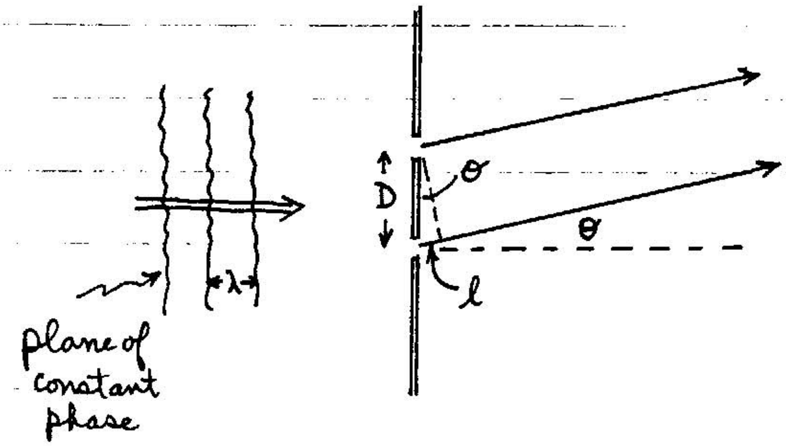
Fraunhofer Diffraction - 8.282

Two Narrow Slits

Consider a plane, monochromatic wave incident on a pair of very narrow slits:



θ = the diffraction angle
D = the slit separation
l = the path difference
λ = the wavelength

For light exiting the two slits and "scattered" in a direction defined by the angle θ (with respect to the incident direction), the path difference, l, between the two waves propagating in this direction is:

$$l = D\sin\theta \simeq D\theta \quad \text{(for small angles)}$$

The phase difference corresponding to l is:

$$\phi = D\sin\theta\left(\frac{2\pi}{\lambda}\right) \simeq \frac{2\pi\theta D}{\lambda} \quad \text{(for small angles)}$$

The total electric field propagating at an angle θ is:

$$E(\theta) \propto \cos\left(\omega t - \frac{2\pi\theta D}{2\lambda}\right) + \cos\left(\omega t + \frac{2\pi\theta D}{2\lambda}\right)$$

frequency of the radiation

With the aid of a trig identity, we find:

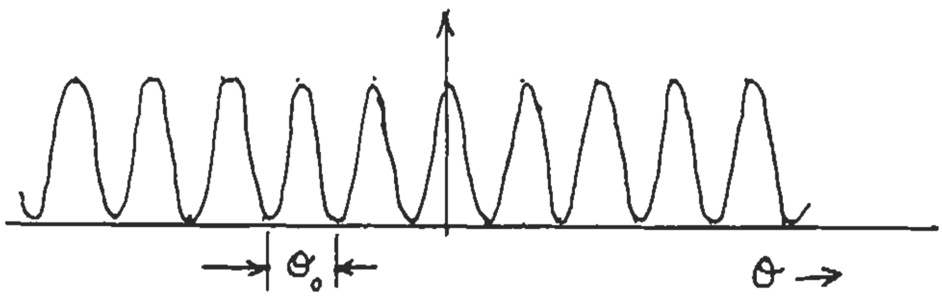
$$E(\theta) \propto \cos(\omega t)\cos\left(\frac{\pi\theta D}{\lambda}\right)$$

However, the light intensity is actually proportional to $|E|^2$. Thus,

$$I(\theta) \propto \cos^2(\omega t)\cos^2\left(\frac{\pi\theta D}{\lambda}\right)$$

The oscillations in intensity ($\sim 10^{15}$ Hz) are too rapid to observe. Thus, averaged over time, the angular diffraction pattern looks like:

$$I(\theta) \propto \cos^2\left(\frac{\pi\theta D}{\lambda}\right)$$



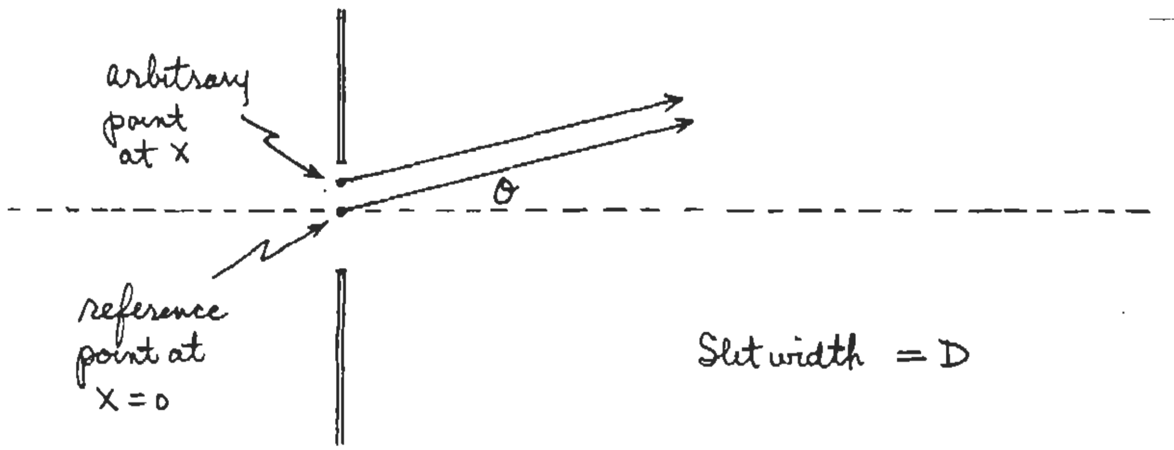
The separation between minima is obtained by allowing the argument of $\cos^2\left(\frac{\pi\theta D}{\lambda}\right)$ to advance by π

$$\frac{\pi\theta_0 D}{\lambda} = \pi$$

$$\boxed{\theta_0 = \frac{\lambda}{D}}$$

A Single Slit of Finite Width

Consider a "Huygens' Construction" where each point within the slit (or aperture) acts as a source of radiation with spherical wavefronts. To find the diffraction pattern, $I(\theta)$, add up the contributions from all points, x, within the aperture.



By the same geometry as used for the case of the narrow slits, we find that the relative phase of the radiation emanating from a point x within the aperture is

$$\phi(x) = \frac{2\pi x \sin\theta}{\lambda} \simeq \frac{2\pi\theta}{\lambda} x \quad \text{(for small angles)}$$

The sum of all contributions to the electric field is given by the integral:

$$E(\theta) \propto \int_{-D/2}^{D/2} \cos\left(\omega t + \frac{2\pi\theta x}{\lambda}\right) dx$$

$$E(\theta) \propto \frac{\sin\left(\omega t + \frac{2\pi\theta x}{\lambda}\right)}{\frac{2\pi\theta}{\lambda}}\Bigg|_{-D/2}^{D/2}$$

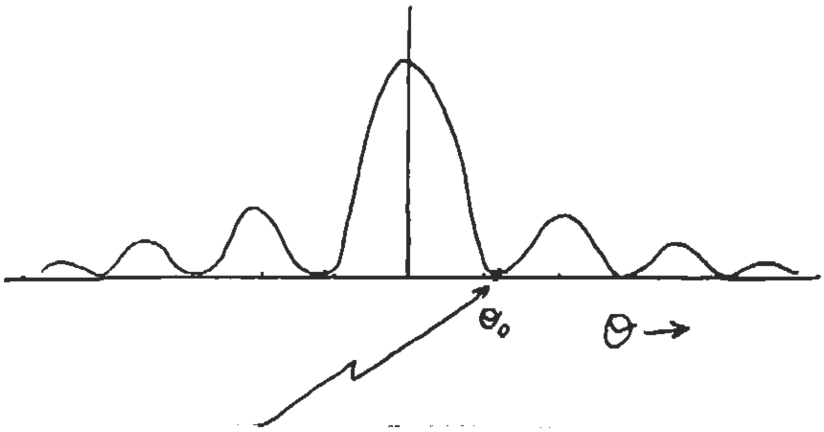
With the aid of a trig identity, we find

$$E(\theta) \propto \frac{\cos\omega t \, \sin\left(\frac{\pi\theta D}{\lambda}\right)}{\left(\frac{\pi\theta D}{\lambda}\right)}$$

[The equation has been divided by the constant D to form a sinc function]

Thus, the intensity $I(\theta)$ is of the form

$$I(\theta) \propto \frac{\sin^2\left(\frac{\pi\theta D}{\lambda}\right)}{\left(\frac{\pi\theta D}{\lambda}\right)^2}$$



The first zero of this diffraction pattern occurs at

$$\frac{\pi\theta_0 D}{\lambda} = \pi$$

$$\boxed{\theta_0 = \frac{\lambda}{D}}$$

Thus, if we image a star with a "one-dimensional" telescope whose "lens" has a width equal to D, we would obtain a blurry image. The first dark point would occur at an angle λ/D away from the center of the stellar image. Thus, in some real sense the angular resolution of such a telescope is limited to $\sim \lambda/D$.

Arbitrary Mask

Consider a mask with an arbitrary transmission $T(x)$. Generalizing the arguments above, we find

$$E(\theta) \propto \int_{-\infty}^{\infty} T(x) \cos\left(\omega t + \frac{2\pi\theta x}{\lambda}\right) dx$$

or, more conveniently,

$$E(\theta) \propto \text{Real} \int_{-\infty}^{\infty} T(x)\, e^{i\omega t + i\frac{2\pi\theta x}{\lambda}}\, dx$$

$$E(\theta) \propto \text{Real}\, e^{i\omega t} \underbrace{\int_{-\infty}^{\infty} T(x)\, e^{i\kappa x}\, dx}_{\text{Fourier transform of the mask}} \qquad \text{where } \kappa \equiv \frac{2\pi\theta}{\lambda}$$

Two-Dimensional Masks

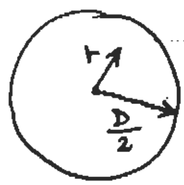
All of the above arguments can be generalized to two dimensional masks (or apertures) $\longrightarrow$ for example, a telescope lens.

For circularly symmetric apertures, integrals (analogous to the ones developed above) for the E field reduce to:

$$E(\theta) \propto \int_0^\infty T(r)\, J_0\left(\frac{2\pi\theta r}{\lambda}\right) r\, dr$$

where $T(r)$ is the transmission as a function of radial position, r, only, and J_0 is a zero order Bessel function. For a circular telescope aperture of diameter D

$$T(r) = 1 \quad \text{for} \quad r \le D/2$$
$$T(r) = 0 \qquad r > D/2$$



$$E(\theta) \propto \int_0^{D/2} J_0\left(\frac{2\pi\theta r}{\lambda}\right) r\, dr$$

You can look this integral up in a book and find:

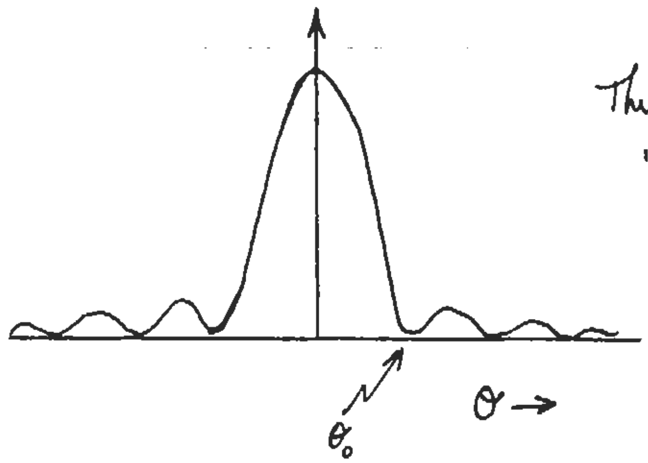
$$E(\theta) \propto \left. \frac{J_1\left(\frac{2\pi\theta r}{\lambda}\right)}{\left(\frac{2\pi\theta r}{\lambda}\right)} \right|_0^{D/2}$$

(where J_1 is a first order Bessel function)

$$E(\theta) = \frac{J_1(\pi\theta D/\lambda)}{(\pi\theta D/\lambda)}$$

The intensity of light from a stellar image in the focal plane of a telescope (of aperture D) is given by

$$I(\theta) = \frac{J_1^2(\pi\theta D/\lambda)}{(\pi\theta D/\lambda)^2}$$



This is referred to as an 'Airy' pattern after its discoverer.

The first zero of a first order Bessel function occurs at an argument equal to $\sim 1.22\pi$. This corresponds to an angle θ_0 given by

$$\frac{\pi\theta D}{\lambda} \cong 1.22\,\pi$$

$$\boxed{\theta_0 = \frac{1.22\lambda}{D}}$$

This is often cited as the angular resolution of a telescope of diameter D.