8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

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Lecture 5

1.4: QUANTIZATION OF NON-ABELIAN GAUGE THEORIES

1.4.1: Gauge Symmetries

Gauge symmetry is not a true symmetry, but a reflection of the fact that a theory possesses redundant degrees of freedom. A gauge symmetry implies the existence of different field configurations which are equivalent. For example, in the case of $U(1)$,

$$\psi \longrightarrow e^{i\alpha(x)}\psi,$$
$$A_\mu \longrightarrow A_\mu + \frac{1}{e}\partial_\mu \alpha(x),$$

the phase of ψ is not a physical degree of freedom. Similarly, nor is the longitudinal part of A_μ. A massless spin-1 representation of the Lorentz group has only two polarizations. A_μ has four components. Thus to have a Lorentz covariant formulation, we require gauge symmetries to get rid of the extra degrees of freedom.

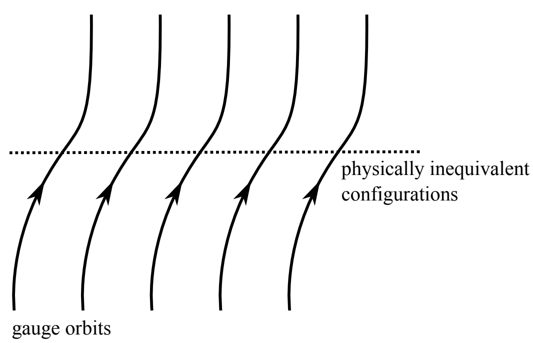

Figure 1: Equivalent gauge orbits in configuration space.

When quantizing the theory, we should separate the redundant and physical degrees of freedom. We need to make sure only physical modes contribute to observables. This leads to complications in dealing with gauge theories. There are two general approaches:

1. Isolate the physical degrees of freedom: fix a gauge and quantize the resulting constrained system. This method is used, for example, in axial gauge quantization in quantum electrodynamics.

2. Retain the unphysical modes, or even introduce additional modes, but make sure that they do not contribute to any physical observables. This method is used, for example, in covariant path integral quantization.

For the first complication in the path integral quantization, consider, for example, the path integral for a scalar field

$$\int \mathfrak{D}\phi\, e^{-\int d^dx \frac{1}{2}\phi^T K\phi + V(\phi) - J^T\phi} = e^{-V(\frac{\delta}{\delta J})} e^{\frac{1}{2}\int d^4x J^T K^{-1} J}, \tag{1}$$

where K is the kinetic operator $(-\partial^2 + m^2)$ and K^{-1} is the propagator for ϕ. For gauge theories, the inverse of K is not defined. For example, in quantum electrodynamics,

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= A_\mu K^{\mu\nu} A_\nu + \text{total derivatives}, \end{aligned}$$

with $K^{\mu\nu} = \partial^2\eta^{\mu\nu} - \partial^\mu\partial^\nu$. We see that $K^{\mu\nu}\partial_\nu\alpha(x) = 0$ for any $\alpha(x)$, and so the matrix is singular. These zero eigenmodes are the configuarations which are gauge-equivalent to 0. Non-Abelian gauge theories have the same quadratic kinetic terms as quantum electrodynamics. In order for K to have an inverse, we need to separate gauge orbits with physically inequivalent configurations.

1.4.2 Fadeev-Popov method:

Example 1: A trivial example

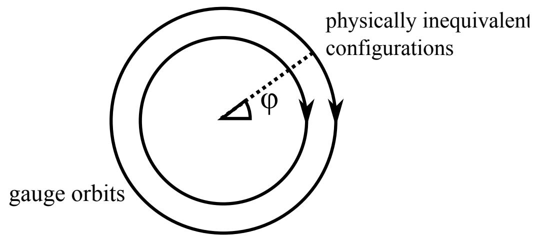

Figure 2: The radial direction gives inequivalent configurations, and the circles of fixed radius are the gauge orbits.

Consider

$$W = \int dxdy\, e^{f(x,y)} \tag{2}$$

and suppose $f(x,y)$ only depends on $r = \sqrt{x^2+y^2}$. Then

$$\begin{aligned} W &= \int drd\phi\, re^{f(r)} \\ &= 2\pi \int dr\, re^{f(r)}, \end{aligned}$$

where the 2π is the factorized orbit volume. Equivalently, we can insert a delta function. More explicitly, we can insert a factor of $\int d\phi_0 \delta(\phi - \phi_0) = 1$. Then

$$W = \int d\phi_0 \int dxdy e^{f(x,y)} \delta(\phi - \phi_0). \tag{3}$$

The $\int d\phi_0$ integrates over the gauge orbit, and the other factor integrates over a section of non-gauge equivalent configurations.

Example 2: Gauge theories

We consider pure-gauge theories only; adding matter fields is trivial.

$$Z = \int \prod_{a=1}^{\dim G} \mathfrak{D} A^a_\mu(x)\, e^{iS[A_\mu]}. \tag{4}$$

We define a set of gauge-fixing conditions:

$$f_a(A) = 0,\ a = 1, \ldots, \dim G, \tag{5}$$

in order to select a section of non-equivalent configurations:

$$1 = \int \prod_a d\Lambda_a(x) \delta(f_a(A_\Lambda)) \det \left[\frac{\delta f_a(A_\Lambda(x))}{\delta \Lambda_b(y)} \right]. \tag{6}$$

Here, the determinant is the determinant of both the color and function space. Inserting (6) into (4), using an abridged notation,

$$Z = \int d\Lambda \int \mathfrak{D} A\, e^{iS[A]} \delta(f(A_\Lambda)) \det \left[\frac{\delta f(A_\Lambda)}{\delta \Lambda} \right]. \tag{7}$$

Now we observe that $\mathfrak{D}A = \mathfrak{D}A_\Lambda$, as gauge transformations correspond to unitary transformations plus shifts, and that $S[A] = S[A_\Lambda]$, because of the defining gauge symmetry. Hence,

$$\begin{aligned} Z &= \int d\Lambda \int \mathfrak{D}A_\Lambda e^{iS[A_\Lambda]} \delta(f(A_\Lambda)) \det \left[\frac{\delta f(A_\Lambda)}{\delta\Lambda}\right] \\ &= \int d\Lambda \int \mathfrak{D}A e^{iS[A]} \delta(f(A)) \det \left[\frac{\delta f(A)}{\delta\Lambda}\right], \end{aligned}$$

as A_Λ is a dummy integration variable. So, again we factor the partition function into the gauge volume $\int d\Lambda$ and a path-integral over gauge-inequivalent configurations which is independent of Λ. We redefine this latter factor to be the new partition function; that is,

$$Z \equiv \int \mathfrak{D}A e^{iS[A]} \delta(f(A)) \det \left[\frac{\delta f(A)}{\delta\Lambda}\right]. \tag{8}$$

Example 3: Axial gauge

$$f_a(A) = A_z^a = 0. \tag{9}$$

From this,

$$f_a(A_\Lambda) = A_z^a + \frac{1}{g}(\partial_z \Lambda_a + g f_{abc} A_z^b \Lambda^c), \tag{10}$$

and hence,

$$\left.\frac{\delta f_a(A_\Lambda(x))}{\delta\Lambda_b(x')}\right|_{\Lambda=0} = \frac{1}{g}\partial_z \delta_{ab}\delta(x-x'). \tag{11}$$

We see that the Jacobian is independent of A_μ: its determinant only gives an overall constant in the partition function. Hence,

$$Z = \int \mathfrak{D}A e^{iS[A]} \prod_a \delta(A_z^a) \tag{12}$$

up to a constant. This is a particularly simple form. However, the drawback is that Lorentz covariance has been broken. In a general covariant gauge, both the determinant and delta-function factors are more difficult to work with. We need to employ additional tricks.

(i) Determinant factor

Recall that

$$\int \prod_a d\Psi d\bar{\Psi} e^{\bar{\psi}_a M_{ab} \psi_b} = \det M_{ab}, \tag{13}$$

where the ψ_a and $\bar{\psi}_b$ are independent Grassman variables. Hence, the Fadeev-Popov determinant is given by

$$\det \left[\left.\frac{\delta f_a(A_\Lambda(x))}{\delta\Lambda_b(y)}\right|_{\Lambda=0}\right] = \int \mathfrak{D}C_a(x)\mathfrak{D}\bar{C}_a(x)\, e^{i\int d^4x d^4y\, \bar{C}_a(x) \left.\frac{\delta f_a(A_\Lambda(x))}{\delta\Lambda_b(y)}\right|_{\Lambda=0} C_b(y)}, \tag{14}$$

where $C_a(x)$ and $\bar{C}_a(x)$, $a = 1, \ldots, \dim G$, are real fermionic fields with no spinor indices. These are the ghost fields.

(ii) Delta-function factor

Again, the method is to write this factor in the form of an exponential. Firstly, generalize

$$\delta(f_a(A)) \longrightarrow \delta(f_a(A) - B_a(x)) \tag{15}$$

where $B_a(x)$ is an arbitrary function. This does not change the Fadeev-Popov determinant. Therefore, Z is independent of $B_a(x)$, and so we can weight the integrand of Z with a Gaussian distribution of $B_a(x)$. That is,

$$Z = \int \prod_a \mathfrak{D}B_a(x) e^{-i\int d^4x\, \frac{1}{2\xi} B_a^2(x)} \times \int \mathfrak{D}A e^{iS[A]} \delta(f(A)) \det \left[\left.\frac{\delta f(A)}{\delta\Lambda}\right|_{\Lambda=0}\right]. \tag{16}$$

Collecting (14) and (16), we find for Z,

$$Z = \int \mathfrak{D}A\mathfrak{D}C_a D\bar{C}_a \, e^{iS_{eff}[A,C,\bar{C}]} \tag{17}$$

with the effective action S_{eff} given by

$$S_{eff}\,[A] = S_{YM}\,[A] - \frac{1}{2\xi}\int d^4x\, f_a^2(A) + \int d^4x d^4y\, \bar{C}_a(x) \left[\left. \frac{\delta f_a(A_\Lambda(x)}{\delta \Lambda_b(y)} \right|_{\Lambda=0} \right] C_b(y). \tag{18}$$

Example 4: Lorentz gauge

$$f_a(A_\mu) = \partial^\mu A_\mu^a. \tag{19}$$

We have that $(A_\mu^a)_\Lambda(x) = A_\mu^a(x) + \frac{1}{g}(\partial_\mu \Lambda_a(x) + g f^{acd} A_\mu^c \Lambda^d)$, and so the Jacobian is given by

$$\left. \frac{\delta f_a(A_\Lambda(x))}{\delta \Lambda_b} \right|_{\Lambda=0} = \frac{1}{g}\left[\partial^\mu \delta_{ab}\partial_\mu + g^{cab} A_\mu^c\right] \delta^{(4)}(x-y), \tag{20}$$

giving

$$\mathscr{L}_{eff} = \mathscr{L}\,[A] + \mathscr{L}_{gf} + \mathscr{L}_{gh}, \tag{21}$$

with

$$\mathscr{L}_{gf} = -\frac{1}{2\xi}(\partial^\mu A_\mu^a), \quad \mathscr{L}_{gh} = \int d^4x\, \bar{C}_a(x)\partial^\mu D_\mu C_a(x), \tag{22}$$

where $D_\mu C_a(x) \equiv \partial_\mu C_a(x) + g f_{abd} A_\mu^b C_d$. From this, we can derive the Feynman rules for the theory.

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