8.13-14 Experimental Physics I & II "Junior Lab"
Fall 2007 - Spring 2008

PRIMER ON ERRORS

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- **Random & Systematic Errors**
- **Distribution of random errors**
- **Binomial, Poisson, Gaussian**
- **Poisson <-> Gaussian relation**
- **Propagation of Errors**
- **Fits -**

Read Bevington ch. 1-4

- No measurement has infinite accuracy or precision”
- *K.F.Gauss:*
- *“Experimental physics numbers must have errors and dimension.”*
- $G = (6.67310 \pm 0.00010)[m^3\ kg^{-1}\ s^{-2}]$
- **Errors = deviations from Truth** (unknown, but approached)

Large error: Result is insignificant.
Small error: Good measurement, it will test theories.

Blunders : **no,no, –** repeat on Fridays correctly.

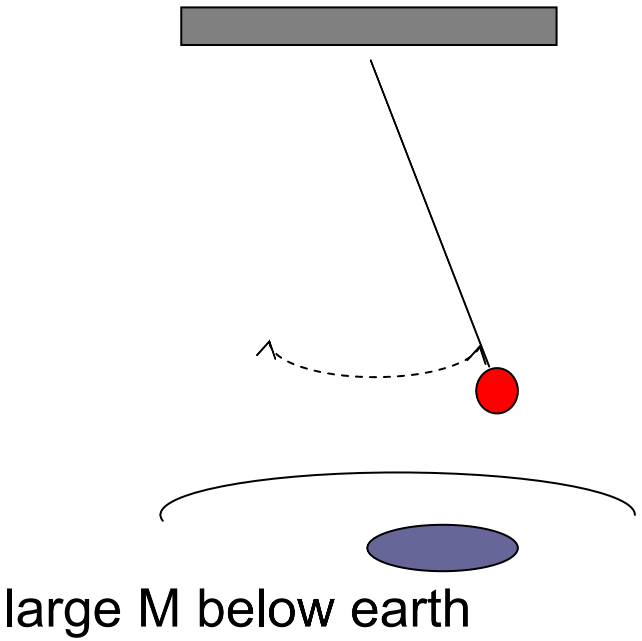
Random Errors or “statistical” - independent, repeated measurements give (slightly) different results

Systematic Errors “in the system”, DVM 2% low,…
correct if you can, otherwise quote separate

Systematic error: inherent to the system or environment

Example:

Measure g with a pendulum: $T = 2\pi\sqrt{\frac{l}{g}}$



neglect m of thread)
equipment (accuracy of scale, watch)
environment (wind, big mass in basement)
thermal expansion t

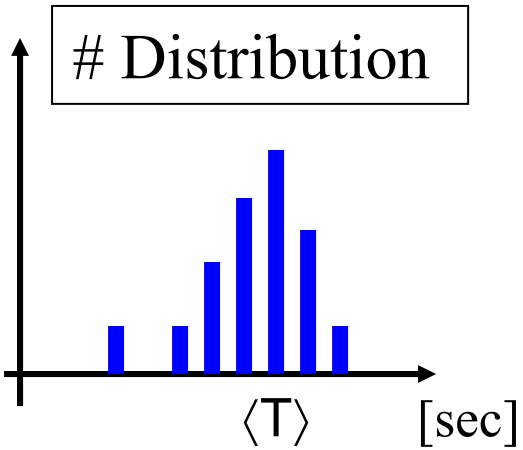
1. correct by $l = l_0(1+\alpha t)$ if t known or
2. give sys.error Δ_t from est. range of t.

$\Rightarrow$ **limits ACCURACY** (=closeness to truth)

Random (statistic) error

example:
measure period T of the pendulum several times:

measurements of T jitter around “truth”.
$\langle T \rangle$ improves with N measurements,
if they are **independent**.



Statistics N:

$\Rightarrow$ limits PRECISION , but lim(N->∞)= truth

Evaluate: **$g = (l \pm \sigma_l)\ \{2\pi / (T \pm \sigma_T)\}^2$** = 9,809 **± ??**

To find the statistical and systematic errors of g, we need error propagation (later) and do it separately- for statistical and for systematic errors. Let's say we found:

$$g = (9.80913 \pm .007_{stat} \pm .o11_{sys})\,m/s^2$$

↑**nonsense digits**

Compare to the "accepted value":

(Physics.nist.gov/physRefData/contents.html) gives'global value'

g = 9.80665 ±(0) by definition !!

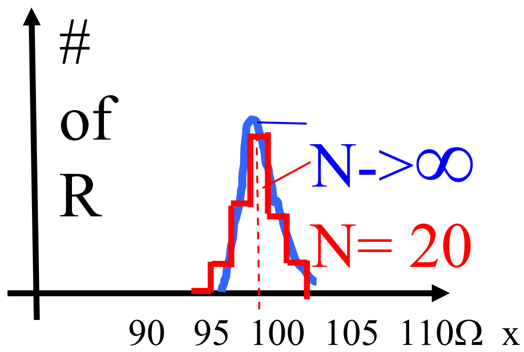
⇒ conclusion: OK within (our) errors

Distribution of Measurements with Random Error:

Limit N->∞ Parent Distribution.

Measure resistors N times:
manufacture 100Ω± 5%
DVM accurate to ± 1%
contact resistance ±0.2Ω

INDEPENDENT !!!



	N measured samples	Parent distribution
Graph	histogram	smooth curve
Average -> Mean	$\langle x \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i$	$\mu = \lim_{N\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)$
Variance -> standard deviation	$s_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \langle x \rangle\right)^2$	$\sigma_x^2 = \lim_{N\to\infty}\left(\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\mu)^2\right)$
Result ± error	$\langle x \rangle \pm s_x/\sqrt{N}$	$\mu \pm 0$

Famous Probability Distibutions:

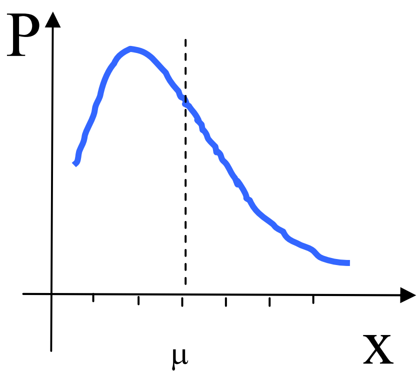
Binomial: Yes(Head): p   No(Tail): q = 1-p

x heads in n trials: $p^x q^{n-x}$ for ppppppp qqqqq seq.,
There are $\binom{x}{n} = \frac{n!}{x!(n-x)!}$ permutations, so that:

$$P(x:n,p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$\text{mean} = n \bullet p =: \mu$$

$$\text{variance} = \sqrt{n \cdot p(1-p)}$$



Seldom used, but grandfather of the following:

Poisson: follows for low p << 1, yet n• p = μ

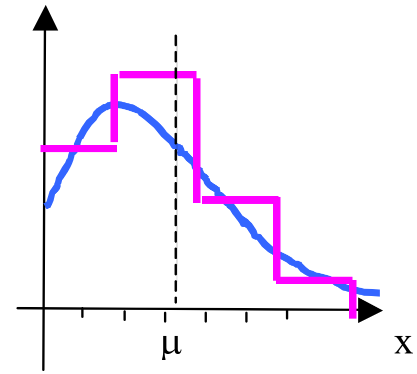
Observe r events in time interval t

$$P(r,\mu) = \frac{\mu^r}{r!} e^{-\mu} = \left(\frac{(\lambda \cdot t)^r}{r!} e^{-\lambda t} \right)$$

mean $= \lambda \cdot \mathrm{t} \quad = \mu$

st. deviation $= \sqrt{\mu}$

Application for low rates r



Intervals between two events: P(0,λ•t) = $e^{-\lambda \bullet t}$ -> t = 0 most likely!!!!

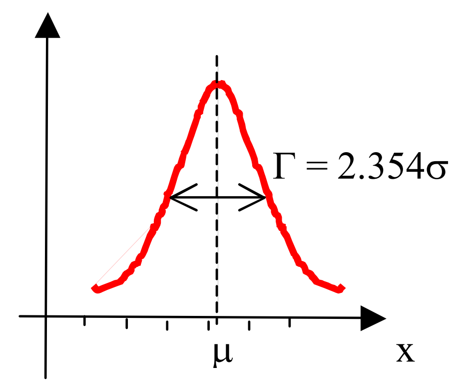
Gaussian.. is the case for n• p >>1 (applicable to 1%, if np >17)

$$P(x;\mu,\sigma) = \frac{1}{2\pi\sigma} e^{\frac{(x-\mu)^2}{2\sigma^2}}$$

mean $= \mu$

St. deviation $\sigma = \sqrt{\mu}$

for counting experiments



Central Limit Theorem:

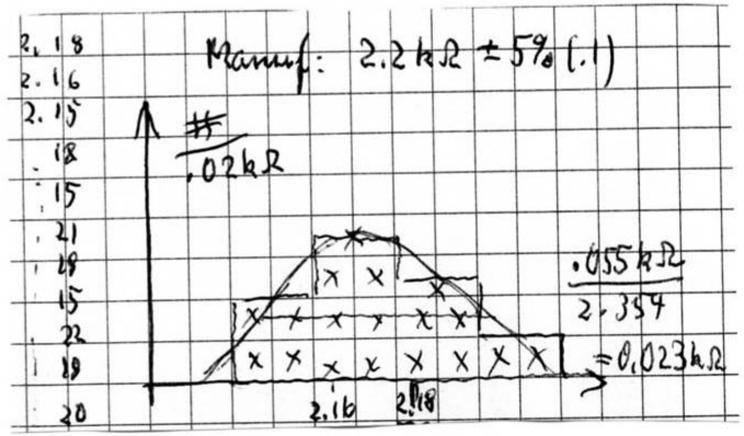
“Folding many different distributions -> always a Gaussian”.

This is the case in most of your measurements.

Distribution of 2200 Ω resistors

$\mu = 2.17 \pm .01$ kΩ

$\sigma = 0.023$ kΩ



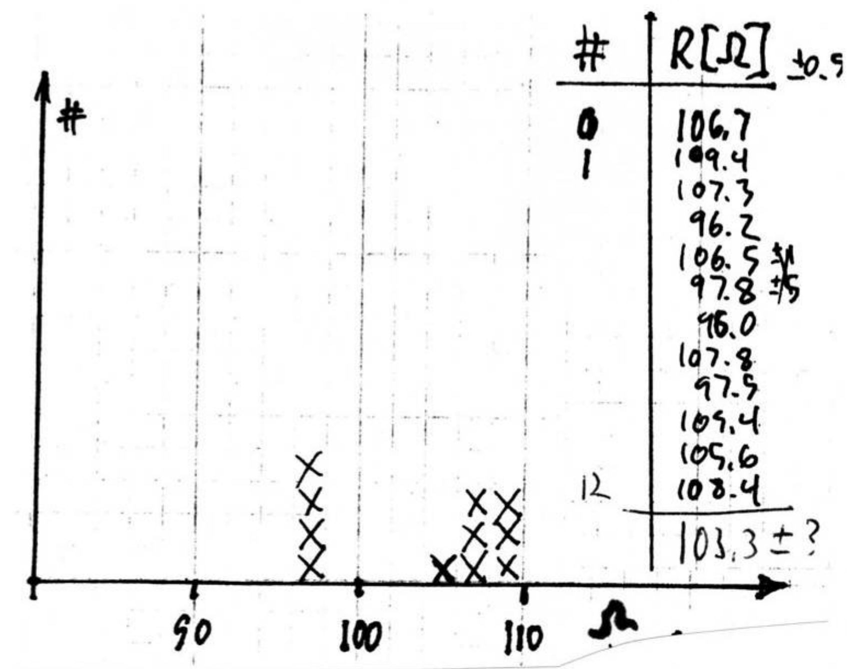
Distribution of 100 ±5 Ω resistors

$\mu = 104 \pm$????

No good fit !!!

Still $\sqrt{s} = 7.5$ Ω

Someone selected out!!



Given a histogram of measurements:

How do you know it is

a) A Poisson distribution?

b) Since you do not know the true mean, check the

c) Sample Variance$^{1/2}$

$$\sqrt{S} = \sqrt{\frac{1}{N-1}\sum_i (x_i - \langle x \rangle)^2}$$

d) Against the standard deviation $\sigma = \sqrt{\mu} \approx \sqrt{\langle x \rangle}$

What doe this mean?

Source $n=10^{10}$ atoms

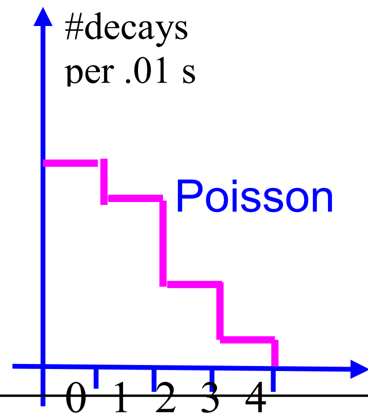
$Fe^{55} \Rightarrow Mn^{55}\gamma$

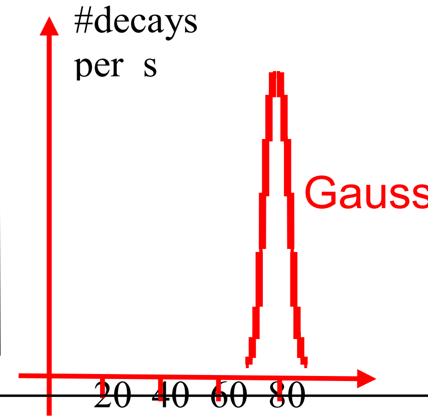
$\tau_{1/2}=2.7y= 8.5\ 10^7 sec$

Expected average

Decays/ Time $=\mu=$ $n\bullet p$

Poisson	**Gaussian**
$p \ll 1$	$n\bullet p \gg 1$
$p=\ln 2/\tau_{1/2} =0.8\ 10^{-8}/s$	$n\bullet p = 80/sec$
0.8 counts/.01s	80 counts/s





As μ increases → Gaussian

For $\mu > 17$ Poisson ≈ Gauss to 1%, Both have variance = $\sqrt{\mu}$

Triva question:"If I make 100 of the left hand measurements and lump together in one distribution, will I see the right hand shape?" YES!!

PROPAGATION OF ERRORS   Bevington ch.3

You measure $u \pm \sigma_u$ and $v \pm \sigma_v$, but evaluate x = u + v or x = u/v
and want the error of x .

Taylor expansion:

$$x_i - \langle x \rangle = (u_i - \langle u \rangle)\frac{\partial x}{\partial u} + (v_i - \langle v \rangle)\frac{\partial x}{\partial v} + \ldots\ldots$$

then

$$\sigma_x^2 = \lim_{N\to\infty}\frac{1}{N}\left((u_i - \langle u \rangle)\frac{\partial x}{\partial u} + (v_i - \langle v \rangle)\frac{\partial x}{\partial v}\right)^2$$

$$= \sigma_u^2\left(\frac{\partial x}{\partial u}\right)^2 + \sigma_v^2\left(\frac{\partial x}{\partial v}\right)^2 + 2\sigma_{uv}^2\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right)$$

= 0 if uncorrelated, otherwise $\sigma_{\mu\nu}$ covariance matrix - difficult

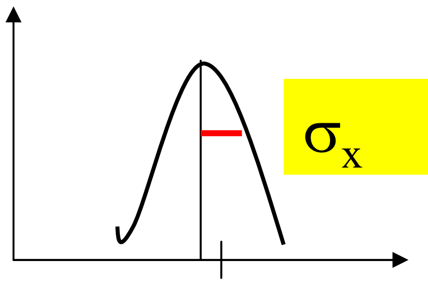
Sum&Diff: x = au + bv -> $\sigma_x^2 = a^2\sigma_u^2 + b^2\sigma_v^2 \pm 2ab\sigma_{uv}^2$

Prod.&Div: x = a.uv or a.u/v -> $\frac{\sigma_x^2}{x^2} = \frac{\sigma_u^2}{u^2} + \frac{\sigma_v^2}{v^2} + \frac{2\sigma_{uv}^2}{uv}$

Powers: $x = a\,T^4$ -> $\frac{\sigma_x}{x} = \pm\, \boxed{4}\, \frac{\sigma_T}{T}$ watch out!

Famous Confusion

Make a measurement, get the mean



What is the error? **NOT σ_x !!!**

$$\sigma_x^2 = s_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \langle x \rangle\right)^2$$

Best estimate = mean: $\langle x \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i \pm \frac{\sigma_x}{\sqrt{N}}$ error of the mean

Bevington eq 4.12

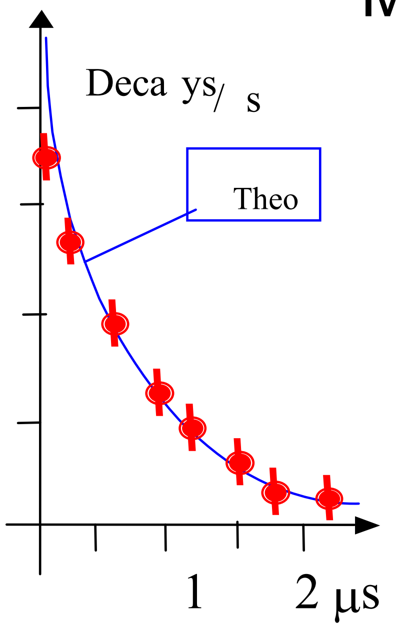
make many measurements!

If measurements x_i have different errors σ_I

Combine: $$\langle x \rangle = \frac{\sum \frac{x_i}{\sigma_i^2}}{\sum \frac{1}{\sigma_i^2}} \quad \text{with} \quad \sigma = \sqrt{\frac{1}{\sum \frac{1}{\sigma_i^2}}}$$

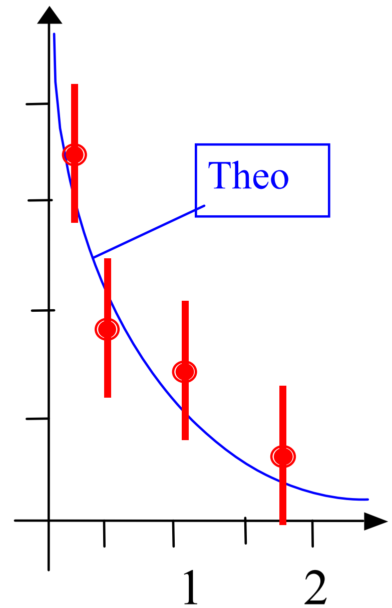
They are called: weighted average ± combined st. dev.

Muon lifetime



Precise data
but inaccurate
T=0 value seems off
Bad fit. Find reason
Delete first point.

$N(t) = (300 \pm 20_{stat} \pm 50_{syst})$
$\exp\{t/(2.05 \pm .03 \mu s\}$
good



Accurate data
but imprecise

More data, finer
intervals needed

$N(t) = (330 \pm 60_{stat} \pm 10_{syst})$
$\exp\{t/(2.3 \pm .6 \mu s\}$
not impressive

The Art of fitting or he χ^2- distribution:

Bevington,ch.4

Assume we know μ_I the true function values in the i-th bin.

We measure n times x_i with random error σ_i in each bin and evaluate

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i} \right)^2$$

Clearly this quantity has a distribution itself, because the next random sample of n measurements will have different χ^2 .

You usually have the reverse case, the true function is not known, but approximated by

a "best fit", based on n measurements.

You want to know: "Is the fit acceptable?"

The χ^2 distribution gives you the probability that your measurement has this particular χ^2. Define χ^2=u and ν = n - #parameters = "degrees of freedom".

(With 2 data points and 2 parameters the curve must go through them, there is no "freedom to fit".)

The distribution is

$$P(u) = \frac{\left(u/2\right)^{\nu/2 - 1} e^{u/2}}{2\Gamma\left(\nu/2\right)} \quad \text{where } \Gamma = \text{Gamma funct.}$$