Quantum Physics III (8.06) — Spring 2016

Assignment 5

Readings

- Scattering Theory: Griffiths, Chapter 11, Cohen-Tannoudji, Ch. VIII, and/or Shankar, Chapter 19.

1. **Which phase? (15 points)**

   In the adiabatic theorem we define $E_n(t)$ and $|\psi_n(t)\rangle$ according to

   $$H(t)|\psi_n(t)\rangle = E_n(t)|\psi_n(t)\rangle. \tag{1}$$

   Unfortunately, this equation does not have a unique solution, even if there is no degeneracy. This is clear because multiplying $|\psi_n(t)\rangle$ by an arbitrary *time-dependent* phase still gives a solution.

   Suppose that Alice solves (1) and obtains solutions $\{|\psi_n^A(t)\rangle\}$ and Bob solves (1) and obtains solutions $\{|\psi_n^B(t)\rangle\}$. Assume that they agree at time $t = 0$, so that

   $$|\psi_n^A(0)\rangle = |\psi_n^B(0)\rangle.$$

   At later times their solutions of (1) may be different. As mentioned above, we may have a time-dependent phase $\alpha_n(t)$ such that

   $$|\psi_n^A(t)\rangle = e^{i\alpha_n(t)}|\psi_n^B(t)\rangle\,, \qquad \text{with} \quad \alpha_n(0) = 0\,.$$

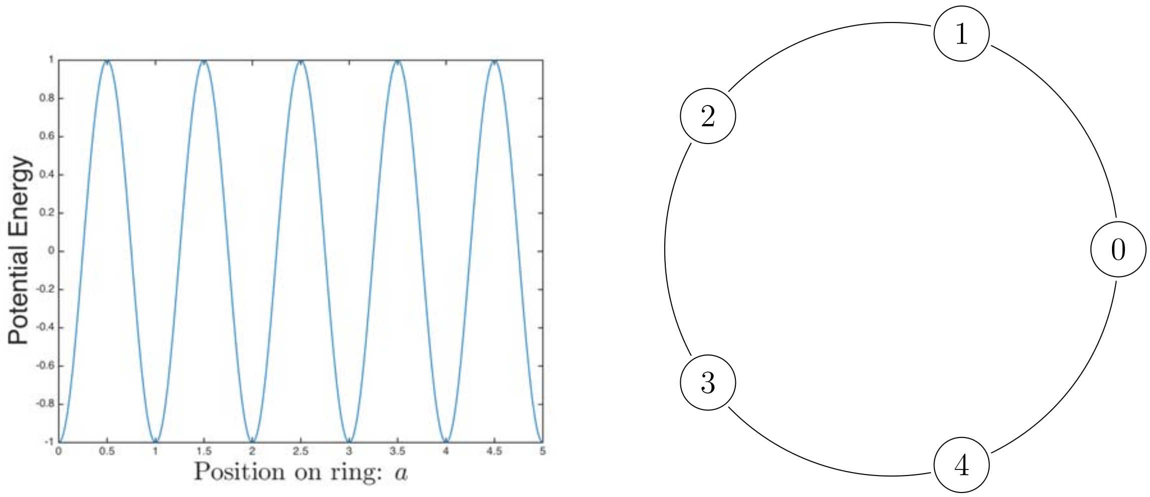
   Will this lead Alice and Bob to get different predictions from the adiabatic theorem? More concretely, suppose that at time $t = 0$ a system is in state

   $$|\Psi(t = 0)\rangle = |\psi_n^A(0)\rangle = |\psi_n^B(0)\rangle\,.$$

   Suppose that for times $0 \le t \le T$ the Hamiltonian changes adiabatically. Both Alice and Bob predict the state at time T using the adiabatic theorem, but each follows their own basis conventions. Write down Alice and Bob's predictions $|\Psi^A(T)\rangle$ and $|\Psi^B(T)\rangle$ for the state at time T. Are these in fact the same state or are they different? Since in the adiabatic theorem we care about phases, equality means equality including phases. Explain your answer.

2. **A static version of the Berry phase (35 points)** The Berry phase applies to not only to dynamic phenomena but also can be seen in time-independent problems. Consider a single electron that is constrained to live on a ring, subject to a periodical potential with N local minima, which we label by $0, 1, 2, \ldots, N-1$. This is depicted in Fig. 1.

   There is also a spatially varying magnetic field which interacts with the spin degree of freedom of the electron. We neglect the interaction of the magnetic field with the



(a) (b)

Figure 1: A periodic potential on a ring (plotted in (a)) forces an electron to sit on one of the N sites in a circle, as depicted in (b). In this example, $N = 5$.

electron's orbital angular momentum. Assume that the field at site a points in the direction

$$\vec{u}(a) \equiv \cos(\theta)\hat{z} + \sin(\theta)\left(\cos\left(\frac{2\pi a}{N}\right)\hat{x} + \sin\left(\frac{2\pi a}{N}\right)\hat{y}\right). \tag{2}$$

Here θ is a fixed parameter, and $a \in \{0, 1 \ldots, N-1\}$. In this problem we will explore how the spatial variation of the spin wavefunction leads to an effect similar to the Berry phase from adiabatic time evolution.

The overall state space is $2N$ dimensional (N dimensions for position, 2 dimensions for spin) and has Hamiltonian:

$$H \equiv T + U \tag{3}$$

$$T \equiv -\Gamma \sum_{a=0}^{N-1} (|a+1\rangle\langle a| + |a\rangle\langle a+1|) \otimes I_2 \tag{4}$$

$$U \equiv \sum_{a=0}^{N-1} |a\rangle\langle a| \otimes \vec{u}(a)\cdot\vec{\sigma} \tag{5}$$

$$= \sum_{a=0}^{N-1} |a\rangle\langle a| \otimes \left(\cos(\theta)\sigma_z + \sin(\theta)\left(\cos\left(\frac{2\pi a}{N}\right)\sigma_x + \sin\left(\frac{2\pi a}{N}\right)\sigma_y\right)\right) \tag{6}$$

(Here the sums "wrap around" so $|N\rangle$ is the same as $|0\rangle$.) The T term resembles a discretized version of the kinetic energy and in fact can be derived from the usual $p^2/2m$ term, although we will not go into the details of that here. The $\otimes I_2$ term indicates that T acts trivially on the spin degree of freedom. The U term is due to the magnetic field. For simplicity we omit the electron's mass, gyromagnetic ratio, etc. and instead use the factor $\Gamma > 0$ to control the relative strengths of T and U.

As a warmup part (a) and (b) of this problem will compute the spectrum of T and U.

(a) Find the eigenvalues and eigenvectors of T. [*Hint: For convenience define $T \equiv T_0 \otimes I_2$ where T_0 is the first term on the RHS of (4). Then each eigenvector of T_0 will correspond to two orthogonal eigenvectors of T. To find the eigenvectors of T_0, it may be helpful to consider states of the form*

$$\frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} e^{ia\varphi} |a\rangle,$$

for appropriate values of φ.]

(b) Define the subspaces V_+, V_- by

$$V_+ = \text{span}\{|a\rangle \otimes |\vec{u}(a); +\rangle : a \in \{0, 1, \dots N-1\}\} \tag{7a}$$

$$V_- = \text{span}\{|a\rangle \otimes |\vec{u}(a); -\rangle : a \in \{0, 1, \dots N-1\}\} \tag{7b}$$

Here the states $|\vec{v}; \pm\rangle$ are the ± 1 eigenstates (respectively) of $\vec{v} \cdot \vec{\sigma}$. Explicitly if $\vec{v} = \cos(\theta)\hat{z} + \sin(\theta)(\cos(\phi)\hat{x} + \sin(\phi)\hat{y})$ then

$$|\vec{v}; +\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \qquad |\vec{v}; -\rangle = \begin{pmatrix} -\sin(\theta/2) \\ e^{i\phi}\cos(\theta/2) \end{pmatrix} \tag{8}$$

Show that the bases for V_+ and V_- from (7) are eigenvectors of U and find the corresponding eigenvalues.

(c) Write T as the block matrix

$$T = \begin{pmatrix} T_{--} & T_{-+} \\ T_{+-} & T_{++} \end{pmatrix},$$

corresponding to breaking the input and output into the V_-, V_+ subspaces. So T_{--} maps V_- to V_-, T_{-+} maps V_+ to V_-, etc. According to degenerate perturbation theory, the low-energy spectrum of H is approximated by T_{--}.

Compute the matrix elements of T_{--} in the V_- basis from (7b); i.e. compute

$$(\langle a_1| \otimes \langle \vec{u}(a_1); -|)\, T\, (|a_2\rangle \otimes |\vec{u}(a_2); -\rangle), \tag{9a}$$

for arbitrary a_1, a_2. Express your answer in the form $(1 + x/N^2)e^{iy/N} + O(1/N^3)$, where x, y are real and independent of N. As a hint, $|\langle \vec{v}; -|\vec{w}; -\rangle|^2 = \frac{1+\vec{v}\cdot\vec{w}}{2}$.

(d) Using the approximation for T_{--} that you computed in (2c), find the eigenvalues and eigenvectors of T_{--}. [*Hint: You should find that as a matrix T_{--} commutes with T_0, although as operators they do act on different spaces.*] Neglecting terms of $O(1/N^3)$, what is the ground state energy? You may find it convenient to define $|\tilde{a}\rangle \equiv |a\rangle \otimes |\vec{u}(a); -\rangle$.

(e) Suppose that $\vec{u}(a)$ is replaced by $\vec{u}(a) = \vec{v}(a/N)$ for some continuous function $\vec{v} : [0,1] \to S^2$ where S^2 denotes the unit sphere in $\mathbb{R}^3$ and $\vec{v}(0) = \vec{v}(1)$. Again the matrix elements of T_{--} are of the form $(1+x/N^2)e^{iy/N} + O(1/N^3)$. This time only calculate y. Write down the eigenstates of T_{--} including the correction up to $O(1/N)$. You do *not* need to calculate the ground-state energy here, but if you did the answer would include a term related to the area swept out by the curve $\vec{v}(s)$ and another term related to its average velocity. [*Hint: It may be helpful to replace* $|\tilde{a}\rangle$ *with* $e^{-i\beta(a)}|\tilde{a}\rangle$ *for some strategically chosen* β *to simplify the form of* T_{--}. *Section 10.2.2 of Griffiths may be helpful to read here.*]

Note that $1/N^2$ is the right scale for the kinetic energy here, as you saw on problem 3 of pset 3. Effects like those in this problem occur in solid-state systems, where the momentum is replaced with a “crystal momentum.” While momenta can take any value in $\mathbb{R}^3$, crystal momenta inhabit a Brillouin zone which will have a periodic topology like the ring. See arXiv:0907.2021 for a review of the role of the Berry phase in this setting.

3. **Scattering from a Reflectionless Potential (20 points)**

Consider a particle of mass m moving in one dimension under the influence of the potential

$$V(x) = -\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax) \ .$$

(a) This potential has a normalizable bound state with wave function $\psi_0(x) \propto \operatorname{sech}(ax)$. What is its energy?

(b) Show that

$$\psi(x) = \left(\frac{k}{a} + i\tanh(ax)\right)\exp(ikx)$$

is a solution to the same problem with energy $E = \hbar^2 k^2/2m$.

(c) Now consider scattering of a particle with energy E from $V(x)$. Explain (should be brief) that the solution of part (b) satisfies the boundary conditions appropriate for this scattering problem, with the particle incident from the left. Use this solution to show that the reflection coefficient is zero, and to determine the transmission coefficient $T(E)$. Show that $|T(E)| = 1$.

(d) Show that $T(E)$ has a pole at the energy of the bound state.

4. **Harmonic oscillator gaining weight (30 points)** Consider a particle moving in 1-d subject to a harmonic potential, and suppose that the mass is slowly increasing; i.e.

$$H(t) = \frac{p^2}{2m(t)} + \frac{1}{2}m(t)\omega^2 x^2, \tag{10}$$

where $m(t) = m_0 e^{\nu t}$ for some constants m_0, ν. Suppose that at time $t = 0$ the particle is in the ground state of $H(0)$.

(a) For small values of ν, the adiabatic theorem guarantees that the particle will remain approximately in the ground state at later times. How small does ν have to be to make this true?

(b) Assuming the adiabatic approximation holds, what are the dynamical and geometric phases at time t?

(c) Suppose that there exists an operator A satisfying

$$[A, p] = \frac{i\hbar\nu p}{2} \qquad \text{and} \qquad [A, x] = \frac{-i\hbar\nu x}{2}. \tag{11}$$

Show that this implies that

$$H(t) = e^{\frac{iAt}{\hbar}} H(0) e^{-\frac{iAt}{\hbar}}. \tag{12}$$

(d) Find a Hermitian operator A satisfying (11). *[Hint: consider products of x and p.]*

(e) Let $|\psi(t)\rangle$ be the solution of the Schrödinger equation. It turns out that

$$|\psi(t)\rangle = e^{\frac{iAt}{\hbar}} e^{-i\frac{H(0)+B}{\hbar}t} |\psi(0)\rangle \tag{13}$$

Find B.

(f) Suppose we omit the B term from (13), so that we have the time-dependent state

$$e^{iAt} e^{-i\frac{H(0)}{\hbar}t} |\psi(0)\rangle. \tag{14}$$

Show that (14) describes the state that would arise if the adiabatic solution were exactly correct. Next, we would like to argue that $|\psi(t)\rangle$ from (13) is close to the ground state at time t. To do so, use time-independent perturbation theory to find the first-order corrections to the ground state of $H(0) + B$, assuming we are in the adiabatic limit (i.e. assuming your bound from part (a)). Show that this implies that (13) remains close to the instantaneous ground state for all t.