PROFESSOR: Before we get started, let me ask you guys if you have any questions, pragmatic or otherwise, about the course so far. Seriously? To those of you reading newspapers, I encourage you to find a slightly different time to do so. I really encourage you find a slightly different time to do so, thanks.

So far we've done basic rules of quantum mechanics. We've done solids. We understand a lot about electrons in atoms, the periodic table, and why diamond is transparent. One thing we did along the way is we talked about spin. We found that when we looked at the angular momentum commutation relations, these guys. We found that the commutators are the same. For these commutators, we can get total angular momentum, l l plus 1 times h bar squared for l squared. And h bar m for a constant and integer m for angular momentum in a particular direction, which we conventionally called z.

But we also found that there were half integer values of the total spin and of the spin in a particular direction which, for example, with Spin s as 1/2, 3/2, 5/2, et cetera. And we discovered that riding a wave function on a sphere that interpreted these states as states with definite probability to be at a particular position on a sphere, a function of theta and phi, was inconsistent. In order for the wave function to satisfy those properties that have those eigenvalues and, in particular, half integer eigenvalues, we discovered that the wave function had to be doubly valued. And thus it would equal to minus itself, it equaled 0.

So the rest of today and tomorrow is going to be an exploration of spin, or tomorrow-- next lecture, is going to be an exploration of spin and the consequences of these spin 1/2 states. Exactly what they are, we already saw that they're important for understanding the structure of the periodic table. So we know they're there. And they're present from the experiment, the Stern-Gerlach experiment that we've discussed many times.

But before I get onto that, I want to improve on the last experiment we did. So in particular, in this last experiment we talked about the effective mass of an object interacting with a fluid or an object interacting with its environment. And why the mass of the object that's moving is not the same as the mass of the object when you put it on a balance. And that's this basic idea of renormalization. And we demonstrated that. I demonstrated that with a beaker of water.

So I had a beaker of water and pulled a ping pong ball under water. We calculated that it should accelerate upward when released at 20 times the acceleration of gravity, depending on the numbers you use, very, very rapidly. And in fact it went glug, glug, glug, but it wasn't a terribly satisfying experiment because it's very hard to get the timing right.

And the basic issue there is that the time scales involved were very short. How long did it take for a ping pong ball to drop from here to the surface? Not much time. And to rise through the water, not a whole lot of time. So I did that experiment, it was sort of comical. But I wanted to improve on it. So over the weekend, I went down to my basement and tweaked the experiment a little bit. And I called up a couple of my friends, and we did an improved version of this experiment.

So this is a diver, I think this one is Kathy. Oh, shoot, we need to turn off the lights. Sorry. I totally forgot. good You'll never see this if we don't. You can do it. There we go. All right. So here you see my friend, I think this one's, actually, is it Kathy? I'm not sure. It's hard to tell when they have their marks on. And we're in the tank at the New England Aquarium, and she's going to perform this experiment for us.

And we're going to film it, as you can see, the bubbles moving rather slowly, with a high speed camera filming at 1,200 frames per second with which we'll be able to analyze the data that results. The camera cost as much as a nice house. And it's not mine, but it's important to have friends who trust you. OK, so here we are. You might notice something in the background.

Before we get started, I just want to emphasize that one should never take casually the dangers of doing an experiment. When you plan an experiment you must, ahead of time, design the experiment, design the experimental parameters. We designed the lighting. We designed everything. We set it up, but there are always variables you haven't accounted for. And a truly great experimentalist is one who has taken account of all the variables. And I just want to emphasize that I'm not a great experimentalist. So here, for example, is a moment.

I probably should have thanked the sharks, it just occurred to me. Anyone who wants to take this experimental data, which, as you can probably guess, filmed for different purposes, I just manage to get the ping pong ball into the tank. Anyone who wants to get this and actually take the data, come to me, I'll give you the raw footage. And you can read off the positions, and hopefully for next lecture we'll have the actual plot of the acceleration as a function of time.

So with that said and done, the moral of the story is you have to account for all variables. The other moral of the story is that you saw this very vividly. In the motion of the ping pong ball up, when it was released, there is that very rapid moment of acceleration when it bursts up very, very rapidly. But it quickly slows in its acceleration. Its acceleration slows down. In fact, a slightly funny thing happens. If you look carefully, and again, anyone who wants this can get access to the video, what you'll see is that the ping pong ball accelerates and then it sort of slows down. It literally decreases in velocity, accelerates and slows down. Can anyone think what's going on in that situation?

AUDIENCE: Boundary layer formation.

PROFESSOR: Sorry?

AUDIENCE: Boundary layer formation.

PROFESSOR: Good. Say that in slightly more words.

AUDIENCE: It's starting to pick up more and more water [INAUDIBLE].

PROFESSOR: Yeah. Exactly. so what's going on is as this guy starts slowly moving along, it's pulling along more and more water, each bit of water around it is starting to drag along the layers of water nearby, and it builds up a sheath of water. Now that water starts accelerating, and the ping pong ball and the bubble of water that it's dragging along need to come to equilibrium with each other. They need to settle down smoothly to a nice uniform velocity.

But it takes a while for that equilibrium to happen. And what actually happens is that the ping pong ball drives up. It pulls up the water. Which then drags with the ping pong ball. So you can see that in the acceleration, which is oscillatory with a slight little oscillation. So it's a damped but not overdamped harmonic oscillator motion.

Any other questions about the effective mass of an electron and solid before moving-- Yeah?

AUDIENCE: Why does it speed up after? That explains why it slows down because it's forming that sheath--

PROFESSOR: Right. Why it speeds back up is it's sort of like a slingshot. As this guy gets going a little ahead of the pack of water, the pack of water has an extra driving force on top of the buoyancy. It has the fact that it's a little bit displaced. So it catches up, but it's going slightly greater velocity than it would be if there were uniform velocity. So this guy catches up with the ping pong ball.

OK so this is an of course of course in fluid mechanics, but I guess we don't actually need this anymore.

Picking up on spin. So the commutation relations for spin are these. And as we saw last time, we have spin states. We have we can construct towers of states because from the sx and s1 we can build s plus-minus is equal to sx plus-minus sy. Sorry. i, thank you. So we can build towers of states using the raising and lowering operators as plus-minus. And those states need to end, they need to terminate. So we find that the spin can have totalling momentum of s squared h bar squared l l plus 1.

And S in some particular direction, which we conventionally called z, is h bar m.

I don't want to call this l. I want to call this s. For orbital angular momentum this would be l and this would be an integer. But for spinning angular momentum these are all the states we could build, all the towers we could build, which were 2n plus 1 over 2, which were not expressible in terms of wave functions, functions of a position on a sphere. These are all the 1/2 integer states. So s could be 1/2, 3/2, 5/2, and so on. And then m sub s is going to go from minus s to s in integer steps, just like m for the orbital angular momentum, l.

So I want to talk about these states in some detail over the rest of this lecture and the next one. So the first thing to talk about is how we describe spin. In principle, this is easy. What we want, is we want to describe the state of a particle that carries this intrinsic angular momentum spin. So that's easy. The particle sits at some point, but the problem is it could be sitting at some point with angular momentum with spin in the z direction, say, plus h bar over 2. And let's focus on the case s is equal to 1/2. So I'll be focusing, for the lecture, for simplicity on the case, total spin is 1/2, which is the two-state ladder, but all of this generalizes naturally. In fact, that's a very good test of your understanding.

So for s as 1/2, we have two states, which I will conventionally call the up in the z direction and the down in the z direction. And I will often omit the subscript. If I omit the subscript it's usually z unless from context you see that it's something else. So the wave function tells us the state of the system. But we need to know now for a spinning particle whether it's in the spin 1/2 up or spin 1/2 down state. And so we could write that as there's some amplitude that it's in the plus 1/2 state of x and at position x. And there's some amplitude that it's at position x and it's in the minus 1/2 state or the down state.

And we again need that the total probability is one. Another way to say this is that the probability that we find the particle to be at x with plus or minus h bar upon 2 being the spin in the z direction. So at x, spin in the z direction is h bar upon 2 plus or minus is equal to norm squared of psi plus or minus of x squared. So this is one way you could talk about spin, and you could develop the theory of spin nicely here. But it's a somewhat cumbersome formalism.

The formulation I want to introduce is one which involves matrices and which presages the study of matrix mechanics which you'll be using in 805. So instead, I want to take these two components, and what we see already is that we can't use a single wave function to describe a particle at spin 1/2. We need to use two functions. And I want to organize them in a nice way. I'm going to write them as psi is equal to-- and I'll call this capital psi of x-- is a two component vector, or so-called spinner, psi up of x and psi down of x. So it's a two component object. It's got a top and a bottom component.

And notice that its conjugate, or its adjoint, psi dagger, is going to be equal to psi up, complex conjugate psi down, a row vector, or a row spinner. And for normalization we'll need that the total probability is 1 which says that psi capital dagger psi, psi capital with psi is equal to 1. But this is going to be equal to the integral dx. And now we have to take the inner product of the two vectors.

So integral dx of psi up squared plus psi down squared. Cool? Yep?

AUDIENCE: What's the coordinate x representing here?

PROFESSOR: The coordinate x is just representing the position. So what I'm saying here is I have, again, we're in one dimension just for simplicity, it's saying, look if I have a particle that carries spin 1/2, it could be anywhere. Let's say it's at position x. So what's the amplitude at position x and spinning up, and I'm not going to indicate spinning down. OK? I like my coffee.

So that's what the of x indicates, and I've just been dropping the of x. So there's some probability that it's at any given point and either spin up or spin down. Now, again, it's important, although I'm going to do this, and we conventionally do this spin up and down, this spin is pointing in a vector space that's two dimensional. It's either plus 1/2 or minus 1/2 h bar upon 2 h bar.

So it's not like the spin is an arrow in three dimensional space that points. Rather, what it is, it's saying, if I measure the spin along some axis, it can take one of two values. And that was shown in the Stern-Gerlach experiment, where if we have a gradient of the magnetic field, dbz, in the z direction, and this is our Stern-Gerlach box, and we send an electron in, the electron always comes out in one of two positions. OK.

Now, this is not saying there is a vector associated with this, that the electron has a angular momentum vector that points in some particular direction. Rather, it's saying that there are two possible values, and we're measuring along the z direction. Cool? So it's important not to make that mistake to think of this as some three dimensional vector. It's very explicitly a vector in a two dimensional vector space. It's not related to regular rotations. Yeah?

AUDIENCE: Where you write psi of x equals psi plus 1 plus psi [INAUDIBLE].

PROFESSOR: Yes.

AUDIENCE: Do we need a 1 over square root of 2 in front of that thing?

PROFESSOR: Yeah, I haven't assumed their normalization, but each one could be independently normalized appropriately.

AUDIENCE: So [INAUDIBLE].

PROFESSOR: Right. The whole thing has to be properly normalized, and writing it this way, this is just the [INAUDIBLE]. Good.

So there's another nice bit of notation for this which is often used, which is probably the most common notation. Which is to write psi of x is equal to psi up at x times the vector 1, 0 plus psi down of x times the vector 0, 1. So this is what I'm going to refer to as the up vector in the z direction. And this is what I'm going to refer to as the down in the z direction vector. And that's going to allow me to write all the operations we're going to need to study spin in terms of simple two by two matrices. Yeah?

AUDIENCE: Will those two psi's not be the same?

PROFESSOR: Yeah, in general, they're not. So for example, here's a situation, a configuration, a quantum system could be in. The quantum system could be in the configuration, the particle is here and it's spinning up. And it could be in the configuration the particles over here, and it's spinning down. And given that it could be in those two configurations it could also be in the superposition over here and up and over here and down. Right? So that would be different spatial wave functions multiplying the different spin wave functions, spin part of the wave function. Make sense? OK.

So these are not the same function. They could be the same function. It could be that you could be either spin up or spin down at any given point with some funny distribution, but they don't need to be the same. That's the crucial thing. Other questions? Yeah.

AUDIENCE: I know that [INAUDIBLE] is a way to call them. So like [INAUDIBLE] something, are they anti-parallel, perpendicular, or are they something?

PROFESSOR: Yeah. So here's what we can say. We know that an electron which carries sz is plus h bar upon 2, and a state corresponding to minus h bar upon 2 are orthogonal because those are two different eigenvectors of the same operator, sz. So these guys are orthogonal. Up in the z direction and down to the z directions are orthogonal.

And thank you for this question, it's a good way to think about how wrong it is to think of up and down being up and down in the z direction. Are these guys orthogonal? These vectors in space?

AUDIENCE: No.

PROFESSOR: No, they happen to be parallel with a minus 1 in the overlap, right? So sz as up sc as down are orthogonal vectors, but this is clearly not sz as down, right? So the direction that they're pointing, the up and down, should not be thought of as a direction in three-dimensional space.

AUDIENCE: It's just [INAUDIBLE].

PROFESSOR: It's just a different thing. What it does tell you, is if you rotate the system by a given amount, how does the phase of the wave function change. But what it does tell you is how the spin operations act on it. In particular, sz acts with a plus 1/2 or minus 1/2. They're just different states. Yeah?

AUDIENCE: Is this similar to what happens when polarizations [INAUDIBLE]?

PROFESSOR: It's similar to the story of polarizations except polarizations are vectors not spinners. It's similar in the sense that they look and smell like vectors in three-dimensional space, but they mean slightly-- technically-- slightly different things. In the case of polarizations of light, those really are honest vectors, and there's a sharp relationship between rotations in space. But that's a sort of quirk. They're both spin and vectors. These are not.

OK so here's a notation I'm going to use. Just to alert you, a common notation that people use in Dirac notation is to say that the wave function is equal to psi up at x times the state up plus psi down at x times the state down. So for those of you who speak Dirac notation at this point, then this means the same thing as this. For those of you who don't, then this means this.

What I want to do is I want to develop a theory of the spin operators, and I want to understand what it means to be a spin 1/2 state. Now, in particular, what I mean by develop a theory of the spin operators, if I was talking about four orbital angular momentum, say the orbital angular momentum in the z direction, I know what operator this is. If you hand me a wave function, I can act on it with lz and tell you what the result is. And that means that I can construct the eigenfunctions. And that means I can construct the allowed eigenvalues, and I can talk about probabilities. Right?

But in order to do all that I need to know how the operator acts. And we know how this operator acts. It acts as h bar upon i dd phi where phi is the angular coordinate around the equator. And so given any wave function, a function of x, y, and z or r, theta, and phi, I can act with this operator, know how the operator acts. And it's true, that again, lx with lx is equal to i h bar lz. It satisfies the same time computation relation as the spin. But we know the spin operators cannot be expressed in terms of derivatives along a sphere. I've harped on that many times.

So what I want to know is what's the analog of this equation for spin? What is a representation of the spin operators acting on the spinners, acting on states that carry half integer spin? We know it's not going to be derivatives. What is it going to be? Everyone cool with the goal?

In order to do that, we need to first decide just some basic definition of spin in the z direction. So what do the angular momentum operators do? Well whatever else is true of the spin in the z direction operator, sz acting on a state up is equal to h bar upon 2 up. And sz actually on a state down is equal to h bar minus upon 2 down. And similarly s squared on up or down--

Oh, by the way, I'm going to relatively casually oscillate between the notations up and plus, up or down, and plus or minus. So sometimes I will write plus for up and minus for down. So I apologize for the sin of this. s squared on plus, this is the plus 1/2 state, is equal to h bar squared ss plus 1, but s is 1/2, so 1/2 times 1/2 plus 1 is 3/4 h bar squared times 3 over 4. And we get the same thing up, ditto down.

Because s squared acts the same way on all states in a tower. Going up and down through a tower of angular momentum states, raising and lowering, does not change the total angular momentum because s plus and s minus commute with s squared because they have exactly the same commutation relations as the angular momentum.

That's an awesome sound. So I want to know what these look like in terms of this vector space notation, up and down. And for the moment I'm going to dispense entirely with the spatial dependence. I'm going to treat the spatial dependence as an overall constant. So we're equally likely to be in all positions. So we can focus just on the spin part of the state.

So again I want to replace up by 1, 0 and down by 0, 1. And I want to think about how this looks. So what this looks like is sz acting on [INAUDIBLE] 1, 0 is equal to h bar upon 2, 1, 0. And sz on 0, 1 is h bar upon 2, 0, 1. And s squared on 1, 0 is equal to 3 h bar squared on 4, 1, 0. And ditto for 0, 1. Yeah?

AUDIENCE: That [INAUDIBLE] should have a line over it.

PROFESSOR: Oh, thank you. Yes, it really should.

AUDIENCE: So how do you get 3/4 there?

PROFESSOR: 3/4, good. Where that came from is that remember that when we constructed the eigenfunctions of l squared, l squared acting on a state lm was equal to h bar squared l l plus 1. l plus 1 on [? file m. ?] Now if we do exactly the same logic, which we actually did at the time. We did in full generality whether the total angular momentum was an integer or a half integer. We found that if we took, I'm just going to use for the half integer states the symbol s, but it's exactly the same calculation. s squared on phi and again sm sub s is equal to h bar squared s s plus 1 phi s m sub s. OK and so for s equals 1/2 then s, s plus 1, is equal to 1/2 times 3/2, which is 3/4. Yeah?

AUDIENCE: [INAUDIBLE] s equals [INAUDIBLE]. Isn't that [INAUDIBLE]?

PROFESSOR: Ah, but remember, does l go negative? Great. Does s go negative? No. s is just labeling the tower. So s is 0, it's 1, it's 2. And so for example, here, these are states where the s is 1/2 and the s in the z direction can be plus 1/2 or minus 1/2. s is 3/2 and then s in the z direction can be m is 3/2, 1/2, minus 1/2, minus 3/2.

OK. Other questions? OK, yeah.

AUDIENCE: What about the lowering and raising of those?

PROFESSOR: Good, we're going to have to construct them, because, what are they? Well, they lower and raise, so we're going to have to build the states that lower and raise.

AUDIENCE: And would lowering on the up will give you down, but raising on up--?

PROFESSOR: Awesome. So what did it mean that we had towers? Let me do that back here. So the question is, look, we're going to have to use the raising and lowering operators at the end of the day, but what happens if I raise the bottom state-- what if I raise down? I'll get up. And what happens if I raise up? You get 0. That's the statement that the tower ends.

On the other hand, if s is 3/2, what happens if I raise 1/2? I get 3/2. And if I raise 3/2, I get nothing, I get 0, identically. And that's the statement that the tower ends. So for every tower labeled by s, we have a set of states labeled by m which goes from minus s to s in integer steps. The raising operator raises us by 1, the lowering operator lowers by 1. The lowering operator annihilates the bottom state, the raising operator kills the top state. Cool? Yeah.

AUDIENCE: Do states like 3/2 and 5/2 have anything akin to up and down?

PROFESSOR: Yeah, so-- do they have anything akin to up and down. Up and down is just a name. It doesn't really communicate anything other than it's shorthand for spin in the z direction 1/2.

So the question could be translated as, are there convenient and illuminating names for the spin 3/2 states? And I don't really know. I don't know. I mean, the states exist. So we can build nuclear particles that have angular momentum 3/2, or 5/2, all sorts of things. But I don't know of a useful name.

For the most part, we simplify our life by focusing on the 1/2 state. And as you'll discover in 8.05, the spin 1/2 states, if you know them very, very well, you can use everything you know about them to construct all of the spin 5/2 8/2-- well, 8/2 is stupid, but-- 9/2, all those from the spin 1/2. So it turns out spin 1/2 is sort of Ur-- in a way that can be made very precise, and that's the theory of Lie algebras. Yeah.

AUDIENCE: Can you just elaborate on what you meant by, you can't really think of spin as an angular momentum vector?

PROFESSOR: Yeah. So OK, good. So the question is, elaborate a little bit on what you mean by, spin can't be thought of as an angular momentum vector. Spin certainly can be thought of as an angular momentum, because the whole point here was that if you have a charged particle and it carries spin, then it has a magnetic moment. And a magnetic moment is the charge times the angular momentum. So if it carries spin, and it carries charge, and thus it carries magnetic moment-- that's pretty much what we mean by angular momentum. That's as good a diagnostic as any.

Meanwhile, on top of satisfying that experimental property, this, just as a set of commutation relations, these commutation relations are the commutation relations of angular momentum. It just turns out that we can have states with total angular momentum little s, which is 1/2 integral-- 1/2, 3/2, et cetera.

Now, the things that I want to emphasize are twofold. First off, something I've harped on over and over again, so I'll attempt to limit my uses of this phrase. But you cannot think of these states with s is 1/2 as wave functions determining position on a sphere. So that's the first sense in which you can't think of it as equivalent to orbital angular momentum.

But there's a second sense, which is that up and down should not be thought of as spin in the z direction being up and spin in the z direction being down meaning a vector in three dimensions pointing up and a vector in three dimensions pointing down, because those states are orthogonal. Whereas these two three-dimensional vectors are not orthogonal-- they're parallel. They have a non-zero inner product.

So up and down, the names we give these spin 1/2 states, should not be confused with pointing up in the z direction and down in the z direction. It's just a formal name we give to the plus 1/2 and minus 1/2 angular momentum in z direction states. Does that answer your question?

AUDIENCE: Yeah, so, when you make a measurement of the value of spin-- so perhaps you do a Stern-Gerlach experiment-- and you get what the spin is, can you not then say, all right, this is spin plus 1/2, spin minus? It's as z is positive 1/2 as z is minus 1/2?

PROFESSOR: Yeah, absolutely. So if you do a Stern-Gerlach experiment, you can identify those electrons that had spin plus 1/2 and those that had spin minus 1/2, and they come out in different places. That's absolutely true. I just want to emphasize that the up vector does not mean that they're somehow attached to the electronic vector that's pointing in the z direction. Good. Yeah. Go ahead, whichever.

AUDIENCE: How do we verify that uncharged particles have spin?

PROFESSOR: Yeah, that's an interesting question. So the question is, how do we know if an uncharged particle has spin? And there are many ways to answer this question, one of which we're going to come to later which has to do with Bell's inequality, which is a sort of slick way to do it. But a very coarse way is this way.

We believe, in a deep and fundamental way, that the total angular momentum of the universe is conserved, in the following sense. There's no preferred axis in the universe. If you're a cosmologist, just stay out of the room for the next few minutes. So there's no preferred axis in the universe and the law of physics should be invariant under rotation.

Now, if you take a system that has a bunch of particles with known angular momentum-- let me give you an example. Take a neutron. A neutron has spin 1/2. Wait, how did I know that? We can do that experiment by doing the following thing. We can take a neutron and bind it to a proton and see that the resulting object has spin 1. So let me try to think of a way that doesn't involve a neutron.

Grant me for the moment that you know that a neutron has spin 1/2. So let's just imagine that we knew that, by hook or by crook. We then do the following experiment.

We wait. Take a neutron, let it sit in empty space. When that neutron decays, it does a very cool thing. It decays relatively quickly into your proton and an electron that you see. You see them go flying away the proton has positive charge, and the electron has negative charge and it goes flying away.

But you've got a problem. Because you knew that the neutron had spin 1/2, which is [INAUDIBLE]. And then you decay one into a proton and an electron. And the total angular momentum there is 1/2 plus 1/2 or 1/2 minus 1/2. It's either 1 or 0. And you've got a problem. Angular momentum hasn't been conserved.

So what do you immediately deduce? That another particle must have also been emitted that had 1/2 integer angular momentum to conserve angular momentum. And it couldn't carry any charge because the electron and the proton were neutral, and the neutron is neutral.

So things like this you can always deduce from conservation of angular momentum one way or the other. But the best way to do it is going to be some version of addition of angular momentum where you have some object like an electron and a proton and you allow them to stick together and you discover it has total spin 1. Yeah. We can talk about that in more detail afterwards. That's a particularly nice way to do the experiment. Yeah.

AUDIENCE: Angular momentum [INAUDIBLE] weird vector since if you reflect your system through [INAUDIBLE]. How does that work?

PROFESSOR: Yeah, OK, good. I don't want to get into this in too much detail, but it's a really good question, so come to my office hours and ask or go to recitations and ask. It's a really good question.

The question is this-- angular momentum has a funny property under parity, under reflection. So if you look in a mirror this way, here's angular momentum and it's got-- right-hand rule, it's got angular momentum up-- if I look in the mirror, it's going this way. So it would appear to have right angular momentum down. That's what it looks like if you reflect in a mirror. Other direction.

So that's a funny property of angular momentum. It's also a true property of angular momentum. It's fine. And what about spin, is the question. Does spin also have this funny property under parity, is that basically the question? Yeah, and it does. And working out exactly how to show that is a sort of entertaining exercise. So again, it's beyond the scope of the lecture, so come ask me in office hours and we can talk about that. Yeah, one more.

AUDIENCE: For orbital angular momentum, say for l equals 1, we had states like m equals plus 1 and minus 1.

PROFESSOR: Yes.

AUDIENCE: And we did think of those as angular momentum vectors.

PROFESSOR: Absolutely.

AUDIENCE: But those states are also orthogonal, are they not?

PROFESSOR: Yeah, those states are also orthogonal.

AUDIENCE: So even though the angular momentum vectors aren't orthogonal, they're still-- it's just a different sense.

PROFESSOR: That's exactly right. So again, even in the case of integer angular momentum, you've got to be careful about talking about the top state and the bottom state corresponding to pointing in some direction, because they're orthogonal states. However, they do correspond to a particular angular momentum vector in three dimensional space. They correspond to a distribution on the sphere.

So there's a sense in which they do correspond to real rotations, real eigenfunctions on a sphere, and there's also a sense in which they don't, because they're still orthogonal. That's exactly right.

So let me move on. I'm going to stop questions at this point. So good.

So these are the properties that need to be satisfied by our operators. And it's pretty easy to see in this basis what these operators must be. Sz has eigenvectors 1, 0 and 0, 1. So Sz should be equal to h bar upon 2 1, 0, 0, minus 1.

So let's just check this on 1, 0 gives me 1, 0, so it gives me the same thing back times h bar upon 2. Cool. And acting on 0, 1, or the down state, we get h bar upon 2 times 0 minus 1. That could be minus 1. Oh, sorry, 1, 0 gives me 0 and 0 minus 1 on 1 gives me minus 1, which is 1 with a minus sign. That's a minus sign. So this works out like a champ. And S squared, meanwhile, is equal to-- well, it's got to give me h bar squared times 3/4 for both of these vectors. So h bar squared-- and meanwhile, these are eigenstates-- h bar squared times 3/4 times 1, 0, 0, 1.

So we know one other fact, which was brought up just a minute ago, which was that if we take S plus and you act on the state 0, 1, what should you get? If you raise your 1-- 1, 0. Great.

So we also worked out the normalization coefficient on the problem set. And that normalization coefficient turns out to be 1. And let's be careful-- we've got an h bar, for dimensional analysis reasons.

So meanwhile, S minus, similarly, on 0, 1, is equal to 0. And S plus on 0, 1, is equal to-- oh, sorry, we already did that. We want S minus on 1, 0. Let's see-- S minus on-- we want S plus on 1, 0 is equal to-- right, 0. And S minus on 1, 0 is equal to h bar 0, 1.

OK, so putting all this together, you can pretty quickly get that S plus is equal to-- we need an h bar and we need it to raise the lower one and kill the top state. So on 1, 0, what does S plus do? That gives us 0 that gives us 0. Good. And on the lowered state, 0, 1, that gives me a 1 up top and that gives me a 0 downstairs, so it works out like this. So we've got h bar. Similarly, S minus is equal to h bar times 0, 0, 1, 0.

So we've got these guys-- so much from just the definitions of raising and lowering. And by taking inner products, you can just derive those two lines from these. But notice that Sx is equal to S plus plus S minus upon 2, and Sy is equal to S plus minus S minus upon 2i.

So this tells us that Sx is equal to h bar upon 2 times S plus-- we're going to get a 1 here-- plus S minus-- we're going to get a 1 here-- 0, 1, 1, 0. And Sy is equal to, again, upon 2i times h bar, h bar upon 2i, times S plus, which is going to give me 1 and minus S minus which is going to give me minus 1, 0, 0. But we can pull this i in, so 1/i is like minus i. So minus i times minus 1 is going to give me i and minus i times 1 is going to give me i. So--

AUDIENCE: Shouldn't it be minus i?

PROFESSOR: Sorry? Yeah, did I write i? That should've been minus i. Thank you.

So now we have a nice representation of these spin operations, of the spin operators. And explicitly we have that Sx is equal to h bar upon 2 0, 1, 1, 0, Sy is equal to h bar upon 2 0, minus i, i, 0. And Sz is equal to h bar upon 2 1, 0, 0, minus 1. So why is Sz the only one that's diagonal? Is it something special about Sz?

AUDIENCE: I mean, we've chosen z as the axis along which to project S squared.

PROFESSOR: Exactly. So the thing that's special about Sz is that at the very beginning of this, we decided to work in a basis of eigenstates of Sz, with definite values of Sz. So if they have definite values, then acting with Sz is just going to give you a number. That's what it is to be a diagonal matrix. You act on a basis vector, you just get a number out.

So we started out by working in the eigenbasis of Sz. And as a consequence, we find that Sz is diagonal. And this is a general truth that you'll discover in matrix mechanics when you work in the eigenbasis of an operator, that operator is represented by a diagonal matrix. And so we often say, rather than to work in an eigenbasis, we often say, to diagonalize. Yeah.

AUDIENCE: Are the signs right for Sy? Because if we had h bar over 2i, and as we initially had a 1 in the top right and minus 1 in the bottom left, shouldn't we just multiply by i?

PROFESSOR: I'm pretty sure-- so originally, we had a downstairs i, right? So let's think about what this looked like. This was 1 and minus 1, right? Agreed? So this is minus 1 over i. So we pull in the i. So that we go from minus 1 to minus 1 over i. And we go from 1 to 1 over i. And I claim that one over i is minus i.

AUDIENCE: Oh, OK.

PROFESSOR: OK? And minus 1 over i is i. That cool? Good. OK, so this is i and minus i. I always get that screwy, but it's useful to memorize these matrices. You might think it's a little silly to memorize matrices. But these turn out to be ridiculously useful and they come up all the time.

This is called sigma x. This is called sigma y. And this is called sigma z. And different people decide whether to put the 1/2 in there or not, the h bar does not go in there. Some people put in the 2, some people don't put in the 1/2, it's a matter of taste. Just be careful and be consistent, as usual.

And these are called the Pauli matrices because A, we really like Pauli and B, Pauli introduced them. Although he didn't actually introduce them in some sense-- this mathematical structure was introduced ages and ages and ages ago. But physicists cite the physicist, not the mathematician. OK I'm not saying that's good. I'm just saying it happens.

So notice a consequence of these. An important consequence of these-- the whole point here was to build a representation of the spin operators. Now whatever else the spin operators do, they had better satisfy that computation relation, otherwise they're not really spin operators. That's what we mean by being spin operators.

So let's check. Is it true that Sx commutator with Sy is equal to i h bar Sz? So this is a question mark. And let's check. Let's do the commutator.

From the Sx, we're going to get an h bar upon 2. From the Sy, from each Sy, we're going to get a factor of h bar upon 2, so I'll just write that as h bar upon 2 squared-- times the commutator of two matrices-- 0, 1, 1, 0 commutator with 0, minus i, i, 0.

This is equal to h bar squared upon 4 times-- let's write this out. The first term is going to be this matrix times this matrix. That's going to be, again, a matrix-- 0, 1, 0, 1. So that first one is a 1. 0, 1, minus i-- oh, sorry, that's an i. 0, 1, that's an i, that's a minus i. So 0, 1, 0, i gives me an i. 0, 1, minus i, 0 gives me a 0. Second row-- 1, 0, 0, i gives me 0. And 1, 0, minus i, 0 gives me minus i.

And then the second term is the flipped order, right? The commutator term. So we get minus the commutator term, which is going to be 0, minus i, 0, 1. That gives me minus i. 0, minus i, 1, 0, that gives me 0. Bottom row-- i, 0, 0, 1-- 0. And i, 0, 1, 0 give me i.

OK, so notice what we get this is equal to h bar squared upon 4. And both of those matrices are the same thing. Those matrices are both i, 0, 0, minus i with minus i, 0, 0, i, giving us i, 0, 0, minus i times 2 from the two terms. The 2's cancel, and this gives me h bar squared upon 2 times i, 0, 0, minus i.

But this is also known as-- pulling out an i and an h bar-- times h bar upon 2 1, 0, 0, minus 1. This is equal to i h bar Sz.

So these matrices represent the angular momentum commutators quite nicely. And in fact, if you check, all the commutators work out beautifully. So quickly, just as a reminder, what are the possible measurable values of Sz for the spin 1/2 system? What possible values could you get if you measured Sz, spin in the z direction?

Yeah, plus or minus h bar upon 2. Now what about the eigenvectors? What are they-- of Sz? In this notation, there are these states. There's one eigenvector, there's the other. But let's ask the same question about Sx. So for Sx, what are the allowed eigenvalues?

Well, we can answer this in two ways. The first way we can answer this is by saying look, there's nothing deep about z. It was just the stupid direction we started with. We could have started by working with the eigenbasis of Sx and we would've found exactly the same story. So it must be plus or minus h bar upon 2. But the reason you make that argument is A, it's slick and B, it's the only one you can make without knowing something else about how Sx acts.

But now we know what Sx is. Sx is that operator. So now we can ask, what are the eigenvalues of that operator? And if you compute the eigenvalues of that operator, you find that there are two eigenvalues Sx is equal to h bar upon 2 and Sx is equal to minus h bar upon 2.

And now I can ask, well, what are the eigenvectors? Now, we know what the eigenvectors are because we can just construct the eigenvectors of Sx plus. And if you construct the eigenvectors of Sx plus-- should I take the time? How many people want me to do the eigenvectors of Sx explicitly? Yeah, that's kind of what I figured. OK, good.

So the eigenvectors of Sx are, for example, on 1, 1 is equal to-- well, Sx on 1, 1, the first term, that 0, 1, is going to give me a 1, the 1, 0 is going to give me a 1. So this is h bar upon 2 coefficient of Sx on 1, 1. And Sx on 1, minus 1 is going to give me h bar upon 2 minus 1, minus 1, because all Sx does is swap the first and second components. So it gives me minus 1, takes it to the top, but that's just an overall minus sign.

So again, we have the correct eigenvalues, plus and minus h bar upon 2, and now we know the eigenvector. So what does this tell us? What does it tell us that up in the x direction is equal to 1 over root 2 if I normalize things properly. Up in the z direction plus down in the z direction. That's what this is telling me. This vector is equal to up in the z direction-- that's this guy-- plus down in the z direction. But this isn't properly normalized, and properly normalizing it gives us this expression.

So what does this tell us? This tells us that if we happen to know that the system is in the state with angular momentum, or spin, angular momentum in the x direction being plus 1/2, then the probability to measure up in the z direction in a subsequent measurement is 1/2. And the probability to measure down is 1/2. If you know it's up in the x direction, the probability of measuring up in the z or down in the z are equal. You're at chance. You're at even odds. Everyone agree with that? That's the meaning of this expression.

And similarly, down in the x direction is equal to 1 over root 2, up in the z direction minus down in the z direction. And we get that from here. This state is explicitly, by construction, the eigenstate of Sx as we've constructed Sx.

And we have a natural expression in terms of the z eigenvectors up and down. That's what this expression is giving us. It gives us an expression of the Sx eigenvector in the basis of Sz eigenvectors.

So for example, this tells you that the probability to measure up in the z direction, given that we measured down in the x direction first-- so this is the conditional probability. Suppose I first measured down in the x direction, what's the probability that I subsequently measure up in the z direction? This is equal to-- well, it's the norm squared of the expansion coefficient.

So first down in the x direction and the probability that we're up in the z direction is 1 upon root 2 squared. Usual rules of quantum mechanics-- take the expansion coefficient, take its norm squared, that's the probability-- 1/2.

And we can do exactly the same thing for Sy. So let's do the same thing for Sy without actually working out all the details. So doing the same thing for Sy, up in the y direction is equal to 1 upon root 2 times up in the z direction plus i down in the z direction. And down in the y direction is equal to 1 upon root 2 up in the z direction minus i down in the z direction.

And I encourage you to check your knowledge by deriving these eigenvectors, which you can do given our representations of Sy.

Now here's a nice thing that we're going to use later. Consider the following. Consider the operator S theta, which I'm going to define as cosine theta Sz plus-- oh, sorry. I'm not even going to write it that way.

So what I mean by S theta is equal to-- take our spherical directions and consider an angle in the-- this is x, let's say, y, x, z-- consider an angle theta down in the zx plane, and we can ask, what is the spin operator along the direction theta? What's the angular momentum in the direction theta? If theta, for example, is equal to pi/2, this is Sx. So I'm just defining a spin operator, which is the angular momentum along a particular direction theta in the xz plane.

Everyone cool with that? This is going to turn out to be very useful for us, and I encourage you to derive the following. And if you don't derive the following, then hopefully it will be done in your recitations. Well, I will chat with your recitation instructors.

And if you do this, then what are we going to get for the eigenvalues of S theta?

What possible eigenvalues could S theta have?

[? AUDIENCE: None. ?]

PROFESSOR: Good. Why?

AUDIENCE: Because it can't have any other redirection [INAUDIBLE] no matter where you start.

PROFESSOR: Fabulous. OK. So the answer that was given is that it's the same, plus or minus h bar upon 2 as Sz, or indeed as Sx or Sy, because it can't possibly matter what direction you chose at the beginning. I could have called this direction theta z. How do you stop me? We could have done that. We didn't.

That was our first wave answering. Second wave answering is what? Well, construct the operator S theta and find its eigenvectors and eigenvalues. So I encourage you to do that. And what you find is that, of course, the eigenvalues are plus or minus h bar upon 2 and up at the angle theta is equal to cosine of theta upon 2 up in the z direction plus sine of theta upon 2 down in the z direction.

And down theta is equal to cosine theta upon 2 up in the z direction minus sine of theta over 2 down in the z direction. Oops, no. I got that wrong. This is sine, and this is minus cosine. Good. That makes more sense.

OK. So let's just sanity check. These guys should be properly normalized. So if we take the norm squared of this guy, the cross terms vanish because up z and down z are orthogonal states. So we're going to get a co squared plus the sine squared. That's 1. So that's properly normalized. Same thing for this guy. The minus doesn't change anything because we norm squared.

Now, I'll check that they're orthogonal. If we take this guy dotted into this guy, everything's real. So we get a cosine from up up, we're going to get a cosine sine. And from down down, we're going to get a minus cosine sine. So that gives us 0. So these guys are orthogonal, and they satisfy all the nice properties we want. So this is a good check-- do this-- of your knowledge. And if we had problem sets allowed this week, I would give you this in your problem set, but we don't.

Yeah, OK. Suppose that I measure. So I want to use these states for something. Suppose that I measure Sz and find Sz is equal to 1/2 h bar plus h bar upon 2, OK, at some moment in time. First question is easy. What's the state of the system subsequent to that measurement?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Man, you all are so quiet today. What's the state of the system subsequent to measurement that Sz is plus h bar upon 2?

AUDIENCE: Up z.

PROFESSOR: Up z. Good. So our state psi is up z upon measurement, OK, after measurement. And I need new chalk. OK. Now, if I measure Sx, so 1, 2, measure Sx, what will I get?

AUDIENCE: [INAUDIBLE]

PROFESSOR: OK. What values will I observe with what probabilities? Well, first off, what are the possible values that you can measure?

AUDIENCE: Plus or minus h bar over 2.

PROFESSOR: Right, the possible eigenvalues, so which is plus or minus h bar upon 2, but with what probability?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Exactly. So we know this from the eigenstate of Sx. If we know that we're in the state up z, we can take linear combinations of this guy to show that up z is equal to 1 over root 2. So let's just check. If we add these two together, up x and down x, if we add them together, we'll get 1/2, 1/2, 2/2, a root 2 times up z, is up x plus down x, up x plus down x, dividing through by the 1/2.

So here we've expressed up in the z direction in a basis of x and y, which is what we're supposed to do. So the probability that I measure a plus 1/2 as x is equal to plus 1/2 h bar upon 2 is equal to 1 over root 2 squared, so 1/2. And ditto, Sx is equal to minus h bar upon 2. Same probability. OK?

What about Sy? Again, we get one of two values. Oops, plus or minus h bar upon 2. But the probability of measuring plus is equal to 1/2, and the probability that we measure minus 1/2 is 1/2. That's h bar upon 2. OK?

So this should look familiar. Going back to the very first lecture, hardness was spin in the z direction, and color was spin in the x direction, I guess. And we added, at one point, a third one, which I think I called whimsy, which is equal to spin in the y direction. OK? And now all of the box operation that we used in that very first lecture, you can understand is nothing other than stringing together chains of Stern-Gerlach experiments doing Sx, Sy, and Sz.

So let's be explicit about that. Let's make that concrete. Actually, let's do that here. So for example, suppose we put a random electron into an Sz color box. OK. Some are going to come out up, and some are going to come out down in the z direction. And if we send this into now an Sx color box, this is going to give us either up in the x direction or down in the x direction. And what we'll get out is 50-50, right? OK.

So let's take the ones that came out down. And if we send those back into an Sz, what do we get?

AUDIENCE: 50-50.

PROFESSOR: Yeah, 50-50, because down x is 1 over 2 up z plus 1 over 2 down z. What was going on in that very first experiment, where we did hardness, color, hardness? Superposition. And what superposition was a hard electron? It was 1 over a 2, white plus black. We know precisely which superposition, and here it is. OK?

And now, let's do that last experiment, where we take these guys, Sx down, and I'm turning this upside down, beam joiners. We take the up in the x direction and the down in the x direction, and we combine them together, and we put them back into Sz, what do we get? Well, what's the state? 1 over root 2 down x plus 1 over root 2 up x?

AUDIENCE: [INAUDIBLE]

PROFESSOR: It's up z. Up z into an Sz box, what do we get? Up z with 100%. That's white with 100%, or I'm sorry, hard with 100%.

AUDIENCE: [INAUDIBLE] down z with a down z.

PROFESSOR: Sorry.

AUDIENCE: You put in down.

PROFESSOR: Oh, I put in down. Shoot, I'm sorry. Well, yeah, indeed, I meant to put in up. Yes, down z with 100% confidence. If we remove the mirror, 50-50. If we add in the mirror, 100%. And the difference is whether we're taking one component of the wave function, or whether we're superposing them back together. All right?

Imagine we know we have a system in this state, and I say, look, this component is also coincidentally very far away, and I'm going to not look at them. So of the ones that I look at, I have 1 over root 2 up x. But if I look at the full system, the superimposed system, that adds together to be an eigenstate of Sz. These are our color boxes. Yeah?

AUDIENCE: But how do you know, when you put the two beams to the beam joiner, it serves to add their two states together?

PROFESSOR: Yeah, excellent. This is a very subtle point. So here we have to decide what we mean by the beam splitter. And what I'm going to mean by the beam splitter, by the mirrors and beam joiners-- so the question is, how do we know that it does this without changing the superposition?

And what I want to do is define this thing as the object that takes the two incident wave functions, and it just adds them together. It should give me the direct superposition with the appropriate phases and coefficients.

So if this was plus up x, then it stays plus up x. If it's minus up x, it's going to be minus up x. Whatever the phase is of this state, when it gets here it just adds together the two components. That's my definition of that adding box.

AUDIENCE: So realistically, what does that look like?

PROFESSOR: Oh, what? You think I'm an experimentalist?

[LAUGHTER]

Look, every time I try an experiment, I get hit by a shark. OK?

[LAUGHTER]

Yeah, no. How you actually implement that in real systems is a more complicated story. So you should direct that question to Matt, who's a very good experimentalist. OK.

So finally, let's go back to the Stern-Gerlach experiment, and let's actually run the Stern-Gerlach experiment. I guess I'll do that here. So let's think about what the Stern-Gerlach experiment looks like in this notation, and not just in this notation, in the honest language of spin. And I'm going to do a slightly abbreviated version of this because you guys can fill in the details with your knowledge of 802 and 803.

OK. So here's the Stern-Gerlach experiment. We have a gradient in the magnetic field. This is the z direction. And we have a gradient where B in the z direction has some B0, a constant plus beta z. OK? So it's got a constant piece and a small gradient. Everyone cool with that? It's just the magnetic field gets stronger and stronger in the z direction, there's a constant, and then there's a rate of increase in the z direction.

And I'm going to send my electron through. Now remember, my electron has a wave function, psi electron, is equal to-- well, it's got some amplitude to be up, a up z, plus some amplitude to be down, b down z. And if this is a random electron, then its state is going to be random, and a and b are going to be random numbers whose norm squared add up to 1, proper normalization. Cool?

So here's our random initial state, and we send it into this region where we've got a magnetic field gradient. And what happens? Well, we know that the energy of an electron that carries some angular momentum is a constant, mu naught, times its angular momentum dotted into any ambient magnetic field. Whoops, sorry, with a minus sign. This is saying that magnets want to anti-align.

Now, in particular, here we've got a magnetic field in the z direction. So this is minus mu 0 Sz Bz. And Bz was a constant, beta 0-- sorry, B0 plus beta z. So the energy has two terms. It has a constant term, which just depends on Sz, and then there's a term that depends on z as well as depending on Sz [INAUDIBLE].

So we can write this as a-- so Sz, remember what Sz is. Sz is equal to h bar upon 2, 1, 0, 0, minus 1. So this is a matrix with some coefficient up a-- do I want to write this out? Yeah, I guess I don't really need to write this out. But this is a matrix, and this is our energy operator. And it acts on any given state to give us another state back. It's an operator.

OK. And importantly, I want this to be only-- this is in some region where we're doing the experiment, where we have a magnetic field gradient. Then outside of this region, we have no magnetic field and no magnetic field gradient. So it's 0 to the left and 0 to the right.

So as we've talked about before, the electron feels a force due to this gradient to the magnetic field. The energy depends on z, so the derivative of the energy with respect to z, which is the force in the z direction, is non-zero. But for the moment that's a bit of a red herring. Instead of worrying about the center of mass motion, let's just focus on the overall phase.

So let's take our initial electron with this initial wave function a up z plus b down z, and let's note that in this time-- so what does this matrix look like? OK. So fine, let's actually look at this matrix.

So Sz is the matrix 1, 0, 0, minus 1 times h bar upon 2. So we have h bar upon 2 minus mu 0. And then B0 plus beta z, which I'll just write as B, which is equal to some constant C, 0, 0, some constant minus C. Everyone agree with that? Where a constant, I just mean it's a number, but it does depend on z, because the z is in here.

Now, here's the nice thing. When we expand the energy on up z, this is equal to C of z up z, because there's our energy operator. And energy on down z is equal to minus Cz of z down to z. So up and down in the z direction are still eigenfunctions of the energy operator. We've chosen an interaction, we've chosen a potential which is already diagonal, so the energy is diagonal. It's already in its eigenbasis.

So as a consequence, this is the energy, energy of the up state is equal to, in the z direction, is equal to plus C. And energy in the down state, energy of the down state, is equal to minus C of z. Everyone agree with that?

So what this magnetic field does it splits the degeneracy of the up and down states. The up and down states originally had no energy splitting. They were both zero energy. We turn on this magnetic field, and one state has positive energy, and the other state has negative energy. So that degeneracy has been split.

Where did that original degeneracy come from? Why did we have a degeneracy in the first place?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Spherical symmetry. And we turn on the magnetic field, which picks out a direction, and we break that degeneracy, and we lift the splitting. OK? So here we see that the splitting is lifted. And now we want to ask, how does the wave function evolve in time?

So this was our initial wave function. But we know from the Schrodinger equation that psi of t is going to be equal to-- well, those are already eigenvectors-- so a e to the minus i e up t upon h bar up z plus b e to the minus i e down t upon h bar down z. All right?

But we can write this as equals a e to the i mu 0 b0 t over 2 plus i mu 0 beta t z upon 2 up z plus b same e to the minus i--oops, that should be, oh, no, that's plus-- that's e to the i mu 0 b0 t over 2. And if we write this as two exponentials, e to the minus i mu 0 beta t z upon 2-- oh no, that's a minus, good, OK-- down z.

OK. So what is this telling us? So what this tells us is that we start off in a state, which has some amplitude to be up and some amplitude to be down, a and b. And at a later time t, sine of t, what we find after we run it through this apparatus is that this is the amplitude.

What do you notice? Well, we notice two things. The first is that the system evolves with some overall energy. So the phase rotates as usual. These are energy eigenstates, but the amount of phase rotation depends on z.

So in particular, this is e to the i some number times z. And what is e to the i some number times z? If you know you have a state that's of the form e to the i some number, which I will, I don't know, call kz times z, what is this telling you about the system? This is an eigenstate of what operator?

AUDIENCE: Momentum.

PROFESSOR: Momentum in the z direction. It carries what momentum?

AUDIENCE: h bar kz.

PROFESSOR: h bar kz, exactly. So what about this? If I have a system in this state, what can you say about its momentum in the z direction?

AUDIENCE: [INAUDIBLE]

PROFESSOR: It's non-zero. Right here's a z. These are a bunch of constants. It's beta, t, mu 0, 2, i. So it's non-zero. In fact, it's got minus some constant times z. Right? So this is a state with negative momentum. Everyone agree? It's got momentum z down in the z direction.

What about this guy? Momentum up. So the states that are up in the z direction get a kick up. And the states that are down in the z direction get a kick down. They pick up some momentum down in the z direction. Yeah?

AUDIENCE: Where did that big T come from?

PROFESSOR: Big T should be little t, sorry. Sorry, just bad handwriting there. That's just t. Yeah?

AUDIENCE: So in this case, the eigenvalues-- it's a function of z?

PROFESSOR: Mhm.

AUDIENCE: Is that allowable?

PROFESSOR: Yeah. So that seems bad, but remember what I started out doing. Oop, did I erase it? Shoot, we erased it. So we started out saying, look, the wave function is up in the z direction, or really rather here, up in the z direction times some constant. Now, in general, this shouldn't be a constant. It should be some function.

So this should be a function of position times up in the z direction. This is a function position times down in the z direction. So what we've done here is we said, under time evolution, that function changes in time, but it stays some linear combination up and down. So you can reorganize this as-- the time evolution equation here is an equation for the coefficients of up z and the coefficient of down z.

AUDIENCE: OK.

PROFESSOR: Other questions? OK. So the upshot of all this is that we run this experiment, and what we discover is that this component of the state that was up z gets a kick in the plus z direction. And any electron that came from this term in the superposition will be kicked up up z. And any electron that came from the superposition down z will have down.

Now, what we really mean is not that an electron does this or does that, but rather that the initial stage of an electron that's here, with the superposition of z up and down, ends up in the state as a superposition of up z being up here and down z being down here. OK? An electron didn't do one, it didn't do the other. It ends up in a superposition, so the state at the end.

So what the Stern-Gerlach experiment has done, apparatus has done, is it's correlated the position of the electron with its spin. So if you find the amplitude, to find it up here and down is very small. The amplitude to find it up here and up is very large. Similarly, the amplitude, to find it down here and up is 0. And the amplitude, to find it down here and down is large. Cool?

So this is exactly what we wanted from the boxes. We wanted not to do something funny to the spins, we just wanted to correlate the position with the spin. And so the final state is a superposition of these guys. And which superposition? It's exactly this superposition. So these calculations are gone through in the notes that are going to be posted. OK. So questions at this point? Yeah?

AUDIENCE: And so when you put the electron through the Stern-Gerlach device, does that count as a measurement of the particle's angular momentum?

PROFESSOR: Excellent. When I put the electrons through the Stern-Gerlach device, do they come out with a definite position?

AUDIENCE: No.

PROFESSOR: No. At the end of the experiment, they're in a superposition of either being here or being here. And they're in a superposition of either being up spin or down spin. Have we determined the spin through putting it through this apparatus? No. We haven't done any measurement.

The measurement comes when we now do the following. We put a detector here that absorbs an electron. We say, ah, yeah, it got hit. And then you've measured the angular momentum by measuring where it came out. If it comes out down here, it will be in the positive. So this is a nice example of something called entanglement. And this is where we're going to pick up next time.

Entanglement says the following. Suppose I know one property of a particle, for example, that it's up here, or suppose I'm in the state psi is equal to up, so up here and up in the z direction plus down there and down in the z direction. OK?

This means that, at the moment, initially, if I said, look, is it going to be up or down with equal probabilities, 1 over root 2 with equal amplitudes, if I measure spin in the z direction, what value will I get? What values could I get if I measure spin in the z direction? Plus or minus 1/2. And what are the probabilities that I measure plus 1/2 or minus 1/2?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Even odds, right? We've done that experiment. On the other hand, if I tell you that I've measured it to be up here, what will you then deduce about what its spin is?

AUDIENCE: Plus.

PROFESSOR: Always plus 1/2. Did I have to do the measurement to determine that it's plus 1/2. No, because I already know the state, so I know exactly what I will get if I do the experiment. I measure up here, and then the wave function is this without the 1 over root 2, upon measurement. But as a consequence, if I subsequently measure up z, the only possible value is up in the z direction. Yeah? And this is called entanglement. And here it's entanglement of the position with the spin.

And next time, what we'll do is we'll study the EPR experiment, which says the following thing. Suppose I take two electrons, OK, take two electrons, and I put them in the state one is up and the other is down, or the first is up and the second is down. All right? So up, down plus down, up. OK? I now take my two electrons, and I send them to distant places.

Suppose I measure one of them to be up in the x direction. Yeah? Then I know that the other one is down in that direction. Sorry. If I measure up in the z direction, I determine the other one is down in the z direction. But suppose someone over here who's causally disconnected measures not spin in the z direction, but spin in the x direction. They'll measure one of two things, either plus or minus.

Now, knowing what we know, that it's in the z eigenstate, it will be either plus 1/2 in the x direction or minus 1/2 in the x direction. Suppose I get both measurements. The distant person over here does the measurement of z and says, aha, mine is up, so the other one must be down. But the person over here doesn't measure Sz, they measure Sx, and they get that it's plus Sx.

And what they've done as a result of these two experiments, Einstein, Podolsky, and Rosen say, is this electron has been measured by the distant guy to be spin z down, but by this guy to be measured spin x up. So I know Sx and Sz definitely. But that flies in the face of the uncertainty relation, which tells us we can't have spin z and spin x definitely.

Einstein and Podolsky and Rosen say there's something missing in quantum mechanics, because I can do these experiments and determine that this particle is Sz down and Sx up. But what quantum mechanics says is that I may have done the measurement of Sz, but that hasn't determined anything about the state over here. There is no predetermined value.

What we need to do is we need to tease out, we need an experimental version of this tension. The experimental version of this tension, fleshed out in the EPR experiment, is called Bell's inequality. We studied it in the very first lecture, and we're going to show it's a violation in the next lecture. See you guys next time.

[APPLAUSE]