8.323 Relativistic Quantum Field Theory I
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8.323: Relativistic Quantum Field Theory I

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LECTURE NOTES 1
QUANTIZATION OF THE FREE SCALAR FIELD

As we have already seen, a free scalar field can be described by the Lagrangian

$$L = \int d^3x \, \mathscr{L} \ , \tag{1}$$

where

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{2a}$$

$$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\nabla_i\phi\,\nabla_i\phi - \frac{1}{2}m^2\phi^2 \ . \tag{2b}$$

Our goal is to "quantize" this theory, in the sense of developing a quantum theory that corresponds to the classical theory described by the above Lagrangian.

1. CANONICAL QUANTIZATION:

Here we will use the method of canonical quantization, which I assume is already familiar to you in the context of quantum mechanics. Specifically, suppose that we were given a Lagrangian with a discrete number of dynamical variables q_i:

$$L = L(q_i, \dot{q}_i, t) \ . \tag{3}$$

The canonical momenta would then be defined by

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \ , \tag{4}$$

and the Hamiltonian would be given by

$$H = \sum_i p_i\,\dot{q}_i - L \ . \tag{5}$$

A quantum theory corresponding to this classical theory could then be constructed by promoting each q_i and p_i to an operator on a Hilbert space, and insisting on the canonical commutation relations

$$[q_i\,,\,p_j] = i\hbar\,\delta_{ij} \ . \tag{6}$$

For most of this course we will use units for which $\hbar \equiv 1$, but for now I will leave the $\hbar$'s in the equations. The Hamiltonian $H(p_i, q_i)$ is then also an operator on the Hilbert space, and in the Schrödinger picture the physical states evolve according to the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \, |\psi(t)\rangle = H \, |\psi(t)\rangle \ . \tag{7}$$

If H is independent of time, Eq. (7) has the formal solution

$$|\psi(t)\rangle = e^{-iHt/\hbar} \, |\psi(0)\rangle \ . \tag{8}$$

Given any operator $\mathcal{O}$, its expectation value in the state $|\psi(t)\rangle$ is then given by

$$\langle \psi(t) \, | \, \mathcal{O} | \, \psi(t) \rangle = \left\langle \psi(0) \left| e^{iHt/\hbar} \, \mathcal{O} \, e^{-iHt/\hbar} \right| \psi(0) \right\rangle \ . \tag{9}$$

This equation leads naturally to the Heisenberg picture description, in which the states are treated as time-independent, and all of the time dependence is incorporated into the evolution of the operators:

$$\mathcal{O}(t) = e^{iHt/\hbar} \, \mathcal{O} \, e^{-iHt/\hbar} \ . \tag{10}$$

2. FIELD QUANTIZATION BY LATTICE APPROXIMATION:

To quantize the classical field theory of Eq. (2), we can begin by quantizing a lattice version of the theory. That is, we can replace the continuous space by a cubic lattice of closely spaced grid points, with a lattice spacing a, and we can truncate the space to a finite region. The system then reduces to one with a discrete number of dynamical variables, exactly like the systems that we already know how to quantize. Then if we can take the limit as the lattice spacing a approaches zero and the volume approaches infinity, the quantization of the field theory can be completed. We will see later that the $a \to 0$ limit is problematic for interacting theories, but we will see here that this program can be carried out easily for the free theory.

When we replace the continuous space by a finite lattice of points, we can label each lattice site with an index k. In a fully detailed lattice description we would probably label each lattice site with a triplet of integers representing the x, y, and z coordinates of the site, but for present purposes it will suffice to imagine simply numbering all the lattice sites from 1 to N, where N is the total number of sites. The field $\phi(\vec{x}, t)$ is then replaced by a set of dynamical variables $\phi_k(t)$, where one can think of $\phi_k(t)$ as representing the average value of $\phi(\vec{x}, t)$ in a cube of size a surrounding the lattice site k. The Lagrangian of Eqs. (1) and (2) is then replaced by

$$L = \sum_k \mathscr{L}_k \, \Delta V \ , \tag{11}$$

where

$$\Delta V = a^3 \tag{12}$$

and

$$\mathcal{L}_k = \frac{1}{2}\dot{\phi}_k^2 - \frac{1}{2}\nabla_i\phi_k\,\nabla_i\phi_k - \frac{1}{2}m^2\phi_k^2\ . \tag{13}$$

Here the lattice derivative $\nabla_i\phi_k$ is defined by

$$\nabla_i\phi_k \equiv \frac{\phi_{k'(k,i)} - \phi_k}{a}\ , \tag{14}$$

where $k'(k,i)$ denotes the lattice site that is a distance a in the ith direction from the lattice site k.

The canonical momenta are then given by

$$p_k = \frac{\partial L}{\partial\dot{\phi}_k} = \frac{\partial \mathcal{L}_k}{\partial\dot{\phi}_k}\,\Delta V = \dot{\phi}_k\,\Delta V\ . \tag{15}$$

Since the canonical momenta are proportional to ΔV, it is natural to define a canonical momentum density π_k by

$$\pi_k \equiv \frac{p_k}{\Delta V} = \frac{\partial \mathcal{L}_k}{\partial\dot{\phi}_k} = \dot{\phi}_k\ . \tag{16}$$

Following Eq. (5), the Hamiltonian is then

$$H = \sum_k p_k\dot{\phi}_k - L = \sum_k \left[\pi_k\dot{\phi}_k\ - \mathcal{L}_k\right]\Delta V\ . \tag{17}$$

The canonical commutation relations become

$$[\phi_{k'}\,,\,\phi_k] = 0\ ,\ \ [p_{k'}\,,\,p_k] = 0\ ,\quad \text{and}\quad [\phi_{k'}\,,\,p_k] = i\hbar\,\delta_{k'k}\ . \tag{18}$$

In terms of the canonical momentum densities,

$$[\phi_{k'}\,,\,\phi_k] = 0\ ,\ \ [\pi_{k'}\,,\,\pi_k] = 0\ ,\quad \text{and}\quad [\phi_{k'}\,,\,\pi_k] = \frac{i\hbar\ \delta_{k'k}}{\Delta V}\ . \tag{19}$$

Although we have not yet constructed the full theory, it is not too early to write down the continuum limit of these defining equations. The continuum canonical momentum density becomes

$$\pi(\vec{x},t) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(\vec{x},t)} = \dot{\phi}(\vec{x},t)\ , \tag{20}$$

and the Hamiltonian becomes

$$H = \int \mathrm{d}^3x \left[\pi\dot{\phi} - \mathscr{L} \right] \; . \tag{21}$$

The trivial canonical commutation relations carry over trivially:

$$[\phi(\vec{x}\,', t)\,,\, \phi(\vec{x}, t)] = 0 \;\; \text{and} \;\; [\pi(\vec{x}\,', t)\,,\, \pi(\vec{x}, t)] = 0 \;, \tag{22}$$

obviously. For the nontrivial commutation relation, the result will be clearest if we first rewrite the last equation in (19) as a sum which will become an integral in the limit. If we let $\mathcal{R}$ denote a region of the lattice, the last equation in (19) becomes

$$\sum_{k \in \mathcal{R}} [\phi_{k'}\,,\, \pi_k]\, \Delta V = i\hbar \sum_{k \in \mathcal{R}} \delta_{k',k} = i\hbar \begin{cases} 1 & \text{if } k' \in \mathcal{R} \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

In the continuum limit

$$\sum_{k \in \mathcal{R}} [\phi_{k'}\,,\, \pi_k]\, \Delta V$$

clearly approaches

$$\int_{\vec{x} \in \mathcal{R}} \mathrm{d}^3x \; [\phi(\vec{x}\,', t)\,,\, \pi(\vec{x}, t)] \;\; ,$$

so Eq. (23) becomes

$$\int_{\vec{x} \in \mathcal{R}} \mathrm{d}^3x \; [\phi(\vec{x}\,', t)\,,\, \pi(\vec{x}, t)] = i\hbar \begin{cases} 1 & \text{if } \vec{x}\,' \in \mathcal{R} \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

This relationship can be expressed more conveniently by introducing the Dirac delta-function $\delta^3(\vec{x})$, which is defined by its integral*:

$$\int_{\vec{x} \in \mathcal{R}} \mathrm{d}^3x \, f(\vec{x})\, \delta(\vec{x} - \vec{x}\,') \equiv \begin{cases} f(\vec{x}\,') & \text{if } x' \in \mathcal{R} \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Given this definition, Eq. (24) can be rewritten as

$$[\phi(\vec{x}\,', t)\,,\, \pi(\vec{x}, t)] = i\hbar\, \delta(\vec{x} - \vec{x}\,') \; . \tag{26}$$

Note that the delta function is symmetric, so $\delta(\vec{x} - \vec{x}\,') = \delta(\vec{x}\,' - \vec{x})$.

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* Note that one often thinks of the Dirac delta function $\delta^3(\vec{x})$ as the limit of a sequence of functions which each integrate to 1, but which become more and more sharply peaked at $\vec{x} = 0$. This approach is useful for intuition, but it is not mathematically rigorous. It can be shown that there exists no **function** that has the properties ascribed to the Dirac delta function. The Dirac delta function is actually a **distribution**, not a function. We will return to the definition of the Dirac delta function later in the course. The bottom line, however, is that the delta function can be defined, and the definition of an integral suitably generalized, so that Eq. (25) becomes exactly true by definition.

3. REVIEW OF SIMPLE HARMONIC OSCILLATOR:

We will soon see that each Fourier component of a scalar field obeys the equations of a harmonic oscillator, so it is useful to review the quantum mechanics of a harmonic oscillator before we proceed.

Consider the Lagrangian for a simple harmonic oscillator, which can be written as

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 \ , \tag{27}$$

where we have adopted units for which the mass m of the harmonic oscillator is one. The canonical momentum is

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q} \ , \tag{28}$$

and the Hamiltonian is

$$H = p\dot{q} - L = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \quad . \tag{29}$$

The canonical commutation relation is

$$[q \, , \, p] = i\hbar \ . \tag{30}$$

Now we can define creation and annihilation operators

$$a = \sqrt{\frac{\omega}{2\hbar}}\, q + \frac{i}{2\hbar\omega}\, p \ , \quad a^\dagger = \sqrt{\frac{\omega}{2\hbar}}\, q - \frac{i}{2\hbar\omega}\, p \ , \tag{31}$$

so that

$$\left[a \, , \, a^\dagger\right] = 1 \ . \tag{32}$$

The Hamiltonian can be rewritten as

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) \ . \tag{33}$$

The commutators of H with the creation and annihilation operators are

$$\left[H \, , \, a^\dagger\right] = \hbar\omega\, a^\dagger \ , \quad [H \, , \, a] = -\hbar\omega\, a \ , \tag{34}$$

from which it follows that the result of applying $a^\dagger$ to an eigenstate of H is to produce an eigenstate of H with an eigenvalue higher than the original by $\hbar\omega$, while a acts on an eigenstate of H to lower the eigenvalue by $\hbar\omega$. The ground state of H therefore satisfies

$$a \left|0\right\rangle = 0 \ , \tag{35}$$

while the (normalized) nth excited state can be written as

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |n\rangle \ , \tag{36}$$

where the eigenvalue (energy) is

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega \ . \tag{37}$$

Eqs. (31) can be solved for q and p, giving

$$q = \sqrt{\frac{\hbar}{2\omega}} \left(a + a^\dagger\right) \ , \quad p = -i\sqrt{\frac{\hbar\omega}{2}} \left(a - a^\dagger\right) \ . \tag{38}$$

In the Heisenberg picture,

$$\begin{aligned} q(t) &= e^{iHt/\hbar}\, q\, e^{-iHt/\hbar} \\ &= \sqrt{\frac{\hbar}{2\omega}} e^{iHt/\hbar} \left(a + a^\dagger\right) e^{-iHt/\hbar} \\ &= \sqrt{\frac{\hbar}{2\omega}} \left(a e^{-i\omega t} + a^\dagger e^{i\omega t}\right) \end{aligned} \tag{39}$$

and

$$p(t) = -i\sqrt{\frac{\hbar\omega}{2}} \left(a e^{-i\omega t} - a^\dagger e^{i\omega t}\right) \ . \tag{40}$$

4. QUANTIZATION OF THE SCALAR FIELD:

We have used the continuum limit of the lattice version of the theory to obtain the key results

$$\pi(\vec{x},t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x},t)} = \dot{\phi}(\vec{x},t) \ , \tag{20}$$

$$H = \int \mathrm{d}^3x \left[\pi\dot{\phi} - \mathcal{L}\right] \ , \tag{21}$$

$$[\phi(\vec{x}\,',t)\,,\, \phi(\vec{x},t)] = 0 \quad \text{and} \quad [\pi(\vec{x}\,',t)\,,\, \pi(\vec{x},t)] = 0 \ , \tag{22}$$

and

$$[\phi(\vec{x}\,',t)\,,\, \pi(\vec{x},t)] = i\hbar\, \delta(\vec{x} - \vec{x}\,') \ . \tag{26}$$

Having done this, we can now proceed with the continuum theory directly.

We view $\phi(\vec{x},t)$ as a collection of dynamical variables, in the classical theory, which have been promoted to operators in the quantum theory, exactly as we did for the discrete system in Section 1. We can then use the Fourier transform to define convenient linear combinations of these operators:

$$\tilde{\phi}(\vec{k},t) \equiv \int \mathrm{d}^3x\, e^{-i\vec{k}\cdot\vec{x}}\, \phi(\vec{x},t) \ , \tag{41}$$

and

$$\tilde{\pi}(\vec{k},t) \equiv \int \mathrm{d}^3x\, e^{-i\vec{k}\cdot\vec{x}}\, \pi(\vec{x},t) \ , \tag{42}$$

so that the Fourier inversion theorem implies that

$$\phi(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\phi}(\vec{k},t) \tag{43}$$

and

$$\pi(\vec{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\pi}(\vec{k},t) \ . \tag{44}$$

The fact that $\phi(\vec{x},t)$ is a real classical variable and hence a Hermitian quantum operator implies that

$$\tilde{\phi}(-\vec{k},t) = \phi^\dagger(\vec{k},t) \ , \tag{45}$$

with a similar relation for $\pi(\vec{k},t)$.

The Heisenberg equations of motion for $\phi(\vec{x},t)$ are the same as the classical equations of motion:

$$\frac{\partial^2\phi}{\partial t^2} - \vec{\nabla}^2\phi + m^2\phi = 0 \ . \tag{46}$$

Since we have not set $\hbar$ to one, we should keep in mind that m in this equation has the units of an inverse length, and not a mass. Eq. (46) implies that the Fourier transform field obeys

$$\frac{\partial^2\tilde{\phi}}{\partial t^2}(\vec{k},t) + (\vec{k}^2 + m^2)\,\tilde{\phi}(\vec{k},t) = 0 \ . \tag{47}$$

The general solution to this equation can be written as

$$\tilde{\phi}(\vec{k},t) = \phi_1(\vec{k})e^{-i\omega_p t} + \phi_2(\vec{k})e^{i\omega_p t} \ , \tag{48}$$

where

$$\omega_p = \sqrt{\vec{k}^2 + m^2} \ . \tag{49}$$

The reality condition (45) implies that

$$\phi_2(\vec{k}) = \phi_1^\dagger(-\vec{k}) \ , \tag{50}$$

so

$$\tilde{\phi}(\vec{k},t) = \phi_1(\vec{k})e^{-i\omega_p t} + \phi_1^\dagger(-\vec{k})e^{i\omega_p t} \; . \tag{51}$$

The relation $\pi = \dot{\phi}$ then implies that

$$\tilde{\pi}(\vec{k},t) = -i\omega_p\phi_1(\vec{k})e^{-i\omega_p t} + i\omega_p\phi_1^\dagger(-\vec{k})e^{i\omega_p t} \; , \tag{52}$$

and then Eqs. (51) and (52) can be solved simultaneously to give an expression for $\phi_1(\vec{k})$:

$$\begin{aligned} \phi_1(\vec{k}) &= \frac{1}{2}\left[\tilde{\phi}(\vec{k},t) + \frac{i}{\omega_p}\tilde{\pi}(\vec{k},t)\right]e^{i\omega_p t} \\ &= \frac{1}{2}\int \mathrm{d}^3x\, e^{-i(\vec{k}\cdot\vec{x}-\omega_p t)}\left[\phi(\vec{x},t) + \frac{i}{\omega_p}\,\pi(\vec{x},t)\right] \; , \end{aligned} \tag{53}$$

where the second line was obtained by using Eqs. (41) and (42). Note that although the right-hand side contains quantities that depend explicitly on t, Eqs. (51) and (52) guarantee that the full expression is independent of time. By comparing with Eq. (31), one sees that $\phi_1(\vec{k})$ bears some resemblance to an annihilation operator. To test this hypothesis, we can compute $\left[\phi_1(\vec{k})\,,\,\phi_1(\vec{q})\right]$ and $\left[\phi_1(\vec{k})\,,\,\phi_1^\dagger(\vec{q})\right]$.

Using Eq. (53) and the canonical commutation relations,

$$\begin{aligned} \left[\phi_1(\vec{k})\,,\,\phi_1(\vec{q})\right] &= \frac{1}{4}\int \mathrm{d}^3x \int \mathrm{d}^3y\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\, e^{-i(\vec{q}\cdot\vec{y}-\omega_q t)} \\ &\qquad \times\left[\phi(\vec{x},t) + \frac{i}{\omega_k}\,\pi(\vec{x},t)\,,\,\phi(\vec{y},t) + \frac{i}{\omega_q}\,\pi(\vec{y},t)\right] \\ &= \frac{1}{4}\int \mathrm{d}^3x \int \mathrm{d}^3y\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\, e^{-i(\vec{q}\cdot\vec{y}-\omega_q t)} \\ &\qquad \times\left\{\frac{i}{\omega_q}\,i\hbar\,\delta^3(\vec{x}-\vec{y}) - \frac{i}{\omega_k}\,i\hbar\,\delta^3(\vec{x}-\vec{y})\right\} \\ &= \frac{\hbar}{4}\int \mathrm{d}^3x\, e^{-i\left((\vec{k}+\vec{q})\cdot\vec{x}-(\omega_k+\omega_q)t\right)}\left\{\frac{1}{\omega_k} - \frac{1}{\omega_q}\right\} \; . \end{aligned} \tag{54}$$

We now use the identity

$$\int \mathrm{d}^3x\, e^{-i(\vec{k}+\vec{q})\cdot\vec{x}} = (2\pi)^3\,\delta^3(\vec{k}+\vec{q}) \; , \tag{55}$$

which implies that the only possible nonzero contribution to Eq. (54) arises when $\vec{q} = -\vec{k}$, but in that case the factor in curly brackets vanishes. Thus,

$$\left[\phi_1(\vec{k})\,,\,\phi_1(\vec{q})\right] = 0 \; , \tag{56}$$

as we would expect if $\phi_1(\vec{k})$ is proportional to an annihilation operator.

Using the same techniques, we can now calculate

$$\begin{aligned}
\left[\phi_1(\vec{k})\,,\,\phi_1^\dagger(\vec{q})\right] &= \frac{1}{4}\int d^3x \int d^3y\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\, e^{i(\vec{q}\cdot\vec{y}-\omega_q t)} \\
&\qquad \times \left[\phi(\vec{x},t) + \frac{i}{\omega_k}\,\pi(\vec{x},t)\,,\,\phi(\vec{y},t) - \frac{i}{\omega_q}\,\pi(\vec{y},t)\right] \\
&= \frac{1}{4}\int d^3x \int d^3y\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\, e^{i(\vec{q}\cdot\vec{y}-\omega_q t)} \\
&\qquad \times \left\{-\frac{i}{\omega_q}\, i\hbar\, \delta^3(\vec{x}-\vec{y}) - \frac{i}{\omega_k}\, i\hbar\, \delta^3(\vec{x}-\vec{y})\right\} \\
&= \frac{\hbar}{4}\int d^3x\, e^{-i\left((\vec{k}-\vec{q})\cdot\vec{x}-(\omega_k-\omega_q)t\right)} \left\{\frac{1}{\omega_k} + \frac{1}{\omega_q}\right\} . \\
&= \frac{\hbar}{4}\,(2\pi)^3\,\delta^3(\vec{k}-\vec{q})\,\frac{2}{\omega_k} = \frac{\hbar}{2\omega_k}\,(2\pi)^3\,\delta^3(\vec{k}-\vec{q}) .
\end{aligned} \tag{57}$$

The standard (continuum) definition of the annihilation operator is then given by

$$a(\vec{k}) = \sqrt{\frac{2\omega_k}{\hbar}}\,\phi_1(\vec{k}) , \tag{58}$$

so the commutation relations become

$$\boxed{\left[a(\vec{k})\,,\,a(\vec{q})\right] = 0 \,,\quad \left[a(\vec{k})\,,\,a^\dagger(\vec{q})\right] = (2\pi)^3\,\delta^3(\vec{k}-\vec{q}) .} \tag{59}$$

Using Eqs. (43), (51), and (58), the field operator can now be expressed in terms of creation and annihilation operators:

$$\boxed{\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\hbar}{2\omega_k}}\left\{a(\vec{k})\,e^{i(\vec{k}\cdot x-\omega_k t)} + a^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right\} .} \tag{60}$$

Note that in the second term I have changed variables of integration, $\vec{k} \to -\vec{k}$, to write in the form that is shown. The canonical momentum density is then given by

$$\boxed{\begin{aligned}
\pi(\vec{x},t) &= \dot{\phi}(\vec{x},t) \\
&= -i\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\hbar\omega_k}{2}}\left\{a(\vec{k})\,e^{i(\vec{k}\cdot x-\omega_k t)} - a^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right\} .
\end{aligned}} \tag{61}$$

Finally, it is also useful to rewrite Eq. (53) in terms of the properly normalized annihilation operator, so that one can express the annihilation operator in terms of the field and canonical momentum density:

$$\boxed{a(\vec{k}) = \sqrt{\frac{\omega_k}{2\hbar}} \int \mathrm{d}^3x \, e^{-i(\vec{k}\cdot\vec{x}-\omega_p t)} \left[\phi(\vec{x},t) + \frac{i}{\omega_p}\,\pi(\vec{x},t)\right] .} \tag{62}$$

We might also want the corresponding formula for the creation operator, which can be found simply by taking the adjoint of the above equation:

$$\boxed{a^\dagger(\vec{k}) = \sqrt{\frac{\omega_k}{2\hbar}} \int \mathrm{d}^3x \, e^{i(\vec{k}\cdot\vec{x}-\omega_p t)} \left[\phi(\vec{x},t) - \frac{i}{\omega_p}\,\pi(\vec{x},t)\right] .} \tag{63}$$

The boxed equations above are the primary results that we will continue to use throughout the course.