Magnetic Dipoles and Magnetic Fields in Matter

Lecture 26, 11/9/12

Magnetic multipole expansion $\Longrightarrow$
(for a locclized current distribution)

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\hat{r}}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)}$$

where

$$\boxed{\vec{m} = \frac{1}{2}\int d^3x\ \vec{r}\times\vec{J}}$$

For wires, $\boxed{\vec{J}\,d^3x = I\,d\vec{\ell}}$

so $\boxed{\vec{m} = \frac{1}{2}I\int \vec{r}\times d\vec{\ell}}$

Does $\vec{m}$ depend on origin? No.

Translated coordinates: $\vec{r}\,' = \vec{r} + \vec{b}$

$$\vec{m}\,' = \frac{1}{2}\int d^3x' \; \vec{r}\,' \times \vec{J}(\vec{r}\,')$$

$$= \frac{1}{2}\int d^3x \, (\vec{r} + \vec{b}) \times \vec{J}$$

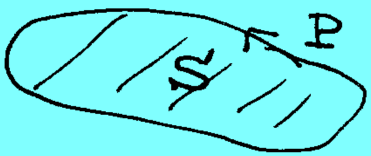
$$= \vec{m} + \frac{1}{2}\vec{b} \times \underbrace{\int d^3x \; \vec{J}}_{=\,0}$$

(by current conservation)

Similarly, $\displaystyle\int d\vec{l} = 0$

Another expression (for wires)

$$\vec{m} = \frac{1}{2} I \int_P \vec{r} \times d\vec{\ell}$$



$$m_i = \frac{1}{2} I \int_P \hat{e}_i \cdot (\vec{r} \times d\vec{\ell})$$

$$= \frac{1}{2} I \int_P d\vec{\ell} \cdot (\hat{e}_i \times \vec{r})$$

$$= \frac{1}{2} I \int_S d\vec{a} \cdot \vec{\nabla} \times (\hat{e}_i \times \vec{r})$$

$$\vec{\nabla} \times (\hat{e}_i \times \vec{r}) = \hat{e}_i (\vec{\nabla} \cdot \vec{r}) - (\hat{e}_i \cdot \vec{\nabla}) \vec{r}$$

$$\vec{\nabla} \cdot \vec{r} = 3$$

$$(\hat{e}_i \vec{\nabla}) \vec{r} = \partial_i \vec{r} = \partial_i (x_j \hat{e}_j)$$

$$= \delta_{ij} \hat{e}_j = \hat{e}_i$$

$$\therefore \vec{\nabla} \times (\hat{e}_i \times \vec{r}) = 2 \hat{e}_i$$

$$m_i = I \int_S d\vec{a} \cdot \hat{e}_i \,, \quad \text{or}$$

$$\vec{m} = I \int_S d\vec{a}$$

$$\boxed{\vec{m} = I\vec{a}\,, \quad \text{where } \vec{a} \equiv \int_S d\vec{a}}$$

$\vec{a} =$ vector area of loop.

If loop is in a plane, $|\vec{a}| =$ area

Magnetic field of a dipole (PS 7, Prob 7)

$$\boxed{\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\,\frac{3(\vec{m}\cdot\hat{r})\,\hat{r} - \vec{m}}{r^3} + \frac{2\mu_0}{3}\,\vec{m}\,\delta^3(\vec{r})}$$

Significance of $\delta^3(\vec{r})$ term:

Hydrogen orbital ground state:

Orbit is $\ell = 0$ (angular momentum $= 0$ — spherically symmetric ~~etc~~ probability density).

Spins interact by dipole-dipole interaction.

2-states: aligned + antialigned

Depends on average $\vec{B}_{proton}$ experienced by electron.

Transition is astronomically crucial.

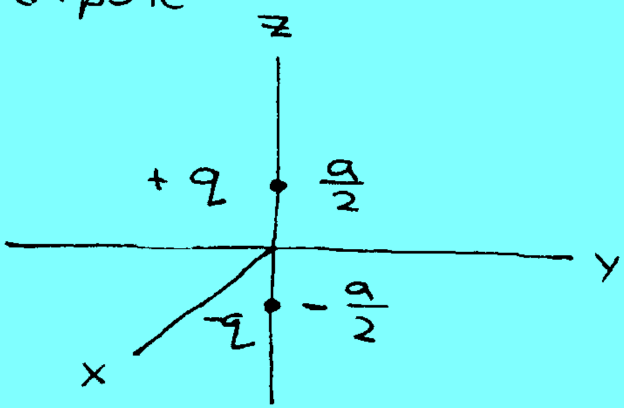
Galaxy is mapped by 21 cm line.

ΔE for transition comes 100% from δ-function term.

Current of a magnetic dipole:

Reminder: charge density of electric dipole:

Physical dipole:



$$\vec{p} = aq\,\hat{z}$$

Ideal dipole is limit of above picture as $a \to 0$ with $\vec{p}$ fixed.

$$\rho(\vec{r}) = q\,\delta(x)\,\delta(y)\,\delta(z-\tfrac{a}{2}) - q\,\delta(x)\,\delta(y)\,\delta(z+\tfrac{a}{2})$$

$$= -qa\,\delta(x)\,\delta(y)\left[\frac{\delta(z+\frac{a}{2}) - \delta(z-\frac{a}{2})}{a}\right]$$

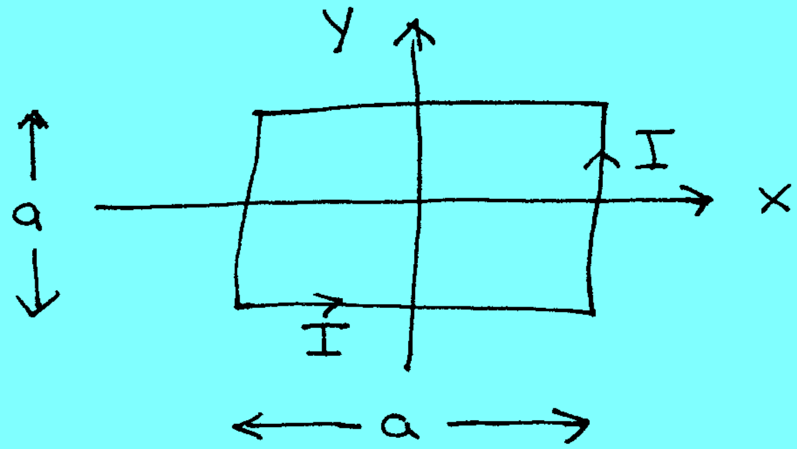
$$= -p\,\delta(x)\,\delta(y)\,\frac{d}{dz}\delta(z)$$

In general,

$$\boxed{\rho_{dip}(\vec{r}) = -\vec{p}\cdot\vec{\nabla}_{\vec{r}}\,\delta(\vec{r}-\vec{r}_d) \qquad \text{where } \vec{r}_d = \text{position of dipole.}}$$

DBADF — Don't be afraid of delta functions (but be careful)

Derivation 1: Square current loop



$$\vec{m} = I\vec{a} = Ia^2\,\hat{z}$$

Ideal dipole $\Longrightarrow$ Take limit as $a \to 0$,

$m = Ia^2$ fixed

For a nonzero,

$$J_x(\vec{r}) = \begin{cases} \text{if } |x| \le \frac{a}{2}, & I\,\delta(z)\left[\delta(y+\frac{a}{2}) - \delta(y-\frac{a}{2})\right] \\ 0 & \text{otherwise.} \end{cases}$$

Set $I = \frac{m}{a^2}$, take limit $a \to 0$:

$$J_x(\vec{r}) = \begin{cases} \text{if } |x| \le \frac{a}{2}, & \frac{m}{a}\,\delta(z)\,\frac{d}{dy}\,\delta(y) \\ 0 & \text{otherwise} \end{cases}$$

$$= m\,\delta(z)\,\frac{d}{dy}\,\delta(y)\,g(x)$$

where $$g(x) = \begin{cases} \text{if } |x| \le \frac{a}{2}, & \frac{1}{a} \\ 0 & \text{otherwise} \end{cases}$$

$$\longrightarrow \delta(x)$$

$$J_x(\vec{r}) = m\,\delta(x)\,\delta(z)\,\frac{d}{dy}\,\delta(y)$$

$$= m\,\frac{\partial}{\partial y}\left[\delta(x)\,\delta(y)\,\delta(z)\right]$$

$$= m\,\frac{\partial}{\partial y}\,\delta^3(\vec{r})$$

$$J_y(\vec{r}) = -m\,\frac{\partial}{\partial x}\,\delta^3(\vec{r})$$

$$\vec{J} = -\vec{m} \times \vec{\nabla}\,\delta^3(\vec{r})$$

In general

$$\boxed{\vec{J} = -\vec{m} \times \vec{\nabla}_{\vec{r}}\,\delta^3(\vec{r} - \vec{r}_d)}$$

where $\vec{r}_d$ = location of dipole.

Arbitrary current distribution:

$$\vec{m} = \frac{1}{2}\int d^3x\ \vec{r}\times\vec{J}(\vec{r})$$

Construct ideal dipole by shrinking while scaling up $\vec{J}$ to keep $\vec{m}$ fixed.

Let λ = scale factor:

$\lambda = 1 \iff$ original size

$\lambda = \frac{1}{2} \iff$ half size.

$$\vec{m} = \frac{1}{2\lambda^4}\int d^3x\ \vec{r}\times\vec{J}\left(\frac{\vec{r}}{\lambda}\right)$$

$$J_i(\vec{r}) = \lim_{\lambda \to 0} \frac{1}{\lambda^4} J_i(\vec{r}/\lambda)$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda^4} \int d^3x' \, J_i(\vec{r}\,') \underbrace{\delta^3\left(\vec{r}\,' - \frac{\vec{r}}{\lambda}\right)}_{\lambda^3 \delta^3(\lambda \vec{r}\,' - \vec{r})}$$

$$= \lambda^3 \delta^3(\vec{r} - \lambda \vec{r}\,')$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda} \int d^3x' \, J_i(\vec{r}\,')$$

$$\times \left[\delta^3(\vec{r}) - \lambda \frac{\partial \delta^3(\vec{r})}{\partial x_j} x_j' + \dots \right]$$

Use $\int d^3x' \, J_i(\vec{r}\,') = 0$

by current conservation.

$$= -\frac{\partial \delta^3(\vec{r})}{\partial x_j} \int d^3x' \, J_i(\vec{r}\,')\, x_j'$$

$$= -\frac{\partial \delta^3(\vec{r})}{\partial x_j} \underbrace{\mathcal{M}^{(1)}_{i;j}}_{m_k \epsilon_{kji}}$$

$$= -\epsilon_{ikj}\, m_k \frac{\partial \delta^3(\vec{r})}{\partial x_j}$$

$$= -\left[\vec{m} \times \vec{\nabla}\delta^3(\vec{r})\right]_i$$

Force on a magnetic dipole—

Recall electric dipole:

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E} \quad \text{or} \quad \vec{\nabla}(\vec{p}\cdot\vec{E})$$

$$\tau = \vec{p}\times\vec{E} + \vec{r}\times\vec{F}$$

$$U = -\vec{p}\cdot\vec{E} \qquad \text{(potential energy)}$$

Magnetic force:

$$\boxed{\begin{aligned} &\vec{F} = q\,\vec{v}\times\vec{B} \\ &d\vec{F} = I\,d\vec{\ell}\times\vec{B} = \vec{J}\times\vec{B}\,d^3x \end{aligned}}$$

Recall $I\,d\vec{\ell} = \vec{J}\,d^3x$ for a wire.

$$\vec{F} = \int \vec{J}\times\vec{B}\,d^3x = -\int \vec{B}\times\vec{J}\,d^3x$$

$$= +\int \vec{B}\times(\vec{m}\times\vec{\nabla}\,\delta^3(\vec{r}))\,d^3x \qquad \text{for dipole.}$$

$$F_i = \int d^3x \left[m_i \left(B_j \partial_j \delta^3(\vec{r}) \right) - \partial_i \delta^3(\vec{r})\, m_j B_j \right]$$

Evaluate derivative of δ-function
by integration by parts.

$$F_i = -\int d^3x \left[\underbrace{\partial_j (m_i B_j)}_{m_i \partial_j B_j = 0} \delta^3(\vec{r}) - \delta^3(\vec{r})\, \partial_i (m_j B_j) \right]$$

$$= \partial_i (\vec{m} \cdot \vec{B})$$

$$\boxed{\vec{F} = \vec{\nabla} (\vec{m} \cdot \vec{B})}$$

Potential energy: Keeping $\vec{m}$ fixed, and defining $U = 0$ at ∞,

$$U(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{F} \cdot d\vec{\ell} = -\int_{\infty}^{\vec{r}} \vec{\nabla}(\vec{m} \cdot \vec{B}) \cdot d\vec{\ell}$$

$$\boxed{U(\vec{r}) = -\vec{m} \cdot \vec{B}(\vec{r})}$$

Torque on a magnetic dipole:

$$\vec{\tau} = \int \vec{r} \times d\vec{F} \qquad d\vec{F} = \vec{J} \times \vec{B}\, d^3x$$

$$= -\int \vec{r} \times (\vec{B} \times \vec{J})\, d^3x \qquad \vec{J} = -\vec{m} \times \vec{\nabla} \delta^3(\vec{r})$$

$$= \int \vec{r} \times [\vec{B} \times (\vec{m} \times \vec{\nabla} \delta^3(\vec{r}))]\, d^3x$$

$$\tau_i = \epsilon_{ijk}\,\epsilon_{klm}\,\epsilon_{mnp}$$

$$\times \int d^3x\, x_j\, B_l\, m_n\, \partial_p\, \delta^3(\vec{r})$$

$$= -\epsilon_{ijk}\,\epsilon_{klm}\,\epsilon_{mnp}$$

$$\times \int d^3x \quad \underbrace{\partial_p\left(x_j\, B_l\, m_n\right)}_{\delta_{pj}\, B_l\, m_n \,+\, x_j(\partial_p B_l)\, m_n}\ \delta^3(\vec{r})$$

$= 0$ due to $\delta^3(\vec{r})$

$$= -\epsilon_{ijk}\,\epsilon_{klm}\,\epsilon_{mnj}\quad B_l(\vec{0})\, m_n$$

$$= -\epsilon_{ijk}\,\epsilon_{mkl}\,\epsilon_{mnj}\quad B_l(\vec{0})\, m_n$$

$$= -\epsilon_{ijk}\left(\delta_{kn}\delta_{lj} - \cancel{\delta_{kj}\delta_{ln}}\right) B_l(\vec{0})\, m_n$$

$$= -\epsilon_{ijk}\, B_j(\vec{0})\, m_k \;=\; (\vec{m}\times\vec{B})_i$$

$$\boxed{\vec{\tau} = \vec{m} \times \vec{B}}$$

More fun with δ-functions: Maxwell Eqs for $\vec{B}_{dip}(\vec{r})$:

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3} + \frac{8\pi}{3}\vec{m}\,\delta^3(\vec{r})\right]$$

Should satisfy

$$\vec{\nabla}\cdot\vec{B}_{dip} = 0$$

$$\vec{\nabla}\times\vec{B}_{dip} = \mu_0\vec{J}_{dip} = -\mu_0\vec{m}\times\vec{\nabla}\delta^3(\vec{r})$$

Can we check?

Recall:

$$\partial_i\left(\frac{x_j}{r^3}\right) = -\partial_i\partial_j\left(\frac{1}{r}\right) = \frac{\delta_{ij}-3\hat{r}_i\hat{r}_j}{r^3} + \frac{4\pi}{3}\delta_{ij}\,\delta^3(\vec{r})$$

Let $\vec{G} \equiv \dfrac{3(\vec{m}\cdot\hat{r})\,\hat{r}-\vec{m}}{r^3}$

$$G_i = -\frac{\delta_{ij}-3\hat{r}_i\hat{r}_j}{r^3}\,m_j$$

$$G_i = m_j\,\partial_i\partial_j\left(\frac{1}{r}\right) + \frac{4\pi}{3}m_i\,\delta^3(\vec{r})$$

Divergence:

$$\partial_i G_i = m_j\,\partial_i\partial_i\partial_j\left(\frac{1}{r}\right) + \frac{4\pi}{3}(\vec{m}\cdot\vec{\nabla})\delta^3(\vec{r})$$

$$\partial_i\partial_i\frac{1}{r} = \nabla^2\frac{1}{r} = -4\pi\,\delta^3(\vec{r})$$

$$\partial_i G_i = -4\pi(\vec{m}\cdot\vec{\nabla})\,\delta^3(\vec{r}) + \frac{4\pi}{3}(\vec{m}\cdot\vec{\nabla})\delta^3(\vec{r})$$

$$\boxed{\vec{\nabla}\cdot\vec{G} = -\frac{8\pi}{3}(\vec{m}\cdot\vec{\nabla})\,\delta^3(\vec{r})}$$

So $\boxed{\vec{\nabla}\cdot\vec{B}_{dip} = 0}$

Curl:

$$\epsilon_{ijk}\,\partial_j G_k = \epsilon_{ijk}\,\partial_j\left[m_\ell\,\partial_k\partial_\ell\left(\frac{1}{r}\right) + \frac{4\pi}{3} m_k\,\delta^3(\vec{r})\right]$$

But $\epsilon_{ijk}\,\underbrace{\partial_j\partial_k}_{\text{symmetric in } j \,\&\, k.}\left[m_\ell\,\partial_\ell\left(\frac{1}{r}\right)\right] = 0$

$$\epsilon_{ijk}\,\partial_j G_k = \frac{4\pi}{3}\epsilon_{ijk}\left[\partial_j\,\delta^3(\vec{r})\right] m_k$$

$$\boxed{\vec{\nabla}\times\vec{G} = -\frac{4\pi}{3}\,\vec{m}\times\vec{\nabla}\,\delta^3(\vec{r})}$$

Then $\quad \vec{\nabla} \times \frac{8\pi}{3} \vec{m}\, \delta^3(\vec{r}) = -\frac{8\pi}{3} \vec{m} \times \vec{\nabla} \delta^3(\vec{r}).$

$$\boxed{\begin{aligned} &\vec{\nabla} \times \vec{B}_{dip} = \mu_0 \vec{J}_{dip} \\ &\qquad \text{where } \vec{J}_{dip} = -\vec{m} \times \vec{\nabla} \delta^3(\vec{r}) \end{aligned}}$$

Can also check:

$$\vec{E}_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3} - \frac{4\pi}{3}\vec{p}\,\delta^3(\vec{r})\right]$$

implies

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{E}_{dip}(\vec{r}) = \frac{\rho_{dip}}{\epsilon_0} \\ &\qquad \text{where} \quad \rho_{dip} = -\vec{p}\cdot\vec{\nabla}\delta^3(\vec{r}) \\ &\vec{\nabla}\times\vec{E}_{dip}(\vec{r}) = 0 \end{aligned}}$$