Polytropic Models for Stars

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1. Integrating the Lane-Emden Equation

1.1. Problem

Integrate the Lane-Emden Equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n \tag{1}$$

for polytropic indices of $n = 1.0$, 1.5, 2.0, 2.5, 3.0, and 3.5.

Break up this second order differential equation into two first-order, coupled equations in $d\phi/d\xi \equiv u$ and $du/d\xi$. Use a 4th-order Runge-Kutta integration scheme or some other equivalent integration method to find $\phi(\xi)$. Recall the boundary conditions at the center: $u(0) = 0$ and $\phi(0) = 1$.

Use the analytic expression for $\phi(\xi)$ near the center:

$$\phi(\xi) = 1 - \frac{\xi^2}{6} \tag{2}$$

to help start the integration. The surface is defined by $\phi(\xi_1) = 0$.

Plot the dimensionless temperature, $\phi(\xi)$, and the dimensionless density, $\phi^n(\xi)$, for all 6 values of n. It would be best to put all the temperature plots on one graph and all the density plots on another.

1.2. Solution

1.2.1. Coupled System of Differential Equations

We take $d\phi/d\xi \equiv u$. Thus we can write

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 u\right) = -\phi^n \tag{3}$$

By the product rule,

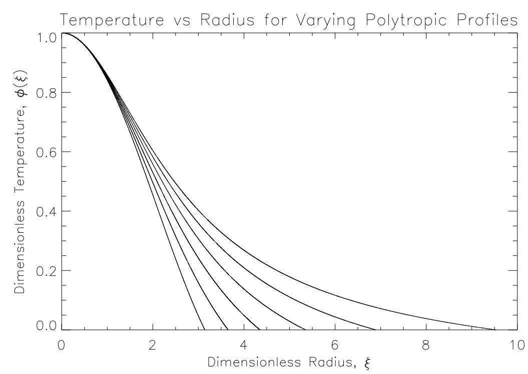
$$\frac{1}{\xi^2}\left(2\xi u + \xi^2\frac{du}{d\xi}\right) = -\phi^n \tag{4}$$

Rearranging, we arrive at the desired coupled system

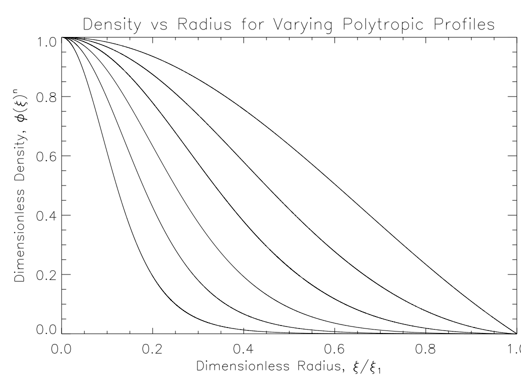
$$\frac{d\phi}{d\xi} = u \tag{5}$$

$$\frac{du}{d\xi} = -\phi^n - \frac{2u}{\xi} \tag{6}$$

Plotting the dimensionless temperature, $\phi(\xi)$ versus the dimensionless radius, ξ, for the n values of interest, we arrive at the graph below. Note that n increases from 1 to 3.5 as we move left to right.



Now we similarly plot the dimensionless density, $\phi(\xi)^n$ versus the normalized dimensionless radius, ξ. Note that n decreases from 3.5 to 1 as we move left to right.



More detailed versions of the plots are in the Appendix.

2. Tabulating Some Physical Properties of Polytropes

2.1. Problem

As the integrations in part 1 are underway, compute for each model the dimensionless potential energy, Ω (in units of $-GM^2/R$), and the dimensionless moment of inertia, k (in units of MR^2). Tabulate ξ_1, $-(d\phi/d\xi)_{\xi_1}$, Ω, and k for each of the 6 polytropic models.

2.2. Solution

2.2.1. Location of Stellar Surfaces

The location of the stellar surface, ξ_1, which is defined by $\phi(\xi_1) = 0$, can be numerically determined using the data from the RK4 integration in section 1. The values of $-(d\phi/d\xi)_{\xi_1}$, which is just $-u(\xi_1)$, can then be found from examining the same RK4 data specifically at ξ_1. These data are tabulated below.

n	ξ_1	$-(d\phi/d\xi)_{\xi_1}$
1.0	3.141	0.318
1.5	3.652	0.203
2.0	4.353	0.127
2.5	5.355	0.0763
3.0	6.896	0.0424
3.5	9.535	0.0208

2.2.2. Dimensionless Potential Energy

The gravitational potential energy of a sphere of radius r is given by the equation

$$U(r) = -4\pi G \int_0^r \frac{M(r')\rho(r')r'^2}{r'} dr' \tag{7}$$

Thus, the first step in determining the potential is to find the mass as a function of ξ. To do so, we want to integrate the density using spherical shells. Let ρ_0 be the central density of the object and let us take the radius of the object to be R. Then we can write

$$M(\xi) = 4\pi\rho_0 \frac{R^3}{\xi_1^3} \int_0^\xi \phi(\xi')^n \xi'^2 d\xi' \tag{8}$$

This allows us to write the expression for potential as

$$U = \frac{-16\pi^2 G\rho_0^2 R^5}{\xi_1^5} \int_0^{\xi_1} \phi(\xi)^n \xi \left[\int_0^\xi \phi(\xi')^n \xi'^2 d\xi' \right] d\xi \tag{9}$$

Ω, the unitless potential (in $-GM^2/R$), will then be given by

$$\Omega = \frac{-UR}{G\left(4\pi\rho_0 \frac{R^3}{\xi_1^3} \int_0^{\xi_1} \phi(\xi')^n \xi'^2 d\xi'\right)^2} \tag{10}$$

Plugging in U, and simplifying, we arrive at the expression

$$\Omega = \frac{\xi_1 \int_0^{\xi_1} \phi(\xi)^n \xi \left[\int_0^\xi \phi(\xi')^n \xi'^2 d\xi'\right] d\xi}{\left(\int_0^{\xi_1} \phi(\xi')^n \xi'^2 d\xi'\right)^2} \tag{11}$$

These integrals do not appear to have a simple closed form. Thus, we set

$$U' = \int_0^{\xi_1} \phi(\xi)^n \xi \left[\int_0^\xi \phi(\xi')^n \xi'^2 d\xi' \right] d\xi \tag{12}$$

and

$$M' = \int_0^{\xi_1} \phi(\xi')^n \xi'^2 d\xi' \tag{13}$$

Numerically evaluating these integrals using a two-point Newton-Cotes method, and then plugging the results into the expression for Ω, we find

n	U'	M'	Ω	Ω_{frac}
1.0	2.355	3.140	0.750	3/4
1.5	1.727	2.713	0.857	6/7
2.0	1.335	2.410	1.000	1
2.5	1.070	2.186	1.200	6/5
3.0	0.885	2.017	1.500	3/2
3.5	0.749	1.889	2.000	2

From these results, we can see that Ω appears to obey the relation

$$\Omega(n) = \frac{3}{5-n} \tag{14}$$

The simplicity of this relation seems to imply that it is possible to find an elegant simplification for the integral formulation of Ω in equation 11.

2.2.3. An Analytical Expression for U

This derivation is due to Chandrasekhar via Cox and Guili.

We begin with the differential of potential energy due to a spherical shell of mass M_r

$$dU = -\frac{GM_r dM_r}{r} \tag{15}$$

From this we can the write

$$dU = d\left(-\frac{GM_r^2}{2r}\right) - \frac{GM_r^2}{2r^2}dr \tag{16}$$

Applying hydrostatic equilibrium,

$$dP/dr = -(GM_r/r^2)\rho$$

we find

$$dU = d\left(-\frac{GM_r^2}{2r}\right) + \frac{1}{2}M_r\frac{dP}{\rho} \tag{17}$$

From the relation

$$P \propto \rho^{(n+1)/n}$$

we can show that

$$\frac{dP}{\rho} = (n+1)d\left(\frac{P}{\rho}\right) \tag{18}$$

which allow us to write equation 17 as

$$dU = d\left(-\frac{GM_r^2}{2r}\right) + \frac{1}{2}(n+1)M_r d\left(\frac{P}{\rho}\right) \tag{19}$$

We can then write this as

$$\begin{aligned} dU =& d\left(-\frac{GM_r^2}{2r}\right) + \frac{1}{2}(n+1)d\left(\frac{M_r P}{\rho}\right) \\ &- \frac{1}{2}(n+1)\frac{P}{\rho}dM_r \end{aligned} \tag{20}$$

We apply the virial theorem, concluding that

$$3\frac{P}{\rho}dM_r - 3\ d\left(P\frac{4}{3}\pi r^3\right) + dU = 0 \tag{21}$$

Now we eliminate $(P/\rho)dM_r$ in equation 20. Solving for dU we find

$$\begin{aligned} dU =& \frac{3}{5-n}\left[d\left(-\frac{GM_r^2}{r}\right) + (n+1)d\left(\frac{M_r P}{\rho}\right)\right. \\ &\left. -(n+1)d\left(P\frac{4}{3}\pi r^3\right)\right] \end{aligned} \tag{22}$$

If we integrate this from $r = 0$ to $r = R$, we can see that the last two terms vanish, giving us

$$U = -\frac{3}{5-n}\frac{GM^2}{R} \tag{23}$$

This is what we found numerically.

2.2.4. *Dimensionless Moment of Inertia*

The moment of inertia of a body is given by the formula

$$I = \int_V \rho(r) r_\perp^2 dV \tag{24}$$

Therefore, the moment of inertia will be proportional to the following integral.

$$I \propto \int_0^{\xi_1}\int_0^{2\pi}\int_0^{\pi} \phi(\xi)^n \xi^4 sin^3(\theta)\ d\theta\ d\phi\ d\xi \tag{25}$$

Simplifying and inserting the correct constants, we arrive at the expression

$$I = \frac{8\pi\rho_0 R^5}{3\xi_1^5}\int_0^{\xi_1} \phi(\xi)^n \xi^4\ d\xi \tag{26}$$

The unitless moment of inertia (in MR^2) will thus be given by

$$k = \frac{\frac{8\pi\rho_0 R^5}{3\xi_1^5}\int_0^{\xi_1}\phi(\xi)^n\xi^4\ d\xi}{\frac{4\pi\rho_0 R^5}{\xi_1^3}\int_0^{\xi}\phi(\xi')^n\xi'^2 d\xi'} \tag{27}$$

Putting M' as above and setting

$$I' = \int_0^{\xi_1} \phi(\xi)^n \xi^4\ d\xi \tag{28}$$

We can write

$$k = \frac{2I'}{3M'\xi_1^2} \tag{29}$$

We again use a two-point Newton-Cotes method to evaluate the integrals and plug in the results to find k.

n	I'	k
1.0	12.152	0.261
1.5	11.116	0.205
2.0	10.607	0.155
2.5	10.511	0.112
3.0	10.843	0.0754
3.5	11.737	0.0456

3. Model of the Sun

3.1. Problem

Use an $n = 3$ polytropic model to represent the internal structure of the Sun. The two parameters to fix are $M = M_\odot$ and $R = R_\odot$.

(a) Plot the physical temperature (in K) vs. radial distance in units of r/R. Plot log T vs. r/R. Do the same for the physical density (g/cm^3). Again, plot log ρ vs. r/R. Instead of using the values for the central density ρ_0, and central temperature T_0, deduced for an $n = 3$ polytrope with $M = M_\odot$ and $R = R_\odot$, take the known values of ρ_0= 158 g/cm^3 and $T_0 = 15.7 \times 10^6$ K.

(b) Compute the nuclear luminosity of the sun using the above temperature and density profiles. Take the nuclear energy generation rate to be

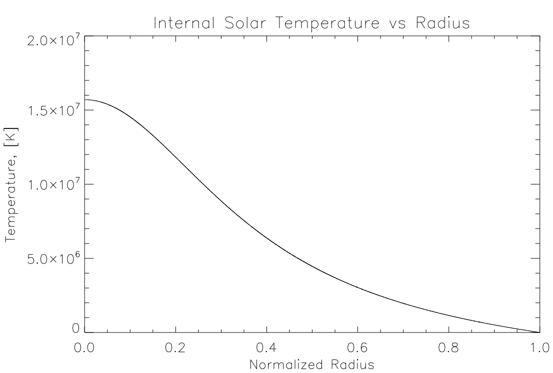
$$\epsilon(\rho, T) = 2.46 \times 10^6 \rho^2 X^2 T_6^{-2/3} e^{(-33.81 T_6^{-1/3})}$$

which is in ergs cm^{-3}sec^{-1}, where ρ is in g/cm^3, T_6 is the temperature in units of 10^6K, and X is the hydrogen mass fraction. (take X = 0.6). Reduce the problem to a dimensioned constant times an integral involving only functions of ϕ and ξ. (There will also appear a T_0 inside the integral for which you can plug in the value of 15.7×10^6) Show the value of your constant and the form of the dimensionless integral. Evaluate the nuclear luminosity of the Sun in units of ergs sec^{-1} .

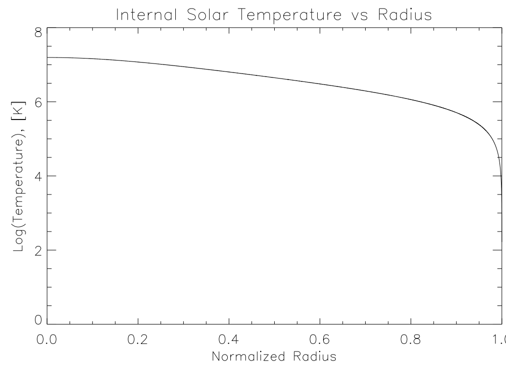
3.2. Solutions

3.2.1. Solar Temperature Plots

We first plot temperature against normalized radius

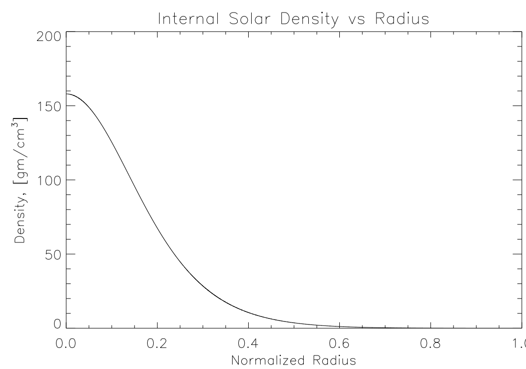


Now we plot the logarithm of temperature

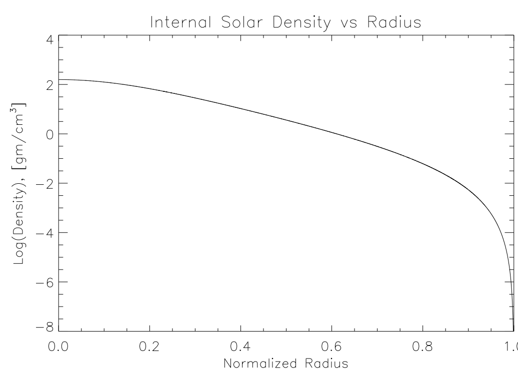


3.2.2. Solar Density Plots

We plot the density against normalized radius



Now we plot the logarithm of density



3.2.3. Solar Nuclear Luminosity

We have both density and temperature for an n = 3 polytrope as a function of ξ. Thus we can write,

$$L_\odot = 4\pi \frac{R_\odot^3}{\xi_1^3} \int_0^{\xi_1} \xi^2 \epsilon(\rho(\xi), T(\xi)) d\xi \qquad (30)$$

Using the unitless, polytropic density and temperature, we can write

$$L_{\odot} = L_c \int_0^{\xi_1} \xi^2 \phi(\xi)^{16/3} e^{-13.5\phi(\xi)^{-1/3}} d\xi \tag{31}$$

where,

$$L_c = 2.46 \times 10^6 \times \rho_0^2 \times X^2 \times T_0^{-2/3} \times 4\pi \frac{R_{\odot}^3}{\xi_1^3} \tag{32}$$

Thus, L_c is our dimensioned constant, which must be in ergs/s, and we can evaluate it, giving

$$L_c = 4.546 \times 10^{40} \tag{33}$$

Numerically evaluating the integral using a two-point Newton-Cotes method, we arrive at the result

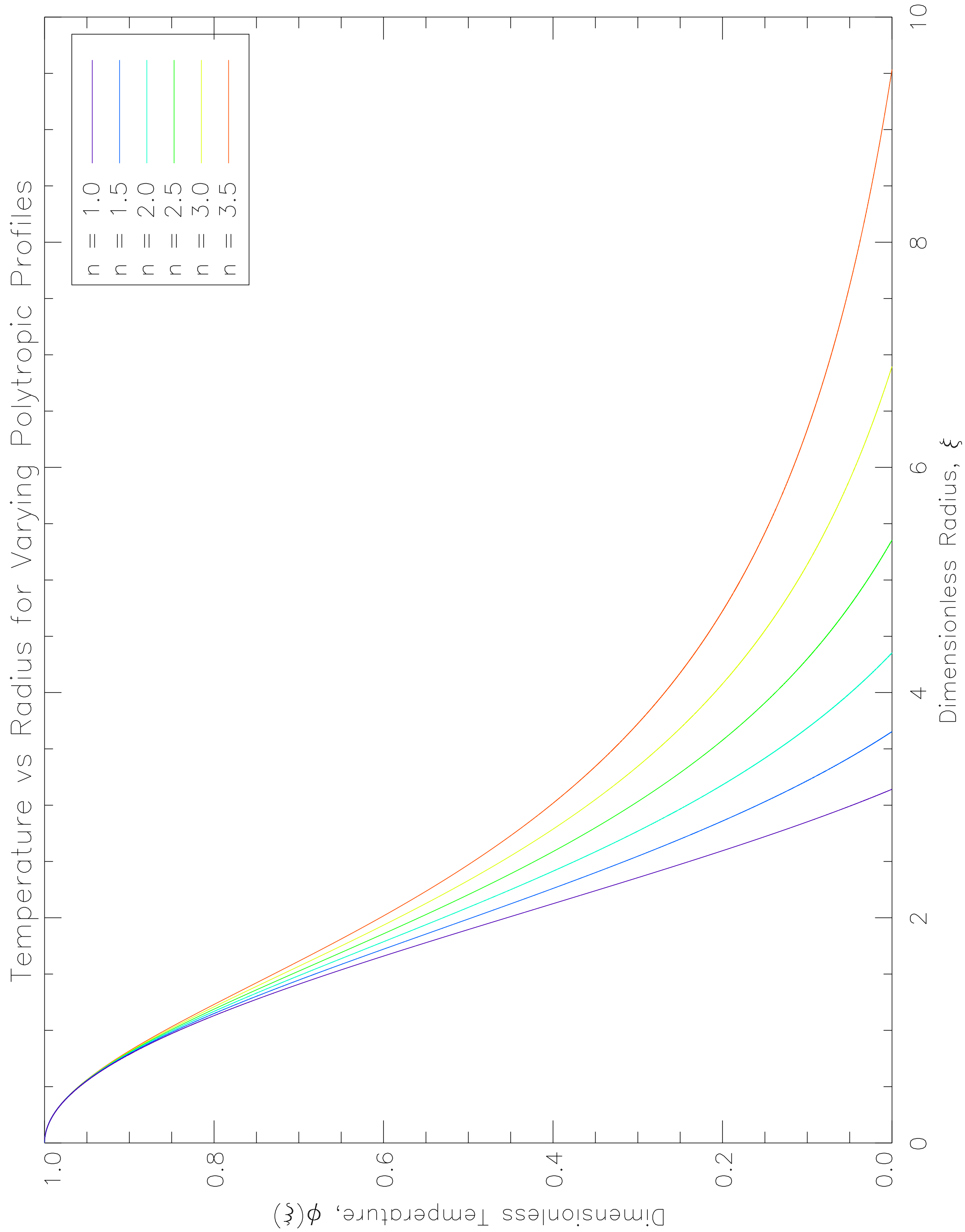
$$\int_0^{\xi_1} \xi^2 \phi(\xi)^{16/3} e^{-13.5\phi(\xi)^{-1/3}} d\xi = 3.26 \times 10^{-7} \tag{34}$$

And thus,

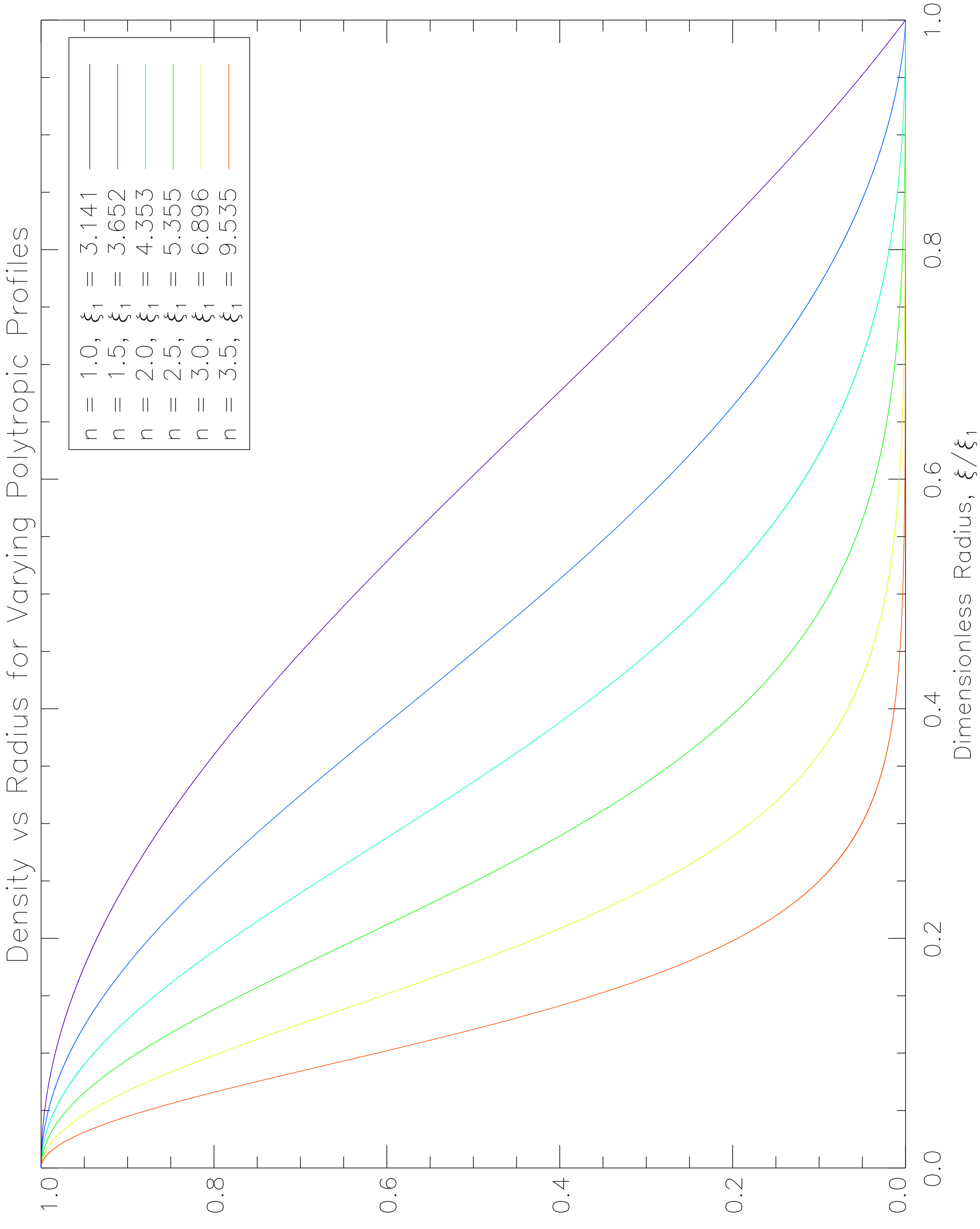
$$L_{\odot} = 1.48 \times 10^{34} ergs/s \tag{35}$$

This is within an order of magnitude of the actual solar luminosity, which seems reasonable for a simple polytropic model.

A. Plots



Temperature vs Radius for Varying Polytropic Profiles
n = 1.0
n = 1.5
n = 2.0
n = 2.5
n = 3.0
n = 3.5
Dimensionless Temperature, $\phi(\xi)$
Dimensionless Radius, ξ
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10


Density vs Radius for Varying Polytropic Profiles
n = 1.0, ξ₁ = 3.141
n = 1.5, ξ₁ = 3.652
n = 2.0, ξ₁ = 4.353
n = 2.5, ξ₁ = 5.355
n = 3.0, ξ₁ = 6.896
n = 3.5, ξ₁ = 9.535
Dimensionless Radius, ξ/ξ₁
Dimensionless Density, φ(ξ)ⁿ
0.0
0.2
0.4
0.6
0.8
1.0