Chapter 4

Boson systems

A boson system is the simplest system in nature. It demonstrates a wide variety of physical phenomena, such as superfluidity, magnetism, crystals, etc. We can also use boson system to study many different phase transitions. Despite (or due to) its simplicity, a boson system may also be the deep fundamental structure that produces all the elementary particles including photons and electrons [Wen 2003*b*]. If this is true, a boson system will actually be a theory of everything.

In this chapter, we will study interacting bosons using a classical picture. We first develop a classical field theory that describes the bosons. Then we consider the collective vibration modes of the field. After quantizing those vibration modes, we gain a understanding of the low energy properties of the quantum interacting bosons.

4.1 A first look at a free boson system

- *n-boson Hamiltonian and n-boson energy eigenstates for a free boson system.*

There are two kinds of particles in nature, bosons and fermions. Photons and Hydrogen molecules are two examples of bosons. Photons hardly interact with each other. So the photon system is a non-interacting boson system, or a free boson system. Hydrogen molecules have a short range interaction. For a dilute Hydrogen gas there is little chance for two molecules to be close to each other. Thus the interaction between the Hydrogen molecules can also be ignored and we can treat the Hydrogen gas as a system of free bosons. In this section, we will study such free boson systems. To simplify our discussion even further, we will consider free bosons in one dimension. The generalization to higher dimension is often straight forward.

To construct a quantum theory for many bosons, let us start with the simplest case: the state with no particle. Such a state is called a vacuum state and is denoted by $|0\rangle$. The energy of such a state is zero.

The next simplest state is a state with one particle. Actually there are many different one-particle states. Those states form a Hilbert space $\mathcal{H}_1$. One set of bases vectors for $\mathcal{H}_1$ is $|x\rangle$ which describe a particle at x. $|x\rangle$'s are normalized according to

$$\langle x|x'\rangle = \delta(x - x').$$

A generic one-particle state $|\psi\rangle$ is described by a complex wave function $\psi(x)$:

$$|\psi\rangle = \int \mathrm{d}x\, \psi(x)|x\rangle$$

Let us assume that the particle is relativistic and is described by the one-particle Hamiltonian

$$\hat{H}_1 = \sqrt{-c^2\partial_x^2 + m^2c^4} \tag{4.1.1}$$

where m is the mass of the particle and c the speed of light. The energy eigenstates of such a Hamiltonian are plane waves

$$|k\rangle = \int \mathrm{d}x \mathrm{e}^{\mathrm{i}kx}|x\rangle$$

with energy $E_k = \sqrt{c^2k^2 + m^2c^4}$. Certainly the statistics is not important here. The one-particle states for a boson or a fermion are identical.

For a particle in three dimensions, $\hat{H}_1$ becomes

$$\hat{H}_1 = \sqrt{c^2(-\partial_x^2 - \partial_y^2 - \partial_z^2) + m^2c^4}$$

The wave vector will have three components $\boldsymbol{k} = (k_x, k_y, k_z)$ and the energy of a 3D plain wave state $|\boldsymbol{k}\rangle$ will be $E_{\boldsymbol{k}} = \sqrt{c^2\boldsymbol{k}^2 + m^2c^4}$. If we take m to be the mass of the Hydrogen molecule, $E_{\boldsymbol{k}}$ will be the energy of of single Hydrogen molecule. For a single massless photon, the energy can be obtained by taking $m = 0$ and is given by $E_{\boldsymbol{k}} = c|\boldsymbol{k}|$.

The two-particle states form a bigger Hilbert space $\mathcal{H}_2$. One set of bases vectors for $\mathcal{H}_2$ is $|x_1x_2\rangle$ with a understanding that $|x_1x_2\rangle$ and $|x_2x_1\rangle$ are the two names for the *same* physical state. So we have

$$|x_1x_2\rangle = |x_2x_1\rangle \tag{4.1.2}$$

The equivalence of $|x_1x_2\rangle$ and $|x_2x_1\rangle$ is very important. It means that there is only one state with one particle at x_1 and one particle at x_2. If $|x_1x_2\rangle$ and $|x_2x_1\rangle$ describe two different quantum states, then there are two different states with one particle at x_1 and one particle at x_2. In this case the system will be a system of non-identical particles. The condition that there is only a single state with one particle at x_1 and one particle at x_2 makes the particles in our system identical particles.

A generic two-particle state is given by

$$|\psi_{\text{two-particles}}\rangle = \int_{x_1 \leqslant x_2} \mathrm{d}x_1\,\mathrm{d}x_2\ \psi(x_1, x_2)|x_1x_2\rangle \tag{4.1.3}$$

Note that the integration is only over the region $x_1 \leqslant x_2$ to avoid double counting, since $|x_1x_2\rangle$ and $|x_2x_1\rangle$ represent the same state. So the the two-particle wave function $\psi(x_1, x_2)$ is only defined for $x_1 \leqslant x_2$.

Using eqn (4.1.2), we can extend the wave function $\psi(x_1, x_2)$ to the region with $x_1 > x_2$ through the relation

$$\psi(x_1, x_2) = \psi(x_2, x_1)$$

This allows us to rewrite eqn (4.1.3) as

$$|\psi_{\text{two-particles}}\rangle = \frac{1}{2}\int \mathrm{d}x_1\,\mathrm{d}x_2\ \psi_{x_1,x_2}|x_1x_2\rangle$$

where the integration is over the whole 2D plane (x_1, x_2). We see that the states of two identical particles can be described by symmetric wave functions $\psi(x_1, x_2) = \psi(x_2, x_1)$.

A careful reader may note that so far we only specified that the two particles are identical particles. We did not specify if the two particles are bosons or fermions. So the above reasoning implies that both bosonic and fermionic identical particles are described by symmetric wave functions.

But what determines the statistics of the identical particles? It turns out that the statistics is not determined by the symmetry or antisymmetry property of the wave function, but by the Hamiltonian that governs the dynamics of the two particles.

If we choose the Hamiltonian that acts on the two-particle state $|\psi_{\text{two-particles}}\rangle$ to be the sum of two one-particle Hamiltonian (4.1.1)

$$\hat{H}_2 = \sqrt{-c^2\partial_{x_1}^2 + m^2c^4} + \sqrt{-c^2\partial_{x_2}^2 + m^2c^4} \tag{4.1.4}$$

then the two identical particles will be bosons. Further more such a Hamiltonian also implies that there is no interaction between the two particles. So $\hat{H}_2$ describes our free 1D boson system with two bosons.

We note that $\hat{H}_2$ is invariant under the exchange $x_1 \leftrightarrow x_2$. So when it acts on a symmetric wave function $\psi(x_1, x_2)$, $\hat{H}_2$ will generate another symmetric wave function. Since the identical particles (bosons or fermions) are always described by symmetric wave functions, the two-particle Hamiltonian for identical particles are always invariant under the exchange, so that the action of the Hamiltonian on the allowed wave functions can only generate allowed wave functions.

The energy eigenstates of $\hat{H}_2$ are plain waves $\psi(x_1, x_2) = \mathrm{e}^{\mathrm{i}(k_1x_1+k_2x_2)} + \mathrm{e}^{\mathrm{i}(k_1x_2+k_2x_1)}$ (which is symmetric under the exchange of x_1 and x_2) or

$$|k_1k_2\rangle = \int_{x_1\leqslant x_2} \mathrm{d}x_1\,\mathrm{d}x_2\ \left(\mathrm{e}^{\mathrm{i}(k_1x_1+k_2x_2)} + \mathrm{e}^{\mathrm{i}(k_1x_2+k_2x_1)}\right)|x_1x_2\rangle$$

We note that $|k_1k_2\rangle = |k_2k_1\rangle$. So $|k_1k_2\rangle$'s are also redundant names: $|k_1k_2\rangle$ and $|k_2k_1\rangle$ are two names for the same plain wave state. The energy of the plain wave state is $E_{k_1k_2} = \epsilon_{k_1} + \epsilon_{k_2}$ where

$$\epsilon_k = \sqrt{c^2k^2 + m^2c^4}. \tag{4.1.5}$$

The above discussion can be easily generalized to n-particles. The n-particle Hamiltonian have a form

$$\hat{H}_n = \sum_{i=1}^{n} \sqrt{-c^2\partial_{x_i}^2 + m^2c^4} \tag{4.1.6}$$

Such a Hamiltonian determines the statistics of the particles to be bosonic. The energy eigenstates are $|k_1k_2\cdots k_n\rangle$ with energy $\sum_{i=1}^{n}\epsilon_{k_i}$. The different orders of $k_1k_2\cdots k_n$ in $|k_1k_2\cdots k_n\rangle$ correspond to the same state, for example $|k_1k_2k_3\rangle = |k_2k_1k_3\rangle = |k_3k_1k_2\rangle$.

4.2 **A brief look at Fermi statistics

- *Identical particles can always be described by symmetric wave functions. The statistics of the identical particles is determined by n-particle Hamiltonians. This provides a unified way to understand Bose, Fermi, and fractional statistics.*

We have stressed that both bosons and fermions can be described by symmetric wave functions. The statistics of the identical particles are determined by the many-particle Hamiltonian. The particular two-particle Hamiltonian (4.1.4) gives rise to Bose statistics. A curious reader may wonder what kind of two-particle Hamiltonian gives rise to Fermi statistics. As an example, let me just write a two-particle Hamiltonian that gives rise to Fermi statistics in 2D (in non-relativistic

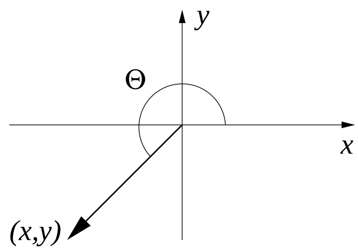

Figure 4.1: The definition of the function $\Theta(x,y)$.

limit):

$$\hat{H}_2^{\text{ferm}} = -\frac{(\partial_{x_1}+\mathrm{i}a_x)^2+(\partial_{y_1}+\mathrm{i}a_y)^2}{2m} - \frac{(\partial_{x_2}-\mathrm{i}a_x)^2+(\partial_{y_2}-\mathrm{i}a_y)^2}{2m}$$
$$a_x = \frac{y_1-y_2}{(x_1-x_2)^2+(y_1-y_2)^2}, \qquad a_y = -\frac{x_1-x_2}{(x_1-x_2)^2+(y_1-y_2)^2}, \tag{4.2.1}$$

We note that $\hat{H}_2^{\text{ferm}}$ is still invariant under the exchange $x_1 \leftrightarrow x_2$. Such a two-particle Hamiltonian when acting on symmetric wave functions $\psi(x_1,y_1,x_2,y_2)=\psi(x_2,y_2,x_1,y_1)$ describes two fermions in two dimensions.

$\hat{H}_2^{\text{ferm}}$ can be simplified by the following transformation

$$\psi(x_1,y_1,x_2,y_2) = \mathrm{e}^{\mathrm{i}\Theta(x_1-x_2,y_1-y_2)}\tilde{\psi}(x_1,y_1,x_2,y_2)$$
$$\hat{H}_2^{\text{ferm}} = \mathrm{e}^{\mathrm{i}\Theta(x_1-x_2,y_1-y_2)}\tilde{\hat{H}}_2^{\text{ferm}}\mathrm{e}^{-\mathrm{i}\Theta(x_1-x_2,y_1-y_2)} \tag{4.2.2}$$

where $\Theta(x,y)$ is the angle between the vector (x,y) and the positive x direction (see Fig. 4.1). For positive x and y, $\Theta(x,y)=\arctan\left(\frac{y}{x}\right)$. Although $\Theta(x,y)$ is discontinuous on the positive x-axis with a discontinuity of 2π, the function $\mathrm{e}^{\mathrm{i}\Theta(x,y)}$ is a smooth function of (x,y) (except at $(x,y)=(0,0)$). Using the relation

$$\mathrm{e}^{\mathrm{i}\Theta(x,y)}\partial_x\mathrm{e}^{-\mathrm{i}\Theta(x,y)} = \mathrm{i}\frac{y}{x^2+y^2}, \qquad \mathrm{e}^{\mathrm{i}\Theta(x,y)}\partial_y\mathrm{e}^{-\mathrm{i}\Theta(x,y)} = -\mathrm{i}\frac{x}{x^2+y^2},$$

we find that the transformed Hamiltonian $\tilde{\hat{H}}_2^{\text{ferm}}$ has a simple form

$$\tilde{\hat{H}}_2^{\text{ferm}} = -\frac{1}{2m}(\partial_{x_1}^2+\partial_{y_1}^2) - \frac{1}{2m}(\partial_{x_2}^2+\partial_{y_2}^2).$$

From $\mathrm{e}^{\mathrm{i}\Theta(x,y)} = -\mathrm{e}^{\mathrm{i}\Theta(-x,-y)}$, we can show that the transformed wave function is antisymmetric $\tilde{\psi}(x_1,y_1,x_2,y_2) = -\tilde{\psi}(x_2,y_2,x_1,y_1)$.

So the simple two-particle Hamiltonian $\tilde{\hat{H}}_2^{\text{ferm}}$ when acting on antisymmetric wave functions $\tilde{\psi}(x_1,y_1,x_2,y_2)$ describes two fermions in two dimensions. This way we recover the usual result that fermions are described by antisymmetric wave functions.

In the standard way to understand fermions, fermions are *defined* as particles described by antisymmetry wave functions. Through the above discussion, we see that this standard understanding of fermions did not capture the essence of Fermi statistics. This is because fermions can be described by both symmetric wave functions (with a complicated many-particle Hamiltonian) or antisymmetric wave functions (with a simpler many-particle Hamiltonian).

I personally believe that symmetric wave functions plus complicated many-particle Hamiltonian is a correct way to understand fermions, at least physically. The standard understanding using

antisymmetric wave function is very formal and misleading despite its mathematical simplicity. The confusion cause by the standard understanding is reflected in the following conversation:
A: I have two fermions. One at $\boldsymbol{x}_1$ and the other at $\boldsymbol{x}_2$. I wonder what is the amplitude of such a state.
B: Well it depends on how you say it. If you say one fermion at $\boldsymbol{x}_1$ and one fermion at $\boldsymbol{x}_2$, the amplitude will be ψ. If you say one fermion at $\boldsymbol{x}_2$ and one fermion at $\boldsymbol{x}_1$, the amplitude will be $-\psi$.
A: This is ridiculous. The two ways of saying mean exactly the same thing. How come it leads to two different results.
B: Well, it is not that ridiculous. You know that two wave functions differ by a total phase factor $\mathrm{e}^{\mathrm{i}\theta}$ actually describe the same quantum state. So the amplitudes ψ and $-\psi$ actually correspond to the same physical state. There is no contradiction.
A: But then why is the minus sign important? Why does the minus sign characterize the Fermi statistics? Saying one particle at $\boldsymbol{x}_2$ and the other at $\boldsymbol{x}_1$ may very well leads to an amplitude $e^{\mathrm{i}\theta}\psi$ instead of $-\psi$. The phase θ should have no physical meaning, less to determine the statistics of the identical particles.
B: Well, we should look at the wave function of identical particles $\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n)$ as a whole. Imposing an arbitrary exchange phase, say $\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n) = \mathrm{e}^{\mathrm{i}\theta}\psi(\boldsymbol{x}_2, \boldsymbol{x}_1, ..., \boldsymbol{x}_n)$, may result in a discontinuous many-particle wave function $\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n)$. Only when $\theta = 0$ or π can we have a continuous wave function.
A: But the continuity of the wave function should not be essential. If the identical particles are defined on a lattice, the continuity of the wave function will be meaningless. On the other hand, identical particles on lattice still have well defined statistics (and even fractional statistics in 2D).

I hope that I have made my point. The statistics of identical particles is a very tricky subject. Using exchange symmetry of many-particle wave function to understand statistics is formal and misleading. It misses the essence of statistics. If we understand the statistics that way, the origin of statistics will appear to be very mysterious. Such a understanding does not tell us how to make identical particles with different statistics. It does not encourage us to think how to make identical particles with different statistics. It suggests that the statistics is fundamental and is given. We just have to accept it.

In contrast, the description of identical particle using symmetric wave function and encoding the statistics in the many-particle Hamiltonian leads to completely different picture. I believe it is a more correct picture that captures more essence of statistics despite its mathematical complexity. Within such a picture, statistics of identical particles is a dynamical property determined by many-particle Hamiltonian. We can change the Hamiltonian to obtain different statistics. We can also naturally obtain fractional statistics in two dimensions. Such an understanding tells us how to make different statistics. We can also have phase transitions that change the statistics of particles. We will have a more detailed discussion of Fermi statistics later.

Problem 4.2.1
By rescaling a_x and a_y in eqn (4.2.1), we can obtain the Hamiltonian that describes two particles with fractional statistics. The following Hamiltonian, when acting on symmetric wave functions, describes two such particles confined by a harmonic potential $\frac{K}{2}(x^2 + y^2)$:

$$\begin{aligned}
\hat{H}_2^{\text{frac}} &= -\frac{1}{2m}(\partial_{x_1} + ia_x)^2 - \frac{1}{2m}(\partial_{y_1} + ia_y)^2 - \frac{1}{2m}(\partial_{x_2} - ia_x)^2 - \frac{1}{2m}(\partial_{y_2} - ia_y)^2 \\
&\quad + \frac{K}{2}(x_1^2 + y_1^2 + x_2^2 + y_2^2) \\
a_x &= \frac{\theta}{\pi}\frac{y_1 - y_2}{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \qquad a_y = -\frac{\theta}{\pi}\frac{x_1 - x_2}{(x_1 - x_2)^2 + (y_1 - y_2)^2},
\end{aligned} \tag{4.2.3}$$

where θ is the statistical angle. $\theta = 0$ correspond to bosons and $\theta = \pi$ correspond to fermions.
(a) Find the ground state energy of $\hat{H}_2^{\text{frac}}$ for $\theta = 0$, $\pi/2$, and π.

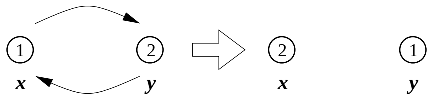

Figure 4.2: Exchanging two particles.

(b) The ground state energy for two bosons/fermions in the harmonic potential $\frac{K}{2}(x^2+y^2)$ can also be obtained by filling energy levels. Check the correctness of your results in (a) against the results obtained by filling energy levels.
(c) Can we still use a transformation similar to eqn (4.2.2) to encode the fractional statistics in the exchange symmetry of the wave function?

Problem 4.2.2
(a) Find a 3-particle Hamiltonian that acts on symmetric wave functions and describes Fermi statistics. (Hint: You may generalize eqn (4.2.1)).
(b) Find a transformation that transform symmetric wave function to antisymmetric wave function. Find the transformed 3-particle Hamiltonian. (Try to choose the 3-particle Hamiltonian in (a) such that the transformed 3-particle Hamiltonian is a sum of three one-particle Hamiltonians.)

4.3 A second look at the free boson system

- *The bosons states with different numbers of bosons can all be labeled by occupation numbers.*
- *A free boson system is equivalent to collection of harmonic oscillators.*

In the section 4.1, we have viewed vacuum as an empty stage. The bosons are actors on the stage. In this picture, the existence and the origin of identical particles are very mysterious. To appreciate this point, let us consider a state with particle 1 at $\boldsymbol{x}$ and particles 2 at $\boldsymbol{y}$. After exchanging the two particles we get another state with particle 1 at $\boldsymbol{y}$ and particles 2 at $\boldsymbol{x}$ (see Fig. 4.2). If the two states are different, then the two particles are not identical. If the two states before and after the exchange are actually the same state, then the two particles are called *identical particles*

But why the two states *have* to be the same? It appears that we can always follow the trajectories of the particles in time history to distinguish the two states before and after the exchange. This is the source of the mystery of identical particles, if we view particles are somethings placed in an empty vacuum. We wonder where do identical particles come from? Why do they have to exist? The mystery of identical particles is one of most fundamental mystery of our nature. It reflects certain deep structures in physics law and the properties of our vacuum. But what is the message? What does the existence of identical particle tell us about the physics laws and the properties of vacuum? In this section, we will try to provide an answer to those fundamental questions. We will use the 1D free boson system to illustrate our points.

We assume the bosons live on circle of length L. In this case, the wave vectors k are quantized:

$$k = \kappa_n \equiv \frac{2\pi}{L} n$$

where n is an integer. The total Hilbert space of arbitrary number of bosons is formed by the no-boson state, one-boson states, two-boson states, etc:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \cdots = \{|0\rangle, |k_1\rangle, |k_1 k_2\rangle, \cdots\}$$

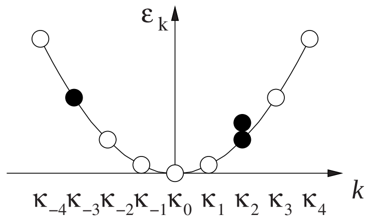

Figure 4.3: The circles on the dispersion curve ϵ_k correspond to allowed k-levels labeled by κ_n. The empty dots represent unoccupied k-levels with $n_{\kappa_n} = 0$, and the filled dots occupied k-levels with $n_{\kappa_n} = 1$. Two filled dots together represent doubly occupied levels with $n_{\kappa_n} = 2$. The state in the above graph is described by $|\cdots n_{\kappa_{-1}} n_{\kappa_0} n_{\kappa_1} n_{\kappa_2} \cdots\rangle$ with $n_{\kappa_{-3}} = 1$, $n_{\kappa_2} = 2$ and other $n_{\kappa_n} = 0$ in the occupation-number notation. It is a three-boson state described by $|k_1 k_2 k_3\rangle$ with $k_1 = -6\pi/L$ and $k_2 = k_3 = 4\pi/L$ in the wave-vector notation.

Instead of using k_i that describes the momentum of each boson, we can use a set of the occupation numbers n_{κ_n} to label each bosonic state in $\mathcal{H}$ $|\cdots n_{\kappa_{-1}} n_{\kappa_0} n_{\kappa_1} n_{\kappa_2} \cdots\rangle$. n_{κ_n} is the number of bosons in the level with momentum $k = \kappa_n$ (see Fig. 4.3). Such an level will be called k-level. The vacuum state $|0\rangle$ is given by the state with all $n_{\kappa_n} = 0$: $|0\rangle = |\cdots 0000 \cdots\rangle$. The one-boson state $|k_1\rangle$ is given by the state with all $n_{\kappa_n} = 0$, except $n_{k_1} = 1$. A more general state is represented in Fig. 4.3. In this way, all the states in $\mathcal{H}$ are labeled by $|\cdots n_{\kappa_{-1}} n_{\kappa_0} n_{\kappa_1} n_{\kappa_2} \cdots\rangle$ with $n_{\kappa_n} = 0, 1, 2, \cdots$. From the relation between the two ways of labeling states, $|k_1 k_2 \cdots\rangle$ and $|\cdots n_{\kappa_{-1}} n_{\kappa_0} n_{\kappa_1} n_{\kappa_2} \cdots\rangle$, one can show that $\sum_i \epsilon_{k_i} = \sum_n n_{\kappa_n} \epsilon_{\kappa_n}$. So if the bosons are free, the state $|\cdots n_{\kappa_{-1}} n_{\kappa_0} n_{\kappa_1} n_{\kappa_2} \cdots\rangle$ is an energy eigenstate with an energy

$$E_{\text{tot}} = \sum_n n_{\kappa_n} \epsilon_{\kappa_n}. \tag{4.3.1}$$

We like to show that the above free boson system can be viewed as a collection of harmonic oscillators. The different oscillators in the collection are labeled by κ_n. The eigenstates of the oscillator κ_n are labeled by n_{κ_n}. The energy of $|n_{\kappa_n}\rangle$ is $(n_{\kappa_n} + \frac{1}{2})\hbar\omega_{\kappa_n}$ where ω_{κ_n} is the oscillation angular frequency of the oscillator κ_n. If we put the oscillators together, a state of the collection with the oscillator κ_n in the n_{κ_n}th excited state can be denoted as $|\cdots n_{\kappa_{-1}} n_{\kappa_0} n_{\kappa_1} n_{\kappa_2} \cdots\rangle$. The energy of such a state is $\sum_n (n_{\kappa_n} + \frac{1}{2})\hbar\omega_{\kappa_n}$. We see that if we choose $\hbar\omega_{\kappa_n} = \epsilon_{\kappa_n}$, then the above energy will reproduce the boson energy $\sum_n n_{\kappa_n} \epsilon_{\kappa_n}$ apart from an overall constant $\frac{1}{2}\sum_n \epsilon_{\kappa_n}$. Also the oscillator states and the many-boson states have an one-to-one correspondence. Thus the free many-boson system can be viewed as a collection of harmonic oscillators.

4.4 A vibrating-string picture of 1D boson system

- *A 1D boson system is equivalent to quantized vibrating string.*
- *A classical vibrating string provides a classical picture of 1D bosons. The vibrating string has a more formal name – 1D field theory.*

The collection of the oscillators with frequency $\omega_{\kappa_n} = \epsilon_{\kappa_n} = \sqrt{c^2\kappa_n^2 + m^2c^4}$ can be shown to be the vibration modes of a string. This leads to a vibrating-string picture or a field theory of the bosons. Let us consider a string whose dynamics is described by the following wave equation

$$\ddot{h}(x,t) = c^2 \partial_x^2 h(x,t) - m^2 c^4 h(x,t) \tag{4.4.1}$$

where $h(x,t)$ is the vibration amplitude of the string and we assume the string form a loop of length L:

$$h(x,t) = h(x+L,t).$$

To show that the vibration modes of the above wave equation reproduce the collection of the oscillators labeled by κ_n, we rewrite the wave equation as

$$\begin{aligned} \dot{h}(x,t) &= \sqrt{-c^2\partial_x^2 + m^2c^4}\; p(x,t) \\ -\dot{p}(x,t) &= \sqrt{-c^2\partial_x^2 + m^2c^4}\; h(x,t) \end{aligned} \tag{4.4.2}$$

Introducing a complex amplitude

$$\phi(x,t) = \frac{h(x,t) + \mathrm{i}p(x,t)}{\sqrt{2}} \tag{4.4.3}$$

the wave equation (4.4.2) becomes

$$\mathrm{i}\dot{\phi}(x,t) = \sqrt{-c^2\partial_x^2 + m^2c^4}\; \phi(x,t) \tag{4.4.4}$$

which has a form of Schrödinger equation! Now we do mode expansion by writing

$$\phi(x,t) = \sum_{\kappa_n} \phi_{\kappa_n}(t) L^{-1/2}\, \mathrm{e}^{\mathrm{i}\kappa_n x}$$

The wave equation (4.4.4) becomes

$$\mathrm{i}\dot{\phi}_{\kappa_n}(t) = \sqrt{c^2\kappa_n^2 + m^2c^4}\; \phi_{\kappa_n}(t) \tag{4.4.5}$$

Let X_{κ_n} and P_{κ_n} be the real and imaginary parts of $\phi_{\kappa_n} = X_{\kappa_n} + \mathrm{i}\,P_{\kappa_n}$. We rewrite eqn (4.4.5) as

$$\begin{aligned} \dot{X}_{\kappa_n}(t) &= \sqrt{c^2\kappa_n^2 + m^2c^4}\; P_{\kappa_n}(t) \\ \dot{P}_{\kappa_n}(t) &= -\sqrt{c^2\kappa_n^2 + m^2c^4}\; X_{\kappa_n}(t) \end{aligned} \tag{4.4.6}$$

We recognize that that eqn (4.4.6) is the equation of motion of an oscillator with X_{κ_n} as the coordinate and P_{κ_n} as the momentum. There is one oscillator for every $\kappa_n = 2\pi n/L$. The mass of the oscillator is $M_{\kappa_n} = 1/\sqrt{c^2\kappa_n^2 + m^2c^4}$ and the spring constant is $K_{\kappa_n} = \sqrt{c^2\kappa_n^2 + m^2c^4}$. So the total (classical) energy of the oscillators is

$$\begin{aligned} E_{\text{tot}} &= \sum_{\kappa_n} \left(\frac{1}{2M_{\kappa_n}} P_{\kappa_n}^2 + \frac{1}{2} K_{\kappa_n} X_{\kappa_n}^2 \right) \\ &= \int \mathrm{d}x\, \phi^*(x) \sqrt{-c^2\partial_x^2 + m^2c^4}\, \phi(x) \\ &= \int \mathrm{d}x\, \left(\frac{1}{2}\dot{h} \frac{1}{\sqrt{-c^2\partial_x^2 + m^2c^4}}\, \dot{h} + \frac{1}{2} h \sqrt{-c^2\partial_x^2 + m^2c^4}\, h \right) \end{aligned} \tag{4.4.7}$$

The expression of the energy and the equation of motion (4.4.1) or (4.4.4) provide a complete description of the oscillators as a classical system. Such a system of vibrating string is also called a classical field theory (in one dimension) where h or ϕ is the field. From the order of the time derivative, we find that eqn (4.4.1) is a coordinate-space equation of motion while eqn (4.4.4) is a phase-space equation of motion.

For oscillator κ_n, the oscillation frequency is $\omega_{\kappa_n} = \sqrt{K_{\kappa_n}/M_{\kappa_n}} = \sqrt{c^2\kappa_n^2 + m^2c^4}$. We see that $\hbar\omega_{\kappa_n} = \epsilon_{\kappa_n}$ (note that $\hbar = 1$). So indeed the vibration modes of the string give rise to the collection

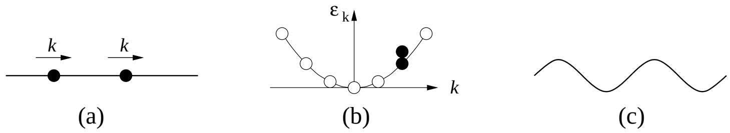

Figure 4.4: Three pictures of a two-boson state where both bosons have the same momentum $k = 2\frac{2\pi}{L}$. (a) The momentum picture, (b) the occupation picture, and (c) the vibrating-string picture.

Figure 4.5: (a) A boson gas and (b) the same boson gas as a vibrating string.

of the oscillators that describes the bosons with dispersion relation $\epsilon_k = \sqrt{c^2k^2 + m^2c^4}$ (see Fig. 4.4).

The above result is quite amazing. The quantum theory of a vibrating string described by eqn (4.4.1) is the same as the quantum theory of bosons described by eqn (4.1.6)! So instead of using the picture Fig. 4.5a to describe a quantum boson gas, we can also use the picture Fig. 4.5b to describe the same quantum boson gas. The picture Fig. 4.5b represents a field theory description of the boson gas.

The vibrating-string picture of the boson gas only works in 1D. In 2D, we need to replace string by membrane: a boson gas in 2D can also be regarded a vibrating membrane. Similarly in d-dimensions, a boson gas is equivalent to a vibrating d-brane (a d-dimensional membrane).

Problem 4.4.1
Following the example of Fig. 4.4, draw three pictures for a two-boson state with one boson carrying momentum $k = 4\pi/L$ and the other $k = 6\pi/L$.

4.5 *The second quantized description of free bosons

- *Quantization of the vibrating string.*
- *A Hamiltonian without fixing the number of bosons.*
- *Boson creation and annihilation operators.*

The vibrating string discussed in the last section is a classical theory. Such a classical theory does not directly describe the quantum boson gas. Only quantized vibrating string describe the quantum bosons. To quantize the vibrating string, we note that a vibrating string is a collection of oscillators described by $(X_{\kappa_n}, P_{\kappa_n})$. Since $(X_{\kappa_n}, P_{\kappa_n})$ is a canonical coordinate-momentum pair, a quantized theory can be obtained by replacing $(X_{\kappa_n}, P_{\kappa_n})$ by a pair of operators $(\hat{X}_{\kappa_n}, \hat{P}_{\kappa_n})$ that satisfy

$$[\hat{X}_{\kappa_n}, \hat{P}_{\kappa_n}] = \mathrm{i}$$

So after quantization, the classical modes ϕ_{κ_n} and the classical field $\phi(x)$ all become operators:

$$\phi_{\kappa_n} \to \hat{\phi}_{\kappa_n} = \frac{1}{\sqrt{2}}(\hat{X}_{\kappa_n} + \mathrm{i}\hat{P}_{\kappa_n})$$
$$\phi(x) \to \hat{\phi}(x) = \sum_{\kappa_n} L^{-1/2}\mathrm{e}^{\mathrm{i}x\kappa_n}\hat{\psi}_{\kappa_n}$$

One can check that $\hat{\phi}_{\kappa_n}$ and $\hat{\phi}(x)$ satisfy the following algebra

$$[\hat{\phi}_{\kappa_n}, \hat{\phi}^\dagger_{\kappa_m}] = \delta_{\kappa_n\kappa_m}, \qquad [\hat{\phi}_{\kappa_n}, \hat{\phi}_{\kappa_m}] = 0. \tag{4.5.1}$$

and

$$[\hat{\phi}(x), \hat{\phi}^\dagger(x')] = \delta(x - x'), \qquad [\hat{\phi}(x), \hat{\phi}(x')] = 0,$$

The Hamiltonian of the quantized vibrating string can be obtained from the classical energy $E_{\text{tot}} = \sum_{\kappa_n} \frac{1}{2}\epsilon_{\kappa_n}(P^2_{\kappa_n} + X^2_{\kappa_n})$ (see eqn (4.4.7))

$$\begin{aligned}\hat{H}_0 &= \sum_{\kappa_n} \frac{1}{2}\epsilon_{\kappa_n}(\hat{P}^2_{\kappa_n} + \hat{X}^2_{\kappa_n}) \\ &= \sum_{\kappa_n} \epsilon_{\kappa_n}(\hat{\phi}^\dagger_{\kappa_n}\hat{\phi}_{\kappa_n} + \frac{1}{2})\end{aligned} \tag{4.5.2}$$

where ϵ_k is given by eqn (4.1.5). Let n_{κ_n} be the eigenvalues of $\hat{\phi}^\dagger_{\kappa_n}\hat{\phi}_{\kappa_n}$. The eigenstates of $\hat{H}_0$ has a form $|\cdots n_{\kappa_{-1}}n_{\kappa_0}n_{\kappa_1}n_{\kappa_2}\cdots\rangle$ which has an eigenvalue $\sum_{\kappa_n} n_{\kappa_n}\epsilon_{\kappa_n}$. So the above Hamiltonian can also be regarded as the Hamiltonian of the free quantum many-boson system. Such a description of the bosons is called the second quantized description.

From eqn (4.5.1) and eqn (4.5.2), we see that $\hat{\phi}^\dagger_{\kappa_n}$ and $\hat{\phi}_{\kappa_n}$ are the raising and the lowing operators of the oscillator κ_n. Since the eigenvalues of $\hat{\phi}^\dagger_{\kappa_n}\hat{\phi}_{\kappa_n}$ correspond to occupation numbers, the total boson number operator is given by

$$\hat{N} = \sum_{\kappa_n} \hat{\phi}^\dagger_{\kappa_n}\hat{\phi}_{\kappa_n} = \int \mathrm{d}x\, \hat{\phi}^\dagger(x)\hat{\phi}(x)$$

Thus we may interpret $\hat{\phi}^\dagger(x)\hat{\phi}(x)$ as the boson number-density operator. One can also show that

$$[\hat{N}, \hat{\phi}^\dagger] = \hat{\phi}^\dagger, \qquad [\hat{N}, \hat{\phi}] = -\hat{\phi}.$$

where $\hat{\phi}$ is $\hat{\phi}(x)$ or $\hat{\phi}_{\kappa_n}$. Thus $\hat{\phi}^\dagger$ increases the boson number by one while $\hat{\phi}$ decreases the boson number by one. For this reason we also call $\phi^\dagger$ the *creation operator* and ϕ the *annihilation operator* In contract to the Hamiltonian (4.1.6) in the particle picture which only acts on n-particle states, the Hamiltonian (4.5.2) in the second quantized description acts on states with any numbers of bosons.

4.6 Vacuum as a dynamical medium and the Casimir effect

- *The differences between the two views of bosons: the particle picture and the vibrating-string picture.*
- *Vacuum is not empty. It is a dynamical medium just like any materials encountered in condensed matter physics.*

We have discussed two ways to view a many-boson system: the particle picture (which include both the momentum picture $|k_1k_2\cdots\rangle$ and the occupation picture $|\cdots n_{\kappa_{-1}}n_{\kappa_0}n_{\kappa_1}n_{\kappa_2}\cdots\rangle$) and the vibrating-brane picture. The essence of the particle picture is the assumption that the vacuum is empty. In this picture, bosons are thing placed on the empty vacuum. However, in the vibrating-brane picture, the vacuum is regarded as a dynamical non-empty medium. The vibration of such a medium gives rise to bosons in the vacuum. In the last section, we have stressed the mathematical equivalence of the two pictures. However, the two pictures are not equivalent in physical sense.

First, the vibrating-brane picture (or the field theory picture) provides an origin and an explanation of identical particles. The particles arising from the vibrations of a d-brane are naturally and always identical bosons. Actually, it is impossible to obtain non-identical particles from the vibrations of the brane. In contrast, particles do not have to be identical particles in the particle picture. In this case, the existence and the appearance of identical particles is very mysterious.

We can also turn our argument around. The very existence of identical particles in our vacuum suggests that we should not view particles as things placed in an empty vacuum. We should instead view our vacuum as a 3-brane and the particles as the vibrations of the 3-brane. Our vacuum is not empty. It is a dynamical medium whose collective motion give rise to elementary particles. So our vacuum just like a material studied on condensed matter physics. The theory of elementary particles is actually a theory about one material – the vacuum material.

Here we would like to remark that the vibrating-brane only reproduce a particular kind of identical particles – scaler bosons. The photons in our vacuum are actually vector bosons (or gauge bosons) due to its two polarizations and electrons are fermions. Those particles cannot arise from vibrating 3-branes. However, as we will see later in this book, the above philosophy is still correct. We should not view our vacuum as an empty stage. We should view it as a dynamical medium whose collective motion can even give rise to photons and electrons. The vacuum material has a more complicated internal structure than the simple 3-brane. It is this more complicated internal structure that leads to photons, electrons, gluons, quarks,[Wen 2002, 2003*b*] and possibly all other elementary particles observed in our vacuum. In this chapter, for simplicity, we treat photons as massless scaler bosons and regard them as vibrations of a 3-brane.

The above discussion about the advantage of the vibrating-brane picture sounds philosophical. Actually, the vibrating-brane picture has a measurable consequence – Casimir effect.[Casimir 1948] The Casimir effect has been observed in our vacuum, confirming that the vibrating-brane picture is a correct picture while the particle picture is an incorrect picture for bosons.

To understand the Casimir effect, we start with the energy of vacuum. In the particle picture, the vacuum is just a empty stage or a reference point. We naturally assign a zero energy to it and define the reference point of energy. In contrast, the vacuum energy in the vibrating-brane picture is naturally non-zero and is given by the zero-energies $\frac{1}{2}\hbar\omega_k = \frac{1}{2}\epsilon_k$ of the oscillators. For 1D free boson system, the vacuum energy is given by

$$U_{\text{vac}} = \sum_{\kappa_n} \frac{1}{2}\epsilon_{\kappa_n}$$

where ϵ_k is the dispersion of the bosons. One may say that U_{vac} is just a constant term in the total energy which define the reference point of the energy. Such a constant term has no measurable effect. Indeed, U_{vac} cannot be measured directly. However, if we change the dispersion ϵ_k and/or change the distribution of quantized momentum κ_n, then the change in U_{vac} has physical effects and can be measured.

Before discussing how to calculate the change of U_{vac}, let us calculate U_{vac} itself. At first sight, the calculation of U_{vac} appear to be very simple since $U_{\text{vac}} = \frac{1}{2}\sum_{\kappa_n}\epsilon_{\kappa_n} = +\infty$. Certainly, such a simple result of infinity is meaningless. The infinite U_{vac} is the famous infinity problem that plague all forms of field theories. It is so annoying that it once led people to abandon field theories. One can use this infinity problem to argue that the vibrating-brane picture for bosons is wrong.

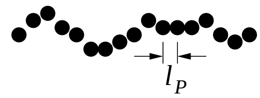

Figure 4.6: According to quantum gravity, a continuous string cannot be a physical reality. A real string may look more like a sequence discrete beads.

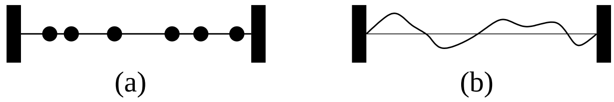

Figure 4.7: (a) A boson gas between two hard walls. (b) The same boson gas is described by a vibrating string with boundary condition $h(0) = h(L) = 0$.

Actually, the problem of the infinity is not the problem of the vibrating-brane picture but the problem of regarding space (or the vibrating-brane) as a continuous manifold. As has been argued in section 2.2, continuous manifold simply does not exist in our universe. It is meaningless and impossible to have two points separated by a distance less the Planck length l_P. Similarly, a wave vector larger than $1/l_P$ is also meaningless. So it is more correct to view the string as a sequence discrete beads (see Fig. 4.6). I would like to stress that this is a quantum gravity effect (see section 2.2). Our vibrating-brane picture is meaningful only for wave vectors less then $\Lambda \sim 1/l_P$ if we take into account the quantum gravity effect. The momentum scale Λ is called the cut-off scale. So we should limit the Wave vector summation $\sum_{\kappa_n}$ to over the meaningful wave vectors $|\kappa_n| < \Lambda$. Therefore, the vacuum energy is really finite

$$U_{\text{vac}} = \sum_{\kappa_n=0,\pm 2\pi/L,\pm 4\pi/L,\cdots,\pm\Lambda} \frac{1}{2}\epsilon_{\kappa_n}$$

Certainly, we are not sure that the energy levels should suddenly disappear for k above Λ as implied by the above formula. We may very well have a softer cut-off

$$U_{\text{vac}} = \sum_{\kappa_n=0,\pm 2\pi/L,\pm 4\pi/L,\cdots} \frac{1}{2}\epsilon_{\kappa_n}\, \mathrm{e}^{-|\kappa_n|/\Lambda}$$

But what is the right way to calculate U_{vac}? The physics at short distance is still unknown to us (since we still do not have a theory of quantum gravity). So it is not clear what is the correct way to cut-off the wave vector summation $\sum_{\kappa_n}$. However, we will see later that the low energy and long distance effects do not depend on how we cut-off the momentum summation. This allows us to make prediction without a complete understanding of the theory. The ignorance of the short distance physics does not prevent us from gaining a (partial) understanding of long distance physics.

Let us consider a 1D mass m boson system between two hard walls (see Fig. 4.7a). One wall is at $x = 0$ and the other at $x = L$. Such a boson system is described by a string (4.4.1) that satisfy the boundary condition $h(0) = h(L) = 0$ (see Fig. 4.7b).[1] In contrast to the periodic boundary condition $h(0) = h(L)$, the string vibration modes for the hard-wall case has a form $h(x) \propto \sin(n\pi x/L)$ and are labeled by $\tilde{\kappa}_n = n\pi/L$, $n = 0, 1, 2, \cdots$. In this case, the vacuum energy

[1]The connection between the hard wall and the boundary condition $h(0) = h(L) = 0$ will be discussed in section 4.7.

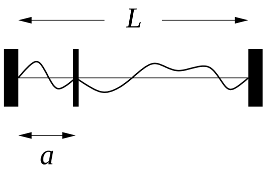

Figure 4.8: The vibrating string with $h(0) = h(a) = h(L) = 0$ describes a boson gas between three hard walls.

is different from that for periodic boundary condition

$$U_{\text{vac}}^{\text{hw}}(L) = \sum_{\tilde{\kappa}_n = 0, \pi/L, 2\pi/L, \cdots} \frac{1}{2} \epsilon_{\tilde{\kappa}_n} \, \mathrm{e}^{-|\tilde{\kappa}_n|/\Lambda}$$

For massless particles, $\epsilon_k = c|k|$. $U_{\text{vac}}^{\text{hw}}(L)$ can be calculated easily. We find

$$U_{\text{vac}}^{\text{hw}}(L) = \frac{c\pi}{8L\sinh^2(\frac{c\pi}{2L\Lambda})} = \frac{\Lambda^2 L}{4c\pi} - \frac{c\pi}{24L} + O(\Lambda^{-2}) \tag{4.6.1}$$

Now let us add the third hard walls at $x = a$ (see Fig. 4.8). The third hard wall modifies the distribution of the vibration modes which changes the vacuum energy. Using the result (4.6.1), we find the modified vacuum energy to be

$$U_{\text{vac}}^{\text{hw}}(a, L) = \frac{\Lambda^2 L}{4c\pi} - \frac{c\pi}{24(L-a)} - \frac{c\pi}{24a} + O(\Lambda^{-2})$$

When L is very large (i.e. $L \gg a$), we find that the total vacuum energy depends on the separation between the two walls at $x = 0$ and a: $U_{\text{vac}}^{\text{hw}}(a, L) = -\frac{c\pi}{24a}$+Const. This causes an attractive force between the two walls separated by a distance a

$$F = \frac{\pi\hbar c}{24a^2}$$

This force between two walls in vacuum is called the Casimir effect.[Casimir 1948] It is interesting to note that the force does not depend on the cut-off.

The above result is for massless bosons in 1D. For massless photons in 3D, the attractive force between two plates separated by a is

$$F = \frac{\pi^2\hbar c}{240a^4} A$$

where A is the area of the plates. Such a force was measured by Spamaay, 1958 and Lamoreaux, 1997, indicating that our vacuum is really a dynamical medium. The study of our vacuum and its elementary particles is really a material science.

4.7 Classical field theory for non-relativistic free bosons

• *Phase-space Lagrangian for the classical field theory that describes free bosons.*

In the section 4.4, we discussed the classical field theory (or the vibrating-string picture) of relativistic bosons. The same calculation also apply to a d-dimensional non-relativistic bosons in a potential U. Such non-relativistic bosons have a dispersion

$$\epsilon_{\boldsymbol{k}} = \frac{\boldsymbol{k}^2}{2m} + U$$

and are described by an n-particle Hamiltonian

$$\hat{H}_n = \sum_{i=1}^{n}(-\frac{1}{2m}\partial_{\boldsymbol{x}_i}^2 + U) \tag{4.7.1}$$

Following the discussion in the last section, we can show that the quantum system (4.7.1) is related to a classical system – a classical field theory or a vibrating brane. The quantized vibrating brane describes the quantum system (4.7.1). The classical system[2] is described by a phase-space equation of motion

$$\mathrm{i}\dot{\phi}(\boldsymbol{x}, t) = \left(-\frac{1}{2m}\partial_{\boldsymbol{x}}^2 + U\right)\phi(\boldsymbol{x}, t) \tag{4.7.2}$$

and a total energy

$$E_{\text{tot}} = \int \mathrm{d}^d\boldsymbol{x}\, \phi^* \left(-\frac{1}{2m}\partial_{\boldsymbol{x}}^2 + U\right)\phi, \tag{4.7.3}$$

where the complex field ϕ encodes both the amplitude and the velocity of the vibration (see eqn (4.4.3) and eqn (4.4.2)). Both the phase-space equation of motion (4.7.2) and the total energy (4.7.3) can be derived from the following phase-space Lagrangian

$$L = \int \mathrm{d}^d\boldsymbol{x}\, \mathrm{i}\phi^*\dot{\phi} - E_{\text{tot}} = \int \mathrm{d}^d\boldsymbol{x} \left(\mathrm{i}\phi^*\dot{\phi} - \phi^*\left(-\frac{1}{2m}\partial_{\boldsymbol{x}}^2 + U\right)\phi\right). \tag{4.7.4}$$

We note that the potential U in the classical energy (4.7.3) or in the phase-space Lagrangian (4.7.4) may have a spatial dependence $U = U(\boldsymbol{x})$. A hard wall in a certain region can be realized by an inifinite potential $U(\boldsymbol{x})$ in that region. In order for the total energy (4.7.3) to be finite, an inifinite potential in a region will force $\phi(\boldsymbol{x})$ to be zero in that region. This is why a hard wall can be represented by the boundary condition $\phi(\boldsymbol{x}) = 0$ or $h(\boldsymbol{x}) = 0$.

Problem 4.7.1
Derive eqn (4.7.2) and eqn (4.7.3) from the phase-space Lagrangian (4.7.4).

4.8 *The second quantized description of interacting bosons

- *The many-body Hamiltonian that describes interacting bosons.*
- *The boson density operator.*

Now let us turn to a more complicated problem of interacting bosons. Let $V(\boldsymbol{x})$ be the potential energy of two bosons separated by $\boldsymbol{x}$. The Hamiltonian that describe n interacting bosons in d-dimension is

$$\hat{H}_n = \sum_i(-\frac{1}{2m}\partial_{\boldsymbol{x}_i}^2 + U) + \sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j) \tag{4.8.1}$$

[2]The classical field theory (the vibrating brane) is also described by a coordinate-space equation of motion

$$\ddot{h} = \quad -\frac{1}{2m}\partial_{\boldsymbol{x}}^2 + U \quad^2 h$$

where h is the amplitude of the vibration and a Lagrangian

$$L = \int \mathrm{d}^d\boldsymbol{x} \quad \frac{1}{2}\dot{h}\frac{1}{-\frac{1}{2m}\partial_{\boldsymbol{x}}^2 + U}\dot{h} - \frac{1}{2}h \quad -\frac{1}{2m}\partial_{\boldsymbol{x}}^2 + U \quad h \quad .$$

We know that the free bosons have a second quantized description (4.5.2) which describes 0-boson, 1-boson, 2-boson, and general n-boson situations with a single Hamiltonian. What is the second quantized description (or the oscillator description) of the interacting bosons?

Again the second quantized description is written in terms of the boson creation/annihilation operators $\hat{\phi}$ and $\hat{\phi}^\dagger$. To obtain the form of the second quantized Hamiltonian for interacting bosons, we first note that the total potential energy can be rewritten in terms of the boson density

$$\sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j) = \int \mathrm{d}^d\boldsymbol{x}\,\mathrm{d}^d\boldsymbol{x}'\, \frac{1}{2} n(\boldsymbol{x}) V(\boldsymbol{x}-\boldsymbol{x}') n(\boldsymbol{x}')$$

Since the boson density operator is given by $\hat{n}(\boldsymbol{x}) = \hat{\phi}^\dagger(\boldsymbol{x})\hat{\phi}(\boldsymbol{x})$ as discussed in section 4.5, the interaction Hamiltonian is given by

$$\begin{aligned}\hat{H}_{\text{int}} &= \int \mathrm{d}^d\boldsymbol{x}\,\mathrm{d}^d\boldsymbol{x}'\, \frac{1}{2} \hat{n}(\boldsymbol{x}) V(\boldsymbol{x}-\boldsymbol{x}') \hat{n}(\boldsymbol{x}') \\ &= \frac{1}{2\mathcal{V}} \sum_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{k}'} \hat{\phi}^\dagger_{\boldsymbol{k}+\boldsymbol{q}} \hat{\phi}_{\boldsymbol{k}} V_{\boldsymbol{q}} \hat{\phi}^\dagger_{\boldsymbol{k}'-\boldsymbol{q}} \hat{\phi}_{\boldsymbol{k}'}\end{aligned}$$

where

$$V_{\boldsymbol{q}} = \int \mathrm{d}^d\boldsymbol{x}\; \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} V(\boldsymbol{x})$$

Here we have assumed that our system is a d-dimensional cube of volume $\mathcal{V} = L^d$. The wave vectors $\boldsymbol{q}, \boldsymbol{k}, \boldsymbol{k}'$ are quantized, i.e. their components are $2\pi/L$ times integers. The summation $\sum_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{k}'}$ sums over those quantize wave vectors. $\hat{\phi}_{\boldsymbol{k}}$ is given by

$$\hat{\phi}(\boldsymbol{x}) = \sum_{\boldsymbol{k}} L^{-d/2} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\phi}_{\boldsymbol{k}} \tag{4.8.2}$$

and satisfies the algebra

$$[\hat{\phi}_{\boldsymbol{k}}, \hat{\phi}^\dagger_{\boldsymbol{k}'}] = \delta_{\boldsymbol{k}\boldsymbol{k}'}, \qquad [\hat{\phi}_{\boldsymbol{k}}, \hat{\phi}_{\boldsymbol{k}'}] = 0.$$

Putting $\hat{H}_{\text{int}}$ and $\hat{H}_0$ in eqn (4.5.2) together, we find the following Hamiltonian

$$\hat{H} = \sum_{\boldsymbol{k}} \tilde{\epsilon}_{\boldsymbol{k}} (\hat{\phi}^\dagger_{\boldsymbol{k}} \hat{\phi}_{\boldsymbol{k}} + \frac{1}{2}) + \frac{1}{2\mathcal{V}} \sum_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{k}'} \hat{\phi}^\dagger_{\boldsymbol{k}+\boldsymbol{q}} \hat{\phi}_{\boldsymbol{k}} V_{\boldsymbol{q}} \hat{\phi}^\dagger_{\boldsymbol{k}'-\boldsymbol{q}} \hat{\phi}_{\boldsymbol{k}'} \tag{4.8.3}$$

describes the interacting bosons. Naively, one expects $\tilde{\epsilon}_{\boldsymbol{k}}$ describes the boson dispersion and should be taken to be $\tilde{\epsilon}_{\boldsymbol{k}} = \frac{\boldsymbol{k}^2}{2m} + U$. As we will see below, this naive expectation is incorrect. We need to choose a different $\tilde{\epsilon}_{\boldsymbol{k}}$ in order to reduce the single-boson dispersion $\frac{\boldsymbol{k}^2}{2m} + U$.

To determine the proper form of $\tilde{\epsilon}_{\boldsymbol{k}}$, let us discuss a few known eigenstates of $\hat{H}$ in eqn (4.8.3). One eigenstate $|0\rangle$ of $\hat{H}$ is defined by the algebraic relation

$$\hat{\phi}_{\boldsymbol{k}} |0\rangle = 0 \tag{4.8.4}$$

Such a state is an eigenstate of the boson number operator with zero eigenvalue $\hat{N}|0\rangle = 0$. Thus $|0\rangle$ is the vacuum state with no bosons. The energy of $|0\rangle$ is $E_0 = \sum_{\boldsymbol{k}} \frac{1}{2}\tilde{\epsilon}_{\boldsymbol{k}}$. Another class of the eigenstates is given by

$$|\boldsymbol{k}\rangle = \hat{\phi}^\dagger_{\boldsymbol{k}} |0\rangle$$

To show $|\boldsymbol{k}\rangle$ is an eigenstate, we commute $\hat{\phi}$ through $\hat{\phi}^\dagger$ in $\hat{H}$ using the commutation relation (4.8.2) to put all a to the right of $a^\dagger$. Such a procedure is called *normal ordering*. This allows us to rewrite $\hat{H}$ as

$$\hat{H} = \sum_{\boldsymbol{k}} (\epsilon_{\boldsymbol{k}} \hat{\phi}^\dagger_{\boldsymbol{k}} \hat{\phi}_{\boldsymbol{k}} + \frac{1}{2}\tilde{\epsilon}_{\boldsymbol{k}}) + \frac{1}{2\mathcal{V}} \sum_{\boldsymbol{q},\boldsymbol{k},\boldsymbol{k}'} V_{\boldsymbol{q}} \hat{\phi}^\dagger_{\boldsymbol{k}+\boldsymbol{q}} \hat{\phi}^\dagger_{\boldsymbol{k}'-\boldsymbol{q}} \hat{\phi}_{\boldsymbol{k}} \hat{\phi}_{\boldsymbol{k}'} \tag{4.8.5}$$

where $\epsilon_{\boldsymbol{k}}$ is given by

$$\epsilon_{\boldsymbol{k}} = \tilde{\epsilon}_{\boldsymbol{k}} + \frac{V(0)}{2} \tag{4.8.6}$$

and $V(0)$ is $V(\boldsymbol{x})$ at $\boldsymbol{x} = 0$. Using eqn (4.8.5), we easily see that $|\boldsymbol{k}\rangle$ is an eigenstate of $\hat{H}$ with eigenvalue $E_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}} + E_0$. $|\boldsymbol{k}\rangle$ is a state with one boson that carries a momentum $\boldsymbol{k}$. The excitation energy of such a one-boson state is $E_{\boldsymbol{k}} - E_0 = \epsilon_{\boldsymbol{k}}$. We see that the single-boson dispersion is given by $\epsilon_{\boldsymbol{k}}$. To reproduce the dispersion $\epsilon_{\boldsymbol{k}} = \frac{|\boldsymbol{k}|^2}{2m} + U$, we need to choose $\tilde{\epsilon}_{\boldsymbol{k}}$ to be $\tilde{\epsilon}_{\boldsymbol{k}} = \frac{|\boldsymbol{k}|^2}{2m} + U - \frac{V(0)}{2}$.

A generic state described by a collection of boson occupation numbers $\{n_{\boldsymbol{k}}\}$ can be created by the boson creation operators $\hat{\phi}^\dagger_{\boldsymbol{k}}$ from the vacuum state $|0\rangle$:

$$|\{n_{\boldsymbol{k}}\}\rangle \propto \prod_{\boldsymbol{k}} (\hat{\phi}^\dagger_{\boldsymbol{k}})^{n_{\boldsymbol{k}}} |0\rangle$$

Such a state is an eigenstate of the total boson number operator $\hat{N} = \sum_{\boldsymbol{k}} \hat{\phi}^\dagger_{\boldsymbol{k}} \hat{\phi}_{\boldsymbol{k}}$. The total boson number of $|\{n_{\boldsymbol{k}}\}\rangle$. The state $|\{n_{\boldsymbol{k}}\}\rangle$ also carry a definite total momentum $\boldsymbol{P} = \sum_{\boldsymbol{k}} \boldsymbol{k} n_{\boldsymbol{k}}$. However, for interacting bosons the state $|\{n_{\boldsymbol{k}}\}\rangle$ is not an energy eigenstate. In general, it is hard to calculate energy eigenstates.

To summarize, if we choose $\epsilon_{\boldsymbol{k}} = \frac{\boldsymbol{k}^2}{2m} + U$, eqn (4.8.5) and eqn (4.8.1) will describe the same interacting boson system! Since $\hat{H}$ is not quadratic in $\hat{\phi}_{\boldsymbol{k}}$ and $\hat{\phi}^\dagger_{\boldsymbol{k}}$, $\hat{H}$ describes a collection of anharmonic quantum oscillators. So interacting bosons are described by anharmonic oscillators. When $\epsilon_{\boldsymbol{k}} = \frac{\boldsymbol{k}^2}{2m} + U$, eqn (4.8.5) can be written in a more compact form

$$\begin{aligned} \hat{H} = & \int \mathrm{d}^d \boldsymbol{x}\, \hat{\phi}^\dagger(\boldsymbol{x}) \left(-\frac{\partial_{\boldsymbol{x}}^2}{2m} + U \right) \hat{\phi}(\boldsymbol{x}) \\ & + \int \mathrm{d}^d \boldsymbol{x}\, \mathrm{d}^d \boldsymbol{x}'\, \frac{1}{2} V(\boldsymbol{x} - \boldsymbol{x}') \hat{\phi}^\dagger(\boldsymbol{x}) \hat{\phi}^\dagger(\boldsymbol{x}') \hat{\phi}(\boldsymbol{x}') \hat{\phi}(\boldsymbol{x}) \end{aligned} \tag{4.8.7}$$

where $\hat{\phi}(\boldsymbol{x})$ satisfies the following algebra

$$[\hat{\phi}(\boldsymbol{x}), \hat{\phi}^\dagger(\boldsymbol{x}')] = \delta(\boldsymbol{x} - \boldsymbol{x}'), \qquad [\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{x}')] = 0,$$

Problem 4.8.1
Show that $[\hat{N}, \hat{H}] = 0$. So the total boson number is conserved and the Hamiltonian is invariant under the $U(1)$ transformation generated by $\hat{N}$: $\hat{H} = \mathrm{e}^{\mathrm{i}\theta\hat{N}} \hat{H} \mathrm{e}^{-\mathrm{i}\theta\hat{N}}$.

Problem 4.8.2
Find the total momentum operator $\hat{\boldsymbol{P}}$ of the bosons in terms of $\hat{\phi}_{\boldsymbol{k}}$. Show that $[\hat{\boldsymbol{P}}, \hat{H}] = 0$ and hence the total momentum is conserved.

Problem 4.8.3
It is too hard to find the eigenstates and eigenvalues of $\hat{H}$ (4.8.5) in the 2-boson sector. Here we simplify the problem by limiting ourselves to 1D and consider only there k-levels: $k = 0, \pm 2\pi/L$. Find the eigenstates and the eigenvalues of $\hat{H}$ in the 2-boson sector with the above simplification.

Problem 4.8.4
Show eqn (4.8.5).

4.9 Classical field theory of interacting bosons

• *A classical picture for interacting bosons.*

The free boson system (4.7.1) is easy to solve. We do not need a classical vibrating brane picture to understand and to visualize the behavior of free bosons. However, an interacting boson system described by

$$\hat{H}_n = \sum_i (-\frac{1}{2m}\partial_{\boldsymbol{x}_i}^2 + U) + \sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j) \tag{4.9.1}$$

(or eqn (4.8.5)) is entirely a different matter. The interacting Hamiltonian eqn (4.9.1) is so hard to solve that we have no clue what do the low energy eigenstates and the eigenvalues look like. So how can we understand the properties of the interacting bosons without being able to diagonalize the Hamiltonian eqn (4.9.1)?

One way to understand interacting bosons is to find the corresponding classical system and study its low energy collective motions. Then we can quantize those low energy classical motions to obtain low energy quantum properties. This way, we can gain a understanding of the low energy properties of the quantum interacting boson system.

To obtain the corresponding classical system for interacting bosons, let us first find out how to represent the boson density in the classical field theory. We know that if the potential U by δU, the change in the total energy of the bosons will be $\delta E_{\text{tot}} = N\delta U$ where N is the total number of the bosons. From eqn (4.7.3) we see that $\delta E_{\text{tot}} = \int \mathrm{d}^d x\, \phi^*\phi\delta U$. So the total number of the bosons is given by $N = \int \mathrm{d}^d x\, \phi^*\phi$ in field theory and the boson density is

$$n(\boldsymbol{x}) = \phi^*(\boldsymbol{x})\phi(\boldsymbol{x})$$

The total potential energy can be rewritten in terms of the boson density

$$\sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j) = \int \mathrm{d}^d\boldsymbol{x}\,\mathrm{d}^d\boldsymbol{x}'\, \frac{1}{2} n(\boldsymbol{x})V(\boldsymbol{x}-\boldsymbol{x}')n(\boldsymbol{x}')$$

This allows us to guess that the classical field theory for interacting bosons to have a modified total energy

$$E_{\text{tot}} = \int \mathrm{d}^d\boldsymbol{x}\, \phi^* \left(-\frac{\partial_{\boldsymbol{x}}^2}{2m} + U\right)\phi + \int \mathrm{d}^d\boldsymbol{x}\,\mathrm{d}^d\boldsymbol{x}'\, |\phi(\boldsymbol{x})|^2 \frac{V(\boldsymbol{x}-\boldsymbol{x}')}{2} |\phi(\boldsymbol{x}')|^2 \tag{4.9.2}$$

and hence a modified Lagrangian

$$\begin{aligned} L = \int \mathrm{d}^d\boldsymbol{x}\, \mathrm{i}\phi^*\dot{\phi} - E_{\text{tot}} &= \int \mathrm{d}^d\boldsymbol{x} \left(\mathrm{i}\phi^*\dot{\phi} - \phi^*\left(-\frac{1}{2m}\partial_{\boldsymbol{x}}^2 + U\right)\phi\right) \\ &\quad - \int \mathrm{d}^d\boldsymbol{x}\,\mathrm{d}^d\boldsymbol{x}'\, |\phi(\boldsymbol{x})|^2 \frac{V(\boldsymbol{x}-\boldsymbol{x}')}{2} |\phi(\boldsymbol{x}')|^2 \end{aligned} \tag{4.9.3}$$

The corresponding equation of motion can be obtained from L and is given by

$$\begin{aligned} \mathrm{i}\dot{\phi}(\boldsymbol{x},t) &= \left(-\frac{1}{2m}\partial_{\boldsymbol{x}}^2 + U_{\text{eff}}(\boldsymbol{x})\right)\phi(\boldsymbol{x},t) \\ U_{\text{eff}}(\boldsymbol{x}) &= U + \int \mathrm{d}^d\boldsymbol{x}' V(\boldsymbol{x}-\boldsymbol{x}')|\phi(\boldsymbol{x}')|^2 \end{aligned} \tag{4.9.4}$$

We note that eqn (4.9.4) is a non-linear equation. So the interacting bosons are described by a non-harmonic field theory.

Problem 4.9.1
Following the discussion in section 2.3.2 to derive the equation of motion (4.9.4) from the Lagrangian (4.9.3).

Problem 4.9.2
Symmetry and conservation:
(a) Derive the equation of motion (4.9.4) from the Lagrangian (4.9.3).
(b) The Lagrangian (4.9.3) is invariant under a $U(1)$ transformation $\phi \to \mathrm{e}^{\mathrm{i}\theta}\phi$. Show that the particle number is conserved for such a system, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^d\boldsymbol{x}\,|\phi|^2 = 0$.
(c) We include a term $\int \mathrm{d}^d\boldsymbol{x}\, g(\phi + \phi^*)$ in the Lagrangian to break the $U(1)$ invariance. Show that the particle number is no longer conserved for the new system.

Problem 4.9.3
In this section we have "guessed" the classical field theory (4.9.3) or (4.9.4) for the interaction bosons. In fact the classical wave equation (4.9.4) can be "derived" using the equation-of-motion approach described in section 2.6.1 from the second quantized Hamiltonian (4.8.7). Find the operator equation of motion for $\hat\phi(\boldsymbol{x},t) \equiv \mathrm{e}^{\mathrm{i}\hat H}\hat\phi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\hat H}$ using $\partial_t\hat\phi(\boldsymbol{x},t) = \mathrm{i}[\hat H, \hat\phi(\boldsymbol{x},t)]$. Replace $\hat\phi$ and $\hat\phi^\dagger$ by $\langle\hat\phi\rangle = \phi$ and $\langle\hat\phi^\dagger\rangle = \phi^*$ to obtain the corresponding classical equation of motion.

4.10 A quantum phase transition in interacting boson system

Instead of directly studying the very difficult quantum interacting boson system (4.9.1), in the next a few sections, we will study the corresponding classical field theory (or the non-harmonic brane) described by eqn (4.9.3). The classical field theory is much easier to deal with. The physical properties of the classical field theory will give us some good ideas about the physical properties of the quantum interacting bosons.

Since it encodes both coordinates and momenta, the complex field ϕ describes a classical state in the classical field theory. That is each different complex function $\phi(\boldsymbol{x})$ correspond to different classical state. So looking for a classical ground state is equivalent to looking for a complex function that minimize the total energy (4.9.2).

We note that a constant function minimize the kinetic energy term $\int \mathrm{d}^d\boldsymbol{x}\,\phi^*\frac{-\partial_{\boldsymbol{x}}^2}{2m}\phi = \int \mathrm{d}^d\boldsymbol{x}\,\frac{\partial_{\boldsymbol{x}}\phi^*\partial_{\boldsymbol{x}}\phi}{2m}$. So if the interaction V is not too large, we may assume the function that minimize the total energy is a constant. In this case the energy density becomes

$$u = U|\phi|^2 + \frac{1}{2}\bar V|\phi|^4$$

where $\bar V \equiv \int \mathrm{d}^d\boldsymbol{x}\, V(\boldsymbol{x})$.

When $\bar V < 0$, we note that the total energy $E_{\text{tot}} = \int \mathrm{d}^d\boldsymbol{x}\, u$ is not bound from below and has no minimum. So the ground state of boson gas with attractive interaction is well defined.[3] When $\bar V > 0$, we find that the total energy is minimized by the following field (or the classical state)

$$\phi_{\text{grnd}} = \begin{cases} 0, & U > 0 \\ \sqrt{-U/\bar V}, & U < 0 \end{cases} \tag{4.10.1}$$

This field corresponds to the classical ground state of the system. The ground state energy density is

$$u = \begin{cases} 0, & U > 0 \\ -\frac{1}{2}U^2/\bar V, & U < 0 \end{cases} \tag{4.10.2}$$

[3]Here we just pointed out a mathematical inconsistence of a particular mathematical description of an attractive boson gas. It is interesting to think physically what really going to happen to a chamber of boson gas with an attractive interaction?

Remember that U is the potential experienced by a boson and $\bar{V}$ characterize the interaction between two bosons. eqn (4.10.2) tells that how the ground state energy density depends on those parameters. Because the ground state is the state at zero temperature, how ground state energy depends on those parameters will tell us if there are quantum phase transitions[4] or not. As we change the parameters that characterize a system, if the ground state energy changes smoothly, we will say that there is no quantum phase transition. If the ground state energy encounter a singularity, the singularity will represent a quantum phase transition. From eqn (4.10.2) we see that there is a quantum phase transition at $U = 0$ if $\bar{V} > 0$. Since it is the second order derivative $\partial^2 u/\partial U^2$ that has a discontinuity, the phase transition is a second order phase transition. The phase for $U > 0$ contains no bosons since it cost energy have a boson. The phase for $U < 0$ has a non-zero boson density $n = -U/\bar{V}$. This is because for negative U the system can lower its energy by having more bosons. But if there are too many bosons, there will be large cost of interaction energy due to the repulsive interaction between the bosons. So the density $n = -U/\bar{V}$ is a balance between the potential energy due to U and the interaction energy due to $\bar{V}$. Later we will see that the phase with non-zero ϕ is a superfluid phase. It is also called the boson condensed phase.

We would like to remark that the eqn (4.10.2) is really the ground state energy density of the *classical* field theory. It is an approximation of the real ground state energy density of the interaction boson system. So we are not sure if the real ground state energy density contains a singularity or not. Even if the singularity does exist, we are not sure if it is the same type as described by eqn (4.10.2). A more careful study indicates that the real ground state energy density of the interaction boson system does have a singularity that is of the same type as in eqn (4.10.2) if the dimensions of the space is 2 or above. In 1D, the real ground state energy density has a singularity at $U = 0$ but the form of the singularity is different from that in eqn (4.10.2).

4.11 Continuous phase transition and symmetry

- *Two mechanisms for phase transitions.*
- *A continuous phase transition is a symmetry breaking transition.*
- *The concept of order parameter.*

We know that ground state energy (4.10.2) is obtained by minimizing the energy functional (4.9.2). U and $\bar{V}$ are the parameters in the energy functional. Can we have a more general and a deeper understanding when the minimum of an energy functional has a singular dependence on the parameters in the energy functional?

Let us consider a simpler question: when the minimum of an energy function has a singular dependence on the parameters in the energy function? To be concrete, let us consider a real function parameterized by a, b, c, d (where $d > 0$):

$$E_{abcd}(x) = ax + bx^2 + cx^3 + dx^4 \tag{4.11.1}$$

Let $E_0(a, b, c, d)$ be the minimum of $E_{abcd}(x)$. How can the minimum $E_0(a, b, c, d)$ to have a singular dependence on a, b, c, d, knowing that the function $E_{abcd}(x)$ itself has no singularity.

One mechanism for generating singularity in $E_0(a, b, c, d)$ is through the "minima switching" as shown in Fig. 4.9. When $E_{abcd}(x)$ has multiple local minima, a singularity in the global minimum $E_0(a, b, c, d)$ is generated when the global minimum switch from a local minimum to another. The singularities generated by "minimum-switching" always correspond to first order phase transitions since the first order derivative of of the ground state energy E_0 is discontinuous at the singularities.

[4] A quantum phase transition, by definition, is a phase transition at zero temperature.

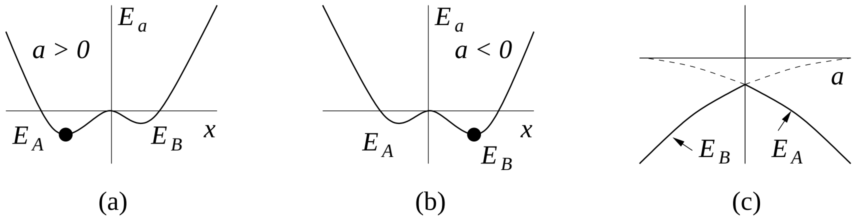

Figure 4.9: The energy function $E_a(x) \equiv E_{a,-1,0,1}(x) = ax - x^2 + x^4$ with $b = -1$, $c = 0$, and $d = 1$. (a) $E_a(x)$ for $a > 0$ and (b) for $a < 0$. E_A and E_B are the energies of the two local minima. The global minimum switch from one local minima to the other as a passes 0. (c) The global minimum $E_0(a)$ has a singularity at $a = 0$.

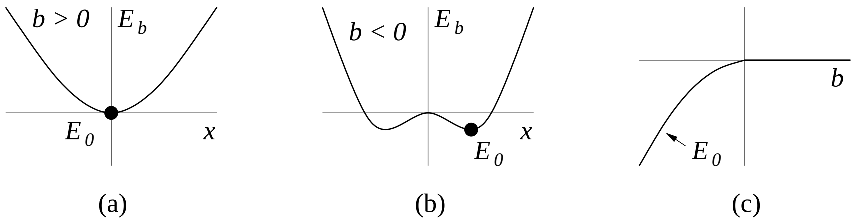

Figure 4.10: The energy function $E_b(x) \equiv E_{0,b,0,1}(x) = bx^2 + x^4$ with $a = 0$, $c = 0$, and $d = 1$ has a $x \to -x$ symmetry. (a) $E_b(x)$ for $b > 0$ and (b) for $b < 0$. (c) The global minimum $E_0(b)$ has a singularity at $b = 0$.

When energy function has a symmetry, there can be another mechanism for generating singularities in the ground state energy. The energy function $E_{abcd}(x)$ has a $x \to -x$ symmetry if $a = c = 0$. For such a symmetric energy function, the single minimum at the symmetric point $x = 0$ for positive b splits into two minima at x_0 and $-x_0$ as b decreases below 0 (see Fig. 4.10). The shifting from the single minimum to one of the two minima generate the singularity in ground state energy $E_0(b)$ at $b = 0$. We will call such a mechanism "minimum-splitting". "Minimum-splitting" always generate continuous phase transitions. This is because the minima before and after the transition are connected continuously (see Fig. 4.11).

We would like to stress that the $x \to -x$ symmetry in the energy function $E_{0b0d}(x)$ is crucial for the existence of the continuous transition caused by the "minimum-slitting". Even a small symmetry breaking term, such as the ax term, will destroy the continuous transition by changing it into a smooth cross-over or a first order phase transition.

When the energy function has a symmetry, one may expect that the minimum (i.e. the ground

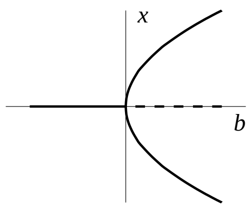

Figure 4.11: Trace of the positions of the minimum/maximum of $E_b(x)$ as we vary b. The solid lines represent minima and the dash line represents the maximum. The single minimum for $b > 0$ splits continuously into two minima when b is lower below zero. The solid curve also shows how the order parameter x becomes non zero after the phase transition.

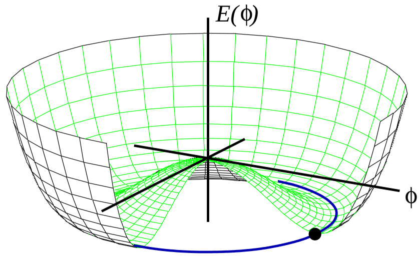

Figure 4.12: The shape of the energy function (4.9.2) for a uniform ϕ when $U < 0$. The dot represents one ground state $\phi = \sqrt{-U/\bar{V}}$ and the think circle represents the infinitely many degenerate ground states $\phi = \mathrm{e}^{\mathrm{i}\theta}\sqrt{-U/\bar{V}}$, $0 < \theta < 2\pi$.

state) also has the symmetry. In our example Fig. 4.10, when $b > 0$ the ground state of E_b (given by $x = 0$) indeed has the $x \to -x$ symmetry (see Fig. 4.10a). However, after the phase transition (i.e. when $b < 0$), the new ground state no longer have the $x \to -x$ symmetry despite the energy function continues to have the same symmetry. Under the $x \to -x$ transformation, the new ground state is changed into another degenerate ground state (see Fig. 4.10b). This phenomenon of ground state having less symmetry than the energy function is called spontaneous symmetry breaking. Our example suggests that the continuous phase transition (caused by the "minimum-splitting") always changes the symmetry of the ground state. So such a transition is also called symmetry breaking transition.

Our simple example (4.11.1) reflects a general phenomenon. The picture described above also applies to more general energy functional such as eqn (4.9.2). It is a deep insight by Landau [Landau 1937; Landau and Lifschitz 1958] that the singularity in the ground energy (or the free energy) is intimately related to the spontaneous symmetry breaking. This leads to a general theory of phase and phase transition based on symmetry and symmetry breaking. Within such a theory, we can introduce an order parameter to characterize different phases. The order parameter must transform non-trivially under the symmetry transformation. In our example Fig. 4.10, we may choose x or x^3 as the order parameter, since they both change signs under $x \to -x$ transformation. In the symmetry unbroken phase, the order parameter $x = 0$. In the symmetry breaking phase, the order parameter is non-zero $x \neq 0$. The continuous phase transition is characterized by the order parameter acquiring a non-zero value. Landua's symmetry breaking theory is so general and so successful that for a long time it was believed that all continuous phase transitions are described by symmetry breaking.

In our field theory description of interacting bosons, the energy functional (4.9.2) has many symmetries, which include a $U(1)$ symmetry $\phi \to \mathrm{e}^{\mathrm{i}\theta}\phi$ and a translation symmetry $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{a}$. The phase $\phi = 0$ (for $U > 0$) is invariant under both the transformations $\phi \to \mathrm{e}^{\mathrm{i}\theta}\phi$ and $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{a}$. Thus the $\phi = 0$ break no symmetries. The phase $\phi = \sqrt{-U/\bar{V}}$ (for $U < 0$) is invariant under the translation $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{a}$ but not the $U(1)$ transformation $\phi \to \mathrm{e}^{\mathrm{i}\theta}\phi$. So the $\phi \neq 0$ phase break the $U(1)$ symmetry spontaneously. Under the $U(1)$ transformation, $\phi = \sqrt{-U/\bar{V}}$ is changed to $\phi = \mathrm{e}^{\mathrm{i}\theta}\sqrt{-U/\bar{V}}$ which corresponds to one of the infinitely many degenerate ground states (see Fig. 4.12). According to Landau's symmetry breaking theory, the two phases, $\phi \neq 0$ and $\phi = 0$, having different symmetries, must be separated by a phase transition.

After understanding the above two mechanism which generate first order phase transitions and continuous phase transitions, one may wonder "is there a third mechanism for the singularity in the ground state energy?" If you do find the third mechanism, it will represent a new type of phase transitions beyond Landau's symmetry breaking theory!

Problem 4.11.1

Show that if we include a term $h\int \mathrm{d}^d\boldsymbol{x}\,(\phi+\phi^*)$ in the energy functional (4.9.2) to explicitly break the $U(1)$ symmetry,[5] the continuous transition cause by changing U will change into a smooth cross-over no matter how small h is.

Problem 4.11.2
Adding $h\int \mathrm{d}^d\boldsymbol{x}\,(\phi+\phi^*)$ term in eqn (4.9.2) completely breaks the $U(1)$ symmetry and destroys the continuous phase transition. Show that adding $h_2\int \mathrm{d}^d\boldsymbol{x}\,(\phi^2+c.c.)$ term in eqn (4.9.2) breaks the $U(1)$ symmetry down to a Z_2 symmetry, i.e. the resulting energy functional is still invariant under $\phi\to-\phi$. Study the phase and the phase transition in the resulting Z_2 symmetric system. Show that the Z_2 symmetry in the energy functional allows a continuous phase transition.

4.12 Collective modes – sound waves

- *Small fluctuations around the ground state have a wave-like dynamics.*
- *The fluctuations around the symmetry breaking ground state have a linear dispersion for small $\boldsymbol{k}$. Those fluctuations are called sound waves.*

For our interacting boson system (4.9.2), the (classical) ground states in the symmetric phase $\phi_{\text{grnd}}=0$ and in the symmetry breaking phase $\phi_{\text{grnd}}=\sqrt{-U/\bar V}$ are very different. As a result, the collective excitations above the ground states are also very different.

The collective fluctuations around the ground state are described $\delta\phi=\phi-\phi_{\text{grnd}}$. The equation of motion for $\delta\phi$ that describes the classical dynamics of the fluctuations can be obtain by substituting $\phi=\delta\phi+\phi_{\text{grnd}}$ into eqn (4.9.4).

To simplify our calculation, we assume the interaction potential $V(\boldsymbol{x}-\boldsymbol{x}')$ is short ranged and approximate it by $V(\boldsymbol{x}-\boldsymbol{x}')=g\delta(\boldsymbol{x}-\boldsymbol{x}')$. The wave equation (4.9.4) and the energy (4.9.2) are simplified to

$$\mathrm{i}\dot\phi(\boldsymbol{x},t)=\left(-\frac{1}{2m}\partial_{\boldsymbol{x}}^2+U+g|\phi(\boldsymbol{x})|^2\right)\phi(\boldsymbol{x},t) \tag{4.12.1}$$

and

$$E_{\text{tot}}=\int \mathrm{d}^d\boldsymbol{x}\,\phi^*\left(-\frac{\partial_{\boldsymbol{x}}^2}{2m}+U+\frac{g}{2}|\phi|^2\right)\phi. \tag{4.12.2}$$

The Lagrangian (4.9.3) is simplified to

$$L=\int \mathrm{d}^d\boldsymbol{x}\,\left(\mathrm{i}\phi^*\dot\phi-\phi^*(-\frac{\partial_{\boldsymbol{x}}^2}{2m}+U)\phi-\frac{g}{2}|\phi|^4\right). \tag{4.12.3}$$

Let us first discuss the equation of motion of $\delta\phi$ in the symmetry breaking phase $\phi_{\text{grnd}}=\sqrt{-U/g}$ for $U<0$ (see eqn (4.10.1) and note that $\bar V=g$). Substituting $\phi=\delta\phi+\sqrt{-U/g}$ into eqn (4.12.1), we find

$$\mathrm{i}\dot{\delta\phi}=\left(-\frac{1}{2m}\partial_{\boldsymbol{x}}^2-U\right)\delta\phi-U\delta\phi^*$$

Since we are interested in low lying fluctuations, we can assume $\delta\phi$ to be small. So in the above equation, we have only kept the terms linear in $\delta\phi$. Separating $\delta\phi$ into real and imaginary part:

[5] Changing the symmetry of the energy functional (or the Hamiltonian) is called explicit symmetry breaking, which should not be confused with the spontaneous symmetry breaking. A spontaneous symmetry breaking refers a change in the symmetry of the ground state with the symmetry of the energy functional (or Hamiltonian) unchanged.

$\phi = \frac{1}{\sqrt{2}}(\delta h + \mathrm{i}\delta p)$, we rewrite the above equation as

$$-\delta\dot{p} = \left(-\frac{1}{2m}\partial_{\boldsymbol{x}}^2 - 2U\right)\delta h$$
$$\delta\dot{h} = -\frac{1}{2m}\partial_{\boldsymbol{x}}^2 \delta p$$

From $\delta\ddot{h} = -\frac{1}{2m}\partial_{\boldsymbol{x}}^2\dot{\delta p}$, we find

$$\delta\ddot{h} = \frac{1}{2m}\partial_{\boldsymbol{x}}^2\left(-\frac{1}{2m}\partial_{\boldsymbol{x}}^2 - 2U\right)\delta h \tag{4.12.4}$$

We see that the equation of motion that describe the weak fluctuations is a linear wave equation. The solutions of eqn (4.12.4) are of form $\delta h = \mathrm{Re}(C\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\omega_{\boldsymbol{k}}t})$. The dispersion relation of the wave is

$$\omega_{\boldsymbol{k}} = \sqrt{\frac{|\boldsymbol{k}|^2}{2m}\left(\frac{|\boldsymbol{k}|^2}{2m} - 2U\right)} = \sqrt{\frac{|\boldsymbol{k}|^2}{2m}\left(\frac{|\boldsymbol{k}|^2}{2m} + 2ng\right)} \tag{4.12.5}$$

where the $n = |\phi_{\text{grnd}}|^2$ is the boson density. For small $\boldsymbol{k}$, we find a linear dispersion

$$\omega_{\boldsymbol{k}} = v|\boldsymbol{k}|, \qquad v = \sqrt{\frac{-U}{m}} = \sqrt{\frac{gn_0}{m}}$$

where v is the velocity of the fluctuating wave. Such a wave in the symmetry breaking phase is called the sound wave in the superfluid.

In the symmetric phase $\phi_{\text{grnd}} = 0$ for $U > 0$, the equation of motion of $\delta\phi = \phi - \phi_{\text{grnd}} = \phi$ is

$$\mathrm{i}\dot{\delta\phi} = \left(-\frac{1}{2m}\partial_{\boldsymbol{x}}^2 + U\right)\delta\phi$$

if we ignore the higher order terms in $\delta\phi$. The solutions have a form $\delta\phi = C\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\omega_{\boldsymbol{k}}t}$. The dispersion relation of the wave is

$$\omega_{\boldsymbol{k}} = \frac{|\boldsymbol{k}|^2}{2m} + U \tag{4.12.6}$$

4.13 Quantized collective modes – phonons

- *Waves = collection of oscillators.*
- *Quantized waves = quantized oscillators = free phonons.*
- *Phonons are a new type of bosons, completely different from the original interacting bosons that form the superfluid.*
- *The emergence of phonons is the simplest example that completely new types of particles can emerge from collective fluctuations.*

We know that a wave with a dispersion $\omega_{\boldsymbol{k}}$ can be viewed as a collection of oscillators.[6] Assuming the space to be a d-dimensional cube of volume $\mathcal{V} = L^d$, then the wave vectors of the wave are quantized: $\boldsymbol{k} = \frac{2\pi}{L}(n_1, n_2, \cdots)$. Each quantized wave vector labels an oscillator. The frequency of the oscillator $\boldsymbol{k}$ is $\omega_{\boldsymbol{k}}$.

[6] We will show explicitly how a wave is related to a collection of oscillators in section 4.14.

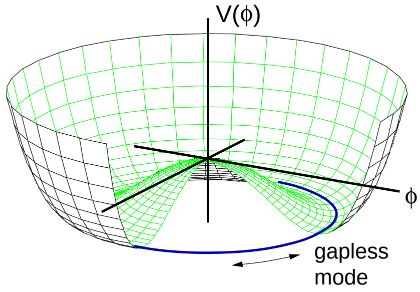

Figure 4.13: The gapless phonons are fluctuations between degenerate ground states.

After identifying the wave (the collections modes) as a collection of oscillators, the quantum theory for the collective modes can then be obtained by quantize those oscillators. The eigenstates of the oscillator $\boldsymbol{k}$ is given by $|n^{\text{ph}}_{\boldsymbol{k}}\rangle$ with energy $n^{\text{ph}}_{\boldsymbol{k}}\omega_{\boldsymbol{k}}$. So the energy eigenstates for the collection of the oscillator are labeled by a set of integers $\{n^{\text{ph}}_{\boldsymbol{k}}\}$: $|\{n^{\text{ph}}_{\boldsymbol{k}}\}\rangle$, where $\boldsymbol{k}$ runs over all quantized wave vectors. The energy of such a state $|\{n^{\text{ph}}_{\boldsymbol{k}}\}\rangle$ is $\sum_{\boldsymbol{k}} n^{\text{ph}}_{\boldsymbol{k}}\omega_{vk}$ where $\sum_{\boldsymbol{k}}$ sums over all the quantized wave vectors.

In the symmetry breaking (the superfluid) phase for $U < 0$, $\omega_{\boldsymbol{k}}$ is given by eqn (4.12.5). In the symmetric phase for $U > 0$, $\omega_{\boldsymbol{k}}$ is given by eqn (4.12.6). The low energy eigenstates are labeled by a set of integers $\{n^{\text{ph}}_{\boldsymbol{k}}\}$. The energies of those eigenstates are $\sum_{\boldsymbol{k}} n^{\text{ph}}_{\boldsymbol{k}}\omega_{vk}$. This way we obtain the low energy eigenstates and their eigenvalues of the very complicated interacting bosons system described by eqn (4.9.1) (or eqn (4.8.5)). Through the classical picture, we are able to obtain the low lying energy eigenstates and their eigenvalues of the Hamiltonian eqn (4.9.1)!

If we interpreted $n^{\text{ph}}_{\boldsymbol{k}}$ as the occupation number of a kind of bosons at the $\boldsymbol{k}$-level, then the collection of the quantum oscillators can also be viewed as the system of free bosons with a dispersion $\omega_{\boldsymbol{k}}$ (see section 4.3). This is quite amazing. We start with an interacting bosons. At the end, we find the low energy excitations of the interacting boson system are described by a free boson system. The distinguish the two kinds of bosons, we will call the free bosons that describe the low lying excitations emergent bosons.

But what are the emergent bosons? In the symmetric phase, the emergent bosons have the dispersion $\omega_{\boldsymbol{k}} = \frac{|\boldsymbol{k}|^2}{2m} + U$, which is exactly the same is the dispersion of the original bosons. In fact, in the symmetric phase, the emergent bosons are the original bosons. This is because the ground state for the symmetry phase, $|\{n^{\text{ph}}_{\boldsymbol{k}} = 0\}\rangle$ is the state with no original bosons. $n^{\text{ph}}_{\boldsymbol{k}}$ in this case is identical to the occupation number $n_{\boldsymbol{k}}$ of the original bosons. For the low energy excitations, only few $n^{\text{ph}}_{\boldsymbol{k}}$'s are non-zero, which correspond to a dilute gas of the original bosons. In this limit, the interactions between the original bosons can be ignore and the original bosons become the free emergent bosons.

However, in the symmetry breaking phase, the emergent bosons have a linear dispersion for small $\boldsymbol{k}$'s, which is very different from the original bosons. The occupation numbers $n^{\text{ph}}_{\boldsymbol{k}}$ of the emergent bosons is not related to the occupation numbers $n_{\boldsymbol{k}}$ of the original bosons. In fact, since $\phi_{\text{grnd}} \neq 0$ and the original bosons have a finite density, the interaction between the original bosons cannot be ignored. In this case, the occupation numbers $n_{\boldsymbol{k}}$ of the original bosons are not even well defined, i.e. the energy eigenstates do not have a definite occupation numbers $n_{\boldsymbol{k}}$ (although they do have a definite occupation numbers $n^{\text{ph}}_{\boldsymbol{k}}$ for the emergent bosons). Since the emergent bosons are completely different from the original bosons, we will give the emergent bosons a new name: phonons.

We note that phonons are *gapless* excitations above the superfluid ground state. We like to pointed out that Gapless excitations are very rare in nature and in condensed matter systems. Therefore, if we see gapless excitations, we should to ask why do they exists?

One mechanism for gapless excitations is spontaneously breaking of a continuous symmetry. As discussed in section 4.11, the superfluid phase spontaneously break the $U(1)$ symmetry: the energy functional has the $U(1)$ symmetry: $E_{\text{tot}}[\phi(\boldsymbol{x})] = E_{\text{tot}}[\mathrm{e}^{\mathrm{i}\theta}\phi(\boldsymbol{x})]$ while the ground state does not: $\phi_{\text{grnd}} \neq \mathrm{e}^{\mathrm{i}\theta}\phi_{\text{grnd}}$. Only in this case, do the gapless excitations exist.

Intuitively if a symmetry (continuous or discrete) is spontaneously broken, then the ground states must be degenerate (see Figs. 4.10b and 4.12). The different ground states are related by the symmetry transformations. So the spontaneous breaking of *continuous* symmetry gives rise to a continuous manifold of degenerate ground states. The fluctuations between the degenerate ground states correspond to the gapless excitations (see Fig. 4.13). Nambu and Goldstone have proved a general theorem: if a continuous symmetry is *spontaneously* broken in a phase, the phase must contain gapless excitations [Nambu 1960; Goldstone 1961]. Those gapless excitations are usually called the Nambu-Goldstone modes. The gapless phonon in the superfluid is a Nambu-Goldstone mode. In the next section, we will give a explicit discussion of the relation between the spontaneous $U(1)$ symmetry breaking and the gapless phonons.

4.14 *An oscillator picture for the sound wave

- *A derivation of the oscillator picture for the sound wave allows us to express the physical quantities of the original bosons, such the boson density, in terms of the oscillator variables. This will allow us to calculate physical properties of interacting bosons using simple oscillators.*

The sound waves in the symmetry breaking phase are described by a collection of oscillators. To show the explicit relation between the sound waves and a collection of oscillators, we start with the field theory Lagrangian (4.12.3) for interacting bosons. Since the sound waves are described by the small fluctuations $\delta\phi = \phi - \phi_{\text{grnd}}$, we can rewrite (4.12.3) in terms of $\delta\phi$ to obtain the Lagrangian for the sound waves. However, to make the $U(1)$ symmetry: $\phi \to \mathrm{e}^{\mathrm{i}\varphi}\phi$ more explicit, we will instead use (θ, ρ) to describe the fluctuations around the ground state $\phi_{\text{grnd}} = \sqrt{-U/g}$. $(\theta, \delta n)$ are defined through

$$\phi(\boldsymbol{x}, t) = \sqrt{n_0 + \delta n(\boldsymbol{x}, t)}\,\mathrm{e}^{\mathrm{i}\theta(\boldsymbol{x},t)}$$

where $n_0 = -U/g$ is the boson density in the ground state.

To the quadratic order in $(\theta, \delta n)$, eqn (4.12.3) becomes

$$L = \int d^d\boldsymbol{x}\ [-(n_0 + \delta n)\partial_t\theta - \frac{n_0(\partial_{\boldsymbol{x}}\theta)^2}{2m} - \frac{(\partial_{\boldsymbol{x}}\delta n)^2}{8mn_0} - \frac{g}{2}\delta n^2]. \tag{4.14.1}$$

The invariance of eqn (4.12.3) under the $U(1)$ transformation $\phi \to \mathrm{e}^{\mathrm{i}\varphi}\phi$ implies that eqn (4.14.1) is invariant under $\theta(\boldsymbol{x}, t) \to \theta(\boldsymbol{x}, t) + \varphi$. This is why the Lagrangian (4.14.1) contains no θ^2 term. The absence of the θ^2 term, as implied by the $U(1)$ symmetry, will leads to gapless excitations.

Introducing[7]

$$\delta\varphi = \frac{1}{2\sqrt{n_0}}\delta n + \mathrm{i}\sqrt{n_0}\theta$$

we can rewrite eqn (4.14.1) as

$$L = \int \mathrm{d}^d\boldsymbol{x}\ \left(\mathrm{i}\delta\varphi^*\delta\dot{\varphi} - \delta\varphi^*\left(-\frac{\partial_{\boldsymbol{x}}^2}{2m} + gn_0\right)\delta\varphi - gn_0\mathrm{Re}(\delta\varphi^2)\right). \tag{4.14.2}$$

[7] $\delta\varphi$ defined this way is equal to $\delta\phi$ up to the linear order in $\delta\phi$: $\delta\varphi = \delta\phi + O(\delta\phi^2)$.

where we have dropped some total time derivative terms. The equal coefficient in the front of the two terms $\delta\varphi^*\delta\varphi$ and $\mathrm{Re}(\delta\varphi^2)$ is a consequence of the $U(1)$ symmetry.

Let us expand the above Lagrangian in terms of $\boldsymbol{k}$-modes. We can expand $\phi(\boldsymbol{x})$ as

$$\delta\varphi(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \mathcal{V}^{-1/2} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \delta\varphi_{\boldsymbol{k}}$$

In terms of $\varphi_{\boldsymbol{k}}$, the Lagrangian for the sound wave takes a form

$$L = \sum_{\boldsymbol{k}} \left(\mathrm{i}\delta\varphi^*_{\boldsymbol{k}}\delta\dot{\varphi}_{\boldsymbol{k}} - \delta\varphi^*_{\boldsymbol{k}} \left(\frac{|\boldsymbol{k}|^2}{2m} + gn_0 \right) \delta\varphi_{\boldsymbol{k}} - \frac{gn_0}{2} (\delta\varphi_{-\boldsymbol{k}}\delta\varphi_{\boldsymbol{k}} + \delta\varphi^*_{-\boldsymbol{k}}\delta\varphi^*_{\boldsymbol{k}}) \right). \tag{4.14.3}$$

If we ignore the term $\frac{gn_0}{2}(\delta\varphi_{-\boldsymbol{k}}\delta\varphi_{\boldsymbol{k}} + \delta\varphi^*_{-\boldsymbol{k}}\delta\varphi^*_{\boldsymbol{k}})$, then each term in the sum $\sum_{\boldsymbol{k}}$

$$L_{\boldsymbol{k}} = \mathrm{i}\delta\varphi^*_{\boldsymbol{k}}\delta\dot{\varphi}_{\boldsymbol{k}} - \delta\varphi^*_{\boldsymbol{k}} \left(\frac{|\boldsymbol{k}|^2}{2m} + gn_0 \right) \delta\varphi_{\boldsymbol{k}} \tag{4.14.4}$$

will describe a decoupled harmonic oscillator where the real part of $\delta\varphi_{\boldsymbol{k}}$ corresponds to the coordinate and the imaginary part of $\delta\varphi_{\boldsymbol{k}}$ corresponds to the momentum of the oscillator. The term $\frac{g}{2}(\delta\varphi_{-\boldsymbol{k}}\delta\varphi_{\boldsymbol{k}} + \delta\varphi^*_{-\boldsymbol{k}}\delta\varphi^*_{\boldsymbol{k}})$ couples the oscillator $\boldsymbol{k}$ to the oscillator $-\boldsymbol{k}$. So eqn (4.14.3) describes a collection of coupled oscillators.

But we can choose a different set of variables to obtain a set of decoupled oscillators. Since the mixing is between the $\boldsymbol{k}$-mode and the $-\boldsymbol{k}$-mode only, let us introduce

$$a_{\boldsymbol{k}} = u_{\boldsymbol{k}}\varphi_{\boldsymbol{k}} + v_{\boldsymbol{k}}\delta\varphi_{-\boldsymbol{k}}$$

One can show that, up to a total time derivative term,

$$\sum_{\boldsymbol{k}} a^*_{\boldsymbol{k}}\dot{a}_{\boldsymbol{k}} = \sum_{\boldsymbol{k}} \varphi^*_{\boldsymbol{k}}\dot{\varphi}_{\boldsymbol{k}}$$

if

$$|u_{\boldsymbol{k}}|^2 - |v_{\boldsymbol{k}}|^2 = 1, \qquad u^*_{\boldsymbol{k}}v_{\boldsymbol{k}} - u_{-\boldsymbol{k}}v^*_{-k} = 0. \tag{4.14.5}$$

One can also show that

$$\sum_{\boldsymbol{k}} E_{\boldsymbol{k}}a^*_{\boldsymbol{k}}a_{\boldsymbol{k}} = \sum_{\boldsymbol{k}} \left(\delta\varphi^*_{\boldsymbol{k}} \left(\frac{|\boldsymbol{k}|^2}{2m} + gn_0 \right) \delta\varphi_{\boldsymbol{k}} + \frac{gn_0}{2} (\delta\varphi_{-\boldsymbol{k}}\delta\varphi_{\boldsymbol{k}} + \delta\varphi^*_{-\boldsymbol{k}}\delta\varphi^*_{\boldsymbol{k}}) \right) \tag{4.14.6}$$

if one chooses

$$u_{\boldsymbol{k}} = \sqrt{\frac{\frac{|\boldsymbol{k}|^2}{2m} + gn_0}{2E_{\boldsymbol{k}}} + \frac{1}{2}}, \qquad v_{\boldsymbol{k}} = \sqrt{\frac{\frac{|\boldsymbol{k}|^2}{2m} + gn_0}{2E_{\boldsymbol{k}}} - \frac{1}{2}},$$
$$E_{\boldsymbol{k}} = \sqrt{\left(\frac{|\boldsymbol{k}|^2}{2m} + gn_0 \right)^2 - (gn_0)^2}. \tag{4.14.7}$$

So in terms of $a_{\boldsymbol{k}}$, the Lagrangian has a form

$$L = \sum_{\boldsymbol{k}} (\mathrm{i}a^*_{\boldsymbol{k}}\dot{a}_{\boldsymbol{k}} - E_{\boldsymbol{k}}a^*_{\boldsymbol{k}}a_{\boldsymbol{k}})$$

which describes a collection of decoupled oscillators.

To show that the term

$$L_{\boldsymbol{k}} = \mathrm{i}a_{\boldsymbol{k}}^* \dot{a}_{\boldsymbol{k}} - E_{\boldsymbol{k}} a_{\boldsymbol{k}}^* a_{\boldsymbol{k}}$$

describes a single harmonic oscillator, we introduce $a_{\boldsymbol{k}} = 2^{-1/2}(X_{\boldsymbol{k}} + \mathrm{i}P_{\boldsymbol{k}})$ and rewrite $L_{\boldsymbol{k}}$ as (up to a total time derivative term)

$$L_{\boldsymbol{k}} = P_{\boldsymbol{k}} \dot{X}_{\boldsymbol{k}} - \left(\frac{P_{\boldsymbol{k}}^2}{2M_{\boldsymbol{k}}} + \frac{K_{\boldsymbol{k}} X_{\boldsymbol{k}}^2}{2} \right)$$

where $M_{\boldsymbol{k}} = 1/E_{\boldsymbol{k}}$ and $K_{\boldsymbol{k}} = E_{\boldsymbol{k}}$. We see that $L_{\boldsymbol{k}}$ is a phase-space Lagrangian that describes an oscillator of mass $M_{\boldsymbol{k}}$ and spring constant $K_{\boldsymbol{k}}$. The oscillation frequency is

$$\omega_{\boldsymbol{k}} = \sqrt{\frac{K_{\boldsymbol{k}}}{M_{\boldsymbol{k}}}} = E_{\boldsymbol{k}} = \sqrt{\left(\frac{|\boldsymbol{k}|^2}{2m} + gn_0 \right)^2 - (gn_0)^2}$$

which agrees with eqn (4.12.5).

We note that in the expression of $E_{\boldsymbol{k}}$ (4.14.7), the first gn_0 comes from the coefficient in front of the $\delta\varphi^*\delta\varphi$ term in eqn (4.14.2). The second gn_0 comes from the coefficient in front of the $\mathrm{Re}(\delta\varphi\delta\varphi)$ term in eqn (4.14.2). We see that the equal coefficient in front of $\delta\varphi^*\delta\varphi$ and $\mathrm{Re}(\delta\varphi^2)$ makes $E_{\boldsymbol{k}} \to 0$ as $\boldsymbol{k} \to 0$. The $U(1)$ symmetry protects the gapless excitations in the superfluid phase.

Problem 4.14.1

Show eqn (4.14.5) and eqn (4.14.6).

Problem 4.14.2

Adding $h_2 \int \mathrm{d}^d\boldsymbol{x}\, (\phi^2 + c.c.)$ term to eqn (4.12.3) breaks the $U(1)$ symmetry down to a Z_2 symmetry. The resulting system still has two phases, one breaks the Z_2 symmetry and the other does not. Show that the collective excitations in both phases have an energy gap. Show that the energy gap of the excitations approach to zero as we approach the continuous phase transition between the two phases.