to the second derivative, the curvature. Everyone see that? So, this should not be an obvious equation, unless you're breathtakingly good at calculus in your head. It's just chain rule, but it does take a little bit of careful thinking, because it's a physical argument, it's not a rigorous mathematical argument. You're gonna do it on your problem set. So, let's

take this expression and let's see what the effective mass is. What is the mass of our particle? Now, here, when I see what is the mass of our particle? I mean a very precise thing. What is the mass of the object moving in this periodic potential? That's what I mean. What is the mass of this thing? And we have a definition for it. Here it is. And let's plot this  $1$  over mass. So, let's plot this  $1$  over mass. Oh, that's not the way I wanna draw it.

Yeah. Yeah, OK, good. So, here-- oh, shoot-- so, let's do it this way. So, I want to plot  $1$  over mass, in the vertical, as a function of  $q$ . And here is  $q$  is equal to  $0$ , and here is  $1$  upon mass is equal to  $0$ . So, at the bottom of the band-- and this is  $\pi$  upon  $L$ , and minus  $\pi$  upon  $L$ . At the bottom of at the band, at  $q$  equals  $0$ , what is the mass? Well, here's the velocity, and there's the band. So, what's the mass at the bottom of the band? Tell me properties about it. Is it  $0$ ? Yeah. There's still some curvature. It's approximated by some parabola, right? It's even. And, in fact, you can take that equation and derive properties about it. But it's some curvy thing. You can see that. It's more obvious if we look at a higher energy band.

So, here's the minimum. Here's what it would have been if we didn't have a potential, right? But here's the minimum, and it's nice and curvy, and at the top, again, it's nice and curvy. But it's got a second derivative there, that's non-zero. Everyone agree with that? Now, here's the thing I want to ask, is that second derivative the same as it would have been if we had  $g$  arbitrarily small?

**AUDIENCE:** [INAUDIBLE]?

**PROFESSOR:** Yeah. Not so much. No, if  $g$  was much lower, the second derivative would be a bit different. OK. So, in particular, this is some funny value. And I'm just gonna give it a name. It's gonna have some value at  $0$ . And, in particular, it is positive. Everyone agree with that? But there's a point here-- there's an inflection point-- where this thing has zero second derivative. And that inflection point is the same point where the velocity becomes a maximum, because the velocity's a first derivative. So, when the velocity's at a maximum, the second derivative has a  $0$ . It's the derivative of the first velocity, zero slope.

And so, at that inflection point, the velocity is 0-- or sorry, the velocity is a maximum-- and  $1$  upon the mass, goes to 0. And then, if we increase  $q$  further, what happens?  $1$  over  $m$  star goes negative. And it reaches a minimum at  $\pi$  over  $L$ , come back, and, again, repeats. So, it's periodic. I'm just artistically challenged. So, how can this possibly be? Right? So, first off, what this is telling us is that we have a point where the mass is 0. It's positive around the origin, where things normally behave intuitively. It goes to 0--  $1$  over the mass-- goes to 0, which tells me that the mass,  $m$  star, is going to infinity.

$1$  over the mass going to 0? Mass must be going to infinity. What does that mean? Well, what does it mean for something to be infinitely massive? It mean  $f$  equals  $ma$ , if it's infinitely massive-- then let's write that as acceleration is force over  $m$  star-- then that tells you that if you apply a force, you get no acceleration. But that's exactly what we saw. We get to this point, we keep applying our force, and the velocity stays constant. There's no acceleration. Yeah?

**AUDIENCE:** [INAUDIBLE] in this equation, that's an external force, right?

**PROFESSOR:** That's an external force.

**AUDIENCE:** So, only--

**PROFESSOR:** Exactly. Yeah. Purely external force. I put a capacitor plate across my piece of metal.

**AUDIENCE:** So, my next question would be is it all that insane then? Because, like, you haven't calculated the net force on the object, so to speak, and so there is the potential. And so, yeah, the net force is zero, sure, you're applying force, but it's not going anywhere.

**PROFESSOR:** Yeah, it's not a complete answer to what's going on here, but it's a very good observation. So, let me rephrase that slightly. So, look, I'm doing a slightly strange thing. I'm applying an external force, and I'm not treating it quantum mechanically, I'm treating this particle quantum mechanically. That's clearly stupid. We should treat everything quantum mechanically, and derive the results in that fashion. It

turns out, in this case, that the negative effective mass doesn't go away if we treat the electromagnetic field quantum mechanically. So, you're absolutely correct, but it's not enough to sort of make us comfortable with the result. It doesn't change this result. But you're absolutely correct that that's an important thing to do if we want to be honest. But this is a perfectly good approximation for our purposes.

So, the mass goes infinite, and then it goes a negative!  $1$  over the mass goes negative, so that's ridiculous, but that's exactly what we wanted. The acceleration is the force divided by the mass, but we found that when we apply a force in this direction, we get an acceleration in this direction. That's a negative coefficient, so, in particular, here's what I want to think about. I want to ask the following question, this value of the mass-- even around the bottom, this should already be disturbing observation-- because this value of the mass is not the same as the mass of our original particle. And the way you can see that is if you go to various strong coupling, then this band shrinks down. It becomes very thin, and the second derivative becomes arbitrarily small, which means the mass becomes arbitrarily large.

So, the mass is getting larger and larger as we crank up the potential-- the effective mass-- of whatever this thing is that's moving. But this is strange. The thing that's moving is our particle. It has mass,  $m$ . Everyone agree with that? What is going on here? And here, I want to do an experiment. And my experiment--

**AUDIENCE:** [MURMURING]

**PROFESSOR:** So, yeah, the seamless pong ball company, in recognition of my contributions to physics and \$3, has given me a series of ping pong balls, with which, I'm going to do the following experiment. OK. Here's the calculation I want to do. The lesson I want you to take away from this, is that you've got to be very careful what you mean by, "the mass" of an object. Take one of these ping pong balls, I'm not gonna do the whole experiment for you, but I'll tell you the set up. Take a ping pong ball, and the first thing I want to do is, I want to measure its mass. Well, what do you mean by measure its mass? Well, I'm gonna take it, and I'm gonna put it in a vacuum, you



know, we could take it to France to where they do the-- And take the ping pong ball--

**ANNOUNCEMENT:** One, two, three, four. One, two, three, four. One, two.

**AUDIENCE:** [LAUGHTER]

**PROFESSOR:** We're just gonna turn that down. OK.

**ANNOUNCEMENT:** One, two, three, four. One, two, three, four.

**PROFESSOR:** Can we kill that? OK, so. So, we take our ping pong ball, and we put some mass over here, with a known mass, and we put it on a scale, and we wait until this goes to the vertical, and we declare the mass. And what we find is that the mass of our ping pong ball-- this is, you know, not the greatest ping pong balls-- but by legislation, this is supposed to be about 2.7 grams, or people get in fistfights. So, the mass of the ping pong ball needs to be about 2.7 grams. And, meanwhile, the radius of this guy is about 20 millimeters, so 2 centimeters. Yeah, that's about right. And yeah. So, this is supposed to-- morally-- it's supposed to be 2 centimeters. And so, if you go through this and you compute the density from these guys-- so, this gives you a volume-- and if you compute the density, the density is about 12.4 by regulation.

Now, I've done this experiment, and I've done this measurement. And I must've used very cheap balls, because I got a very, very low density. I got-- oh sorry, this is the wrong way to do it-- the density is equal to 1, in usual CGS units, is 1 upon 12.4. And I can write that as 1 upon 12.4 times the density of water. OK? Now, when I did this experiment, I got 1 over 20. So, I was probably using very cheap ping pong balls. But if you look at the regulations on Wikipedia, this is what you're supposed to get. Let's assume these are perfect. So, you do this. There's the mass. That is the mass. That's what I mean by the mass. You take a scale, you weigh the thing! Right? Does anyone object to my definition of the mass? It's an observational, empirical process for determining the mass.

If I take my mass, OK? And now I ask the following question, if I take some water, and I pull my ping pong ball, and I put it under water, and I put a spring scale here,

and they measure the force on this ping pong ball, what will the force be? Well, the force will be-- and let's just call this ten, because the calculation's going to be horrible otherwise-- well, whatever, let's call it 12. So, the force-- and another way to write this is that  $\rho_{\text{water}}$  is equal to 12  $\rho_{\text{ping pong ball}}$ -- I did math there. So, the force-- we know this from Archimedes-- the force is a combination of the weight of the ping pong ball down, so, it's going to be the density times the volume. So, the volume times  $\rho_{\text{ping pong ball}}$  down, so, minus. Plus the volume times  $\rho_{\text{water}}$  up, because that's the weight of the water displaced, right? So, your weight down, minus the weight of the water displaced. Everyone cool with that? Archimedes. It's pretty well known at this point.

So, this is equal to the volume times  $\rho_{\text{water}}$  minus  $\rho_{\text{ping pong ball}}$ . And I can write this as the volume times  $\rho_{\text{water}}$  is 12  $\rho_{\text{ping pong ball}}$ , so that's 12 minus 1  $\rho_{\text{ping pong ball}}$ . That's 11  $\rho_{\text{ping pong ball}}$ . But this is equal to 11 times the weight of the ping pong ball-- oh sorry-- 11 times the mass of the ping pong ball. The volume times the density. So, mass ping pong ball. Now, there's an important thing that I forgot to add here, what did I forget to add? G. Right? It's the force is the mass times g. OK, g, g, and g. So, we get that the force is equal to 11 m ping pong ball g. Everyone agree with that? I haven't done anything sneaky.

And so, as a consequence, what this predicts is that if I take the force and I divide it by the mass of the ping pong ball, I get 11 times the acceleration of gravity. And here's what that predicts, that predicts that when I let go of this thing, what should happen? It should accelerate upward through the water at 11 times the acceleration of gravity, which is roughly 10 meters per second squared. So, 110 meters per second squared. Does that sound right? Just intuitively, does that sound right? If you pull something like a ping pong ball, or a beach ball or something underwater, what happens when you let go of it?

Well, at this point, I'm going to remind you that physics is an empirical science. So, let's see if we can do this. OK, now, what's supposed to happen is this thing is supposed to shoot out at nine times-- or 11 times-- the acceleration of gravity. So, it just shoots through this thing much faster than equivalently. So which one's going to

hit first?

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** Should it come out, or should it hit the ground first?

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** Yeah, the ping pong ball. Great, that's helpful. Uh-huh. So, what's gonna happen first? Is the ping pong ball gonna hit the surface, or is it gonna come flying out of the water? This is accelerating with the acceleration of gravity. And this it's accelerating up with 11 times the acceleration of gravity. So, let's do the experiment. And this is what they pay me for. Oh. This would be much more satisfying if we had a nice long-- how do I want to phrase this-- cylinder. Oh, shoot! Stop. So, it's best done with a nice, long cylinder, but let's do this. So, which one wins?

**AUDIENCE:** [MURMURS]

**PROFESSOR:** Gravity wins, right? And why? Intuitively, why does this work?

**AUDIENCE:** [MURMURING]

**PROFESSOR:** I am neglecting the drag effect, but I'm not really heating up the water all that much. I mean, it's not like the dissipation is all that significant. The dissipation of the momentum, which is drag-- the friction-- is not the most important thing for our purposes here. That water hasn't heated up at all. So, what's going on?

**AUDIENCE:** [INAUDIBLE].

**PROFESSOR:** OK, so here's the question. Has a mass of the ping pong ball, my empty little ping pong ball changed? So, something like that has to be true. But what's the right answer? The right answer is this, the thing that's moving, is not the ping pong ball. The ping pong ball is interacting with the fluid. And as a ping pong ball accelerates, it drags the fluid along with it. And as it goes faster, and faster, it drags more and more of the fluid along with it. And as it's dragging more and more of that fluid, the effective mass of the entire object that's moving-- which is now the ping pong ball,

and all of its associated water-- is increasing.

And now, if you take that force, which was constant, and divide by this growing effective mass, you find that the acceleration rapidly falls off. And you get a terminal velocity. And you know this works, because if you blow a bubble under water, it doesn't shoot upwards with an infinite acceleration. It goes glurg, glurg, glurg. And this idea, this idea of the effective mass of a particle changing with its state of motion, is an idea called renormalization. And it plays an essential role, not just in particle physics, but in all of modern condensed matter physics. And we'll pick up at this point next time.

**AUDIENCE:** [APPLAUSE]