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Computational Problem Set 3
Models for White Dwarf Stars

The following notes describe how to construct simple models for white dwarf stars.

Section 1 describes a simple approximation to the equation of state, $P(\rho)$, for a completely degenerate, but arbitrarily relativistic electron gas. Note that the gas is taken to have a negligibly small temperature so that thermal contributions to the pressure are unimportant. There is an analytic expression for $P(\rho)$ which involves the $\sinh^{-1}$ function which is not very intuitive, so we start with a simpler approximate expression (but see §5 below). In any case, an equation of state that includes only electron degeneracy pressure is already approximate in that it neglects such effects as the Coulomb energy of an ion lattice, the Thomas-Fermi deviations from a uniform charge distribution of electrons, the exchange energy between electrons, and the spin-spin interactions between electrons, all of which make small corrections to $P(\rho)$.

Section 2 reminds you how we set up a Lane-Emden type equation.

Section 3 steps you through setting up the differential equation for the white dwarf case where $P(\rho)$ is not quite as simple as for the Lane-Emden equation.

You are asked to generate models for white dwarfs whose central densities range from 10^4 to over 10^{10} g cm^{-3}.

After the models have been generated, you will be asked (in Section 4) to:
(a) Plot the mass of your models vs. the central density. Find the maximum stable mass of a white dwarf.
(b) Plot the radius of your models vs. the central density.
(c) Plot the radius of your models vs. the mass.
(d) For two particular model white dwarfs, $1.0\,M_\odot$ and $1.3\,M_\odot$, plot density vs. radius.

An alternate approach to the problem, using the exact equation of state for a completely degenerate, arbitrarily relativistic Fermi gas is suggested in Section 5.

1. Equation of State at Low Temperatures

a. <u>Non-relativistic Fermi Pressure</u>

$$P_{\text{FN}} = \frac{h^2}{20m_e} \left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3} \tag{1}$$

$$= \frac{h^2}{20m_e} \left(\frac{3}{\pi}\right)^{2/3} \frac{1}{(2m_p)^{5/3}} \rho^{5/3} \tag{2}$$

for He, C, or O, where ρ is the total mass density. Thus, for the non-relativistic case, we can write:

$$P_{\text{FN}} = K_N \rho^{5/3} \quad , \tag{3}$$

where $K_N \simeq 3.166 \times 10^{12}$ (cgs units).

b. <u>Relativistic Fermi Pressure</u>

$$P_{\text{FR}} = \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} n_e^{4/3} \tag{4}$$

$$= \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} \frac{1}{(2m_p)^{4/3}} \rho^{4/3} \tag{5}$$

for He, C, or O, where ρ is the total mass density. Thus, for the completely relativistic case, we can write:

$$P_{\text{FR}} = K_R \rho^{4/3} \quad , \tag{6}$$

where $K_R \simeq 4.936 \times 10^{14}$ (cgs units).

If we graph P_{FN} and P_{FR} on a $\log P$ vs. $\log \rho$ plot, they will each be straight lines with slopes 5/3 and 4/3, respectively. The two curves intersect at a density, ρ_0, defined by:

$$P_{\text{FN}} = P_{\text{FR}} \tag{7}$$
$$K_N \rho_0^{5/3} = K_R \rho_0^{4/3} \tag{8}$$

This defines

$$\rho_0 \equiv \left(\frac{K_R}{K_N}\right)^3 = 3.789 \times 10^6 \text{ g cm}^{-3} \tag{9}$$

The exact expression for a completely degenerate, but arbitrarily relativistic electron gas is dealt with in §5. However, here, we merely "patch" the non-relativistic and relativistic expression together in an approximate way. A formulation which clearly goes to the correct limits for both high and low ρ is:

$$P = \frac{P_{\text{FN}} P_{\text{FR}}}{\sqrt{P_{\text{FN}}^2 + P_{\text{FR}}^2}} \tag{10}$$

You should verify that this has the correct asymptotic behaviors. We can then express the above equation as:

$$P = \frac{K_N \rho^{5/3} K_R \rho^{4/3}}{\sqrt{K_N^2 \rho^{10/3} + K_R^2 \rho^{8/3}}} \tag{11}$$

which can be simplified to:

$$P = \frac{K_N \rho^{5/3}}{\sqrt{1 + (\rho/\rho_0)^{2/3}}} \quad . \tag{12}$$

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2. Structure Equation

Combine the equation of hydrostatic equilibrium:

$$\vec{\nabla} P = \vec{g}\rho \ , \tag{13}$$

and Poisson's equation:

$$\vec{\nabla} \cdot \vec{g} = -4\pi G\rho \ , \tag{14}$$

to obtain:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{1}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \ . \tag{15}$$

Since $P = P(\rho)$ only, we can solve this second order differential equation directly for $\rho(r)$.

3. Suggested Steps

(a) Define $\rho(r) \equiv \rho_0\, \theta(r)$, where θ is a dimensionless density, and ρ_0 is the "matching density" defined above. Note that the pressure now takes the form:

$$P = \underbrace{K_N \rho_0^{5/3}}\frac{\theta^{5/3}}{\sqrt{1 + \theta^{2/3}}} \ , \tag{16}$$

where the bracketed quantity is a constant.

(b) Now define $r = sa$, where s is a dimensionless radial distance, and a is a length scale-factor to be formed from the constants K_N, ρ_0, and G. Show that $a = 1.557 \times 10^8$ cm.

(c) Write out the dimensionless differential equation for $\theta(s)$.

(d) Reduce the second-order differential equation to two coupled, first-order equations by defining:

$$V(s) = \frac{d\theta}{ds} \ . \tag{17}$$

The equations are then of the form:

$$\frac{d\theta}{ds} = V \tag{18}$$

$$\frac{dV}{ds} = \text{function}(\theta, V, s) \tag{19}$$

(e) Start the integration from the center of the star with the boundary conditions:

$$\theta(0) = \theta_c \text{ where you choose } \theta_c \tag{20}$$
$$V(0) = 0 \tag{21}$$

Note that θ_c is the central density in units of ρ_0.

Before trying to integrate your differential equations, you should check your algebra with a symbolic software package such as *Mathematica*. *Mathematica* can even translate the output to Fortran or C by using the commands "FortranForm[%]" or "CForm[%]".

(f) Integrate these first-order coupled equations with a 4$^{\text{th}}$ order Runge-Kutta algorithm (or other equivalent integration scheme) as you did for the polytrope calculations (Computational Problem set #2).

(g) Integrate the coupled differential equations until the density (θ) goes to zero. This defines the surface of the star. To avoid numerical difficulties, stop the integration when θ/θ_c reaches 0.1% – 0.2%.

(h) As you are integrating the stellar structure equation, keep track of the mass of the star. That is, do the integral:

$$\mathcal{M} = \int_0^{s_{max}} \theta(s)s^2 ds \quad , \tag{22}$$

where $\mathcal{M}$ is the dimensionless mass of the star.

The actual mass of the star is then:

$$
\begin{aligned}
M &= 4\pi\rho_0 a^3 \mathcal{M} \tag{23}\\
M &= 1.797 \times 10^{32} \mathcal{M} \ \text{ grams} \tag{24}\\
M &= 0.090\mathcal{M} \ M_\odot \tag{25}
\end{aligned}
$$

The radius is given by:

$$R = s_{max} a \tag{26}$$

(i) Repeat the calculation for a range of central densities covering the interval 10^4 g cm^{-3} to 10^{12} g cm^{-3}. (This corresponds to θ_c in the range of $\sim 3 \times 10^{-3} \to 3 \times 10^5$.) A few values of θ_c per logarithmic decade will be sufficient. Thus, you might end up calculating somewhere between 15 and 30 white dwarf models.

4. Results

(a) Make a plot of the masses of your white-dwarf models (in $M_\odot$) vs. the logarithm of their central density. Identify the maximum stable mass of a white dwarf. [$M_\odot = 2 \times 10^{33}$ gm]

(b) Make a plot of the radii of your white-dwarf models (in units of $R_\odot/100$) vs. the logarithm of their central density. [$R_\odot = 7 \times 10^{10}$ cm]

(c) Make a plot of the radii of your white-dwarf models (in units of $R_\odot/100$) vs. their mass (in $M_\odot$).

(d) For the $1\,M_\odot$ and $1.3\,M_\odot$ models, plot density vs. radial distance from the stellar center.

5. Alternate Approach (in place of the above)

An exact expression for the Fermi pressure of a completely degenerate gas that is arbitrarily relativistic can be obtained from:

$$
\begin{aligned}
P_F &= \frac{8\pi}{3m_e h^3} \int_0^{p_F} p^4 \left[1 + \left(\frac{p}{m_e c^2}\right)^2\right]^{-1/2} dp \tag{27}\\
n_e &= \frac{8\pi}{h^3} \int_0^{p_F} p^2 dp \tag{28}
\end{aligned}
$$

where p is the electron momentum, and the integrals are from 0 to p_F, the Fermi momentum. From (28) we find

$$\frac{p_F}{m_e c} = \frac{h}{m_e c}\left(\frac{3n_e}{8}\right)^{1/3} = 0.01009\left(\frac{\rho}{\mu_e}\right)^{1/3} \quad , \tag{29}$$

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where μ_e is the mean mass of the white dwarf material per electron. The integral in (27) can be done exactly, and yields:

$$P_F = \frac{m_e^4 c^5}{3h^3} f(x) = 6.003 \times 10^{22} f(x) \quad , \tag{30}$$

where

$$f(x) = x \left(2x^2 - 3\right) \sqrt{x^2 + 1} + \sinh^{-1}(x) \quad , \tag{31}$$

and

$$x \equiv \frac{p_F}{m_e c} \quad , \tag{32}$$

While this expression for the degeneracy pressure is substantially more complicated than that given above in equation (10), it has the distinct advantage that the dP_F/dr term that appears in equation (15) is closely related to the integrand in equation (27), i.e., the derivative of the integral. The relation is:

$$\frac{dP_F}{dr} = \left(\frac{dP_F}{dp_F}\right) \left(\frac{dp_F}{d\rho}\right) \left(\frac{d\rho}{dr}\right) \quad . \tag{33}$$