PROFESSOR: Let's go back to our discussion of spectral broadening. And what I started to derive for you is perturbation theory of spectral broadening, which is a very general framework. I like it, because it really provides you insight into spectral broadening. But I will also hope it provides new insight for you for time-dependent perturbation theory [INAUDIBLE].

So what we did in this lecture is we pretty much did standard type perturbation theory. We just did it by assuming that we have a general time-dependent perturbation. I'm not yet telling you what the time-dependent perturbation is. It has actually fluctuating. It has inhomogeneous. It has everything in it which will later lead to line-broadening line shifts.

And the important quantity which is now describing everything we are interested is the correlation function between the perturbation at time t pi and t. And we call this correlation function G. And it is either the correlation function G of t, or it's Fourier transform, which tells us what the excitation rate of the system is.

I want to give you sort of two summaries now. And they're both general to perturbation theory. But I have to say, I myself learned something about time evolution of quantum systems from those examples. So what we get is a very generic thing, that when we look at the probability to be in the final state, the amplitude V of t squared, we start out quadratic, a very general behavior of any quantum system, because your linear equation, which linearly puts amplitude into the excited state is quadratic.

And also, this is the beginning of a Rabi oscillation. The Rabi oscillation starts out with 0 slope. But then you have some de-coherence type. The fields are no longer

driving the system in a coherent way. And then that's where we enter the regime of Fermi's golden rule, the probability to be in the excited state only goes linear. And this is when we have rate equations.

Let me sort of show how it comes about in equations, which I think is really nice. The differential equations is that in a time delta t, we add amplitude delta t to the excited state. And so the amplitude we build up is-- I may have called an h bar. Or maybe I measured [INAUDIBLE] frequency units. It's just the matrix element times delta t. And we usually call the matrix element the Rabi frequency over 2.

So you should think about it, that we add sort of this amplitude. And as long as we are coherent, I have to take the amplitude 1/2 the Rabi frequency delta, then square it. And here, you get the quadratic behavior. This behavior you also have constructive interference. All those delta [INAUDIBLE] are added in a phase-coherent way.

But this behavior ends, of course, at the coherence time. Let me just get my notes. So when the time becomes comparable to the coherence time, then we are adding amplitude not as constructive interference-- we are adding B squared [INAUDIBLE] becomes sort of a [INAUDIBLE], and we add things in quadrature.

So what happens is at the coherence time, we have created an amplitude which is given by this expression. But now if time goes by, we add in quadrature t over tc chunks of that. So therefore, our B square, which we build up with time, is linear in time and is omega Rabi squared over 4, the matrix element times the coherence time tc.

And this here now is our rate in the rate equation. And this is also what we exactly got out of the correlation function formalism. So this is how you should think about it. You go from constructive interference in amplitude and adding things in quadrature.

Let me maybe add one more discussion to it, which I hope will help you to see the big picture. We have a matrix element V. And if I just ask you, think about Fermi's golden rule, you would find out that the rate is V square times the density of states

at the resonance. And sometimes instead of the density of states, you write a delta function, which is just a placeholder that you should do an energy integral. And then you get the density of states. OK. So this is sort of Fermi's golden rule, and it should be old hat to you.

But what I told you now is that what is involved here from the time integration is the correlation function, V of 0, V of t as two different types. But then because of the type integration of Schrodinger's equation, we had a time integral dt. But V of 0, V of t, remember, we have a correlation between the field V of 0. And then it decays with time. This here can be written V of 0 squared times the correlation time tau c.

So therefore, the correlation function, it's a general formalism. It allows us to deal with all time dependencies. But in essence, the time-integrated correlation function is nothing else than your operator, your perturbation operator, time t equals 0 squared times the correlation time. And the correlation time, the inverse of the correlation time, is the spectral widths of your drive field. And this is what the spectral widths is in Fermi's golden rule.

On the other hand, if you do the integration V of t dt, think about it as Fourier transform, it gives us actually some V of omega. It gives us the Fourier transform. Well, I see it omega. If you just do V of t dt, you get the Fourier transform at 0 frequency. But if you look through the derivation we did that was an e to the i omega resonant term, which we were just spitting out. So we have sort of shifted the origin of frequency.

So therefore, the Fourier transform, if I correct for this just offset infrequency which I introduced for simplicity, this is nothing else than asking whether the drive field has a Fourier component which can resonate the drive to transition. And we talked last class about the convolution. This gives actually the power spectrum. It keeps the Fourier transform of it squared.

But the power spectrum is, of course-- when you take a field and fully analyze it, [INAUDIBLE] is the power spectrum, the whole power and the whole intensity of your laser, of your field is spread out over the banquets of the source. So when you

say, the power spectrum, the power spectrum is automatically intensity divided by spectral widths. And that's where the delta function and the spectral widths in the normal formulation of Fermi's golden rule come in.

So I'm just emphasizing I haven't really done anything new than just giving you a general notation. And I would actually say-- whenever you were asking yourself about Fermi's golden rule, this is the full [INAUDIBLE]. This is about a fully time-dependent arbitrary time field. And you realize, what was re-squaring Fermi's golden rule is actually just the simple case, the more general case-- and the more general case is it's a correlation or it's a power spectrum which creates the rate in Fermi's golden rule. Any questions about that? OK.

So we have this powerful formalism. So all we have to do is when we want to understand what is the rate and the spectral widths of the spectroscopic features is we have to understand what is the field driving the atom. And then we take the Fourier transform, or we figure out what the correlation is. We take V of 0 squared times the correlation time, and we know what the rate is. And this will allow us major insight into spectral broadening. OK.

But after introducing that in general, I think it's time for a simple example. And as a simple example, I just thought I'd show you how we get the natural line widths. And of course, the natural line widths-- I've mentioned several times, we need Optical Bloch equations.

We have to really kind of capture the interplay between the coherent drive of the atom and the spontaneous decay. But just phenomenologically, if I say the excited state has a decay rate of gamma over 2, I mentioned it before, we can capture some aspects of spontaneous decay.

And that means now, remember, our spectral widths comes from the correlation function of the matrix element. And now the operator is constant, but the state is decaying. So therefore, the matrix element has this exponentially decaying function. And if you then ask, what is the correlation function at time 0 and time tau, we get this.

Of course, the Fourier transform of an exponentially decaying function is a Lorentian. And I've shown you now with a different formalism that yes, if you have dampening in the excited state, for instance, you do spontaneous decay, we get the Lorentian.

Of course, we know there should be power broadening, too. But of course, don't expect power broadening from a perturbative approach, because a perturbative approach is only valid when we have weak drive fields. Questions? OK.

So now I want to-- after this simple example, I want to explain to you Doppler broadening. I know I will present it in a way and draw some conclusions which are usually not done in the normal presentation of Doppler broadening.

So what we have to bring in now is that we have moving atoms. That's what Doppler broadening is about. We drive the field with an electromagnetic wave. And we have usually ignored the spatial dependence, assuming the atom was clamped down in c equals 0. But now we have to allow for motion.

So the relevant matrix element for which we want to calculate the correlation function has now a spatial dependence. And our correlation function, G of ba, involves now-- I call the Rabi frequency now x for, well, simplicity of-- [INAUDIBLE] took this material, because they use it. OK.

So now we have the correlation function. We have the temporal part, t prime minus t. But then we have a spatial part. And this is now the new part which has to account for that the atom is moving from one position to another one. And when we calculate the correlation function, we have to average now over the velocity distribution of the atom.

So this new part, which will account for Doppler broadening, I called this part I. And z of t prime minus z of t is simply the velocity of the atom times tau. Tau is t prime minus t. So atoms move by that. And v is the z component of the velocity.

So now we have to calculate that. And of course, we assume that we have a

Maxwell-Boltzmann distribution. We assume a Maxwell-Boltzmann distribution. And then our expression I, we have to convolute-- we have this term, e to the kv tau.

But now we have to convolute it with the one-dimensional Maxwell-Boltzmann distribution, where alpha is the most probable velocity. And since the velocity distribution is normalized, we have this p [? vector. ?] Alpha is the most probable speed, and mu is the mass of the-- sorry, it's not mu, it's M. And M is the mass of the atom.

This integral, of course, can easily be solved. And we find that provides-- [INAUDIBLE] Gaussian envelope with time tau, so it's exponential-decayed with time tau. And our rate, which is the matrix element, involves now the temporal integral over-- let me just scroll up. In our correlation function, we had a temporal part and a spatial part. And the rate-- this was Fermi's golden rule-- is the time we've taken over the correlation function.

So therefore, we have the exponential [? vector ?] in time. After convolution with the Maxwell-Boltzmann distribution, we have this exponentially decaying term. And the result of that is, well, hooray, we have re-derived the Gaussian profile for a Doppler broad light. OK.

But yes, we want to look at this result with some new eyes, because until now, you would have said, OK, that's really trivial. Each atom has a velocity. Galilean transformation into the moving frame means the frequency has shifted, and everything falls together. And yes, that's one way to look at it. In inhomogeneous broadening, each atom has its own velocity.

But now we want to look at it from the viewpoint of a correlation function describing the whole system. So we had calculated this correlation function, which the atomic ensemble experiences. And this correlation function here decays as a function of time. And it decays with a characteristic time. Tau c, which is 1 over k alpha. And this is nothing else than the reduced wavelengths lambda bar divided by the most probable velocity. OK.

So we want to relate the line widths to some form of coherence. And what we realize is the coherence time of the correlation function, which needs Doppler broadening, is the time it takes an atom with the most probably velocity to move one wavelength. But wait. What we have is, in the Maxwell-Boltzmann distribution, the most probably velocity is also the widths of the velocities. So therefore, what we can say is, if all the atoms would start at one position, after the correlation time tc, the atoms have spread out over one wavelength.

So therefore, the correlation time tells us how long is the whole ensemble driven coherently. But once the atoms, due to their motion, spread out compared to their initial position, while extra wavelengths, each atom experiences now a different phase of the drive field.

And that sets a limit to the coherence. And this is the point where when we ask how many atoms are getting excited, we can no longer add amplitude in a constructive, linear way. We are adding amplitudes in quadrature. And this is exactly what I explained to you at the beginning of this lecture.

So let me just summarize it. Alpha is the thermal spread of velocities. So the keyword is here. Atoms, in a random way, spread out by lambda in the coherence time tc.

Well, a question which should come to your mind is now, but what happens when the atoms are in an atom [INAUDIBLE] and they cannot spread out? That's what you want to discuss in a few minutes.

But before I do that, let me give you another interpretation, which is helpful. You can regard-- if you have a thermal ensemble, you can say, OK, I have a box. Each atom is a plain wave with a perfect velocity, and now an ensemble of that. But very often, especially if you do localized physics, you want to regard each atom as a wave packet.

And if you want to use a localized description of your gas, where atoms are wave packets, then you assume, in a consistent description, that the atoms are spread

out due to the momentum spread in the Maxwell-Boltzmann distribution. This is nothing else than h bar divided by the mass and the most [INAUDIBLE] velocity. And this is foregoing [? vectors ?] on the order of unity, nothing else than the thermal de Broglie wavelengths. OK.

So now we have the picture that the atom is a wave packet in the ground state. But now we excite it with a laser. Well, if we excite it with a laser, part of the wave packet goes to the excited state. But the atoms in the excited state, because they have absorbed the photon recall of the photon, are now moving away from the ground state part of the wave packet, with the recall velocity, with this h bar k over M.

So if I regard the atom as a wave packet, the natural question is, when does the ground state of the wave packet lose overlap with the excited state part of the wave packet? Lost overlap after time-- well, I derive it for you. But it can only be the coherence time.

So the time is the size of the wave packet divided by the recall velocity. The mass cancels out. H bar cancels out. And this is 1 over k alpha. And this was exactly our coherence time.

So therefore, when I'm telling you you should understand this picture, Doppler broadening is a loss of coherence for the ensemble. You have now two ways to describe it. One is you can say, in a more quantum mechanical way, after the coherence time, the recall velocity has separated the grounded, excited part of your wave packet.

Or you can say, when the atoms in the ensemble have a velocity spread of alpha, then they have spread out by the optical wavelengths. So these are two equivalent picture to understand why this ensemble is no longer coherently driven. Yes?

AUDIENCE: [INAUDIBLE] in both of the curves, like, why is, in the first picture, the optical wavelength not relevant; in the second one, [INAUDIBLE] not relevant? [INAUDIBLE] distance.

PROFESSOR: Well, because these are two different pictures, but the results agree. I mean, the

wavelength comes into the picture with the wave packets through the recall velocity, because the recall velocity is h bar k over M, and k is the inverse wavelengths.

AUDIENCE: Right. But when we write delta t there, we write [INAUDIBLE] the wavelengths off the packet here, and we could have also just thought of it as, when does the atomic ensemble become bigger than the optical rate? But here, we are [INAUDIBLE].

PROFESSOR: I think for consistency-- I'll give you a quick answer. I should think about it longer, but what we usually assume when we describe atoms by wave packets, we assume the atoms are not cooled below the so-called recoil limit. So we assume that the thermal de Broglie wavelengths is shorter than the optical wavelengths.

And so then if you would say, you would expect also that there would be something happening when the wave packet spreads out by an optical wavelength, I want to think about it more. But the quick answer is, just assume what is usually the semi-classical limit of these kind of pictures, where we assume that we have a hierarchy that the thermal de Broglie wavelength is much larger than the size of the atoms but smaller than the optical wavelengths.

AUDIENCE: That makes sense.

PROFESSOR: A few things happen, really, in intuitive pictures when you cool atoms before the recoil limit. OK.

There's one reason why I would like to express it to you. Armed with that knowledge, if I would now ask you-- you have a trapped Bose-Einstein condensate, and you take the spectrum. What is the Doppler widths of the spectrum of a Bose-Einstein condensate?

[INAUDIBLE] extra Maxwell-Boltzmann distribution. No. The condensate is different. It's in one quantum state. But now you can choose your picture. One picture you can take is you can say, the de Broglie wavelengths here has to be-- you know, the wave packet loses overlap when the excited state moves one de Broglie wavelength.

But the condensate is fully coherent. So you would now say, maybe I should think about if-- if the condensate part of it is coupled to the excited state, and with a recoil, the excited state component has moved the size of the condensate, replacing the de Broglie wavelengths by the size of the condensate. And this is a correct answer.

You would then find out what is, quotation mark, the Doppler broadening of a condensate. Of course, you could have also said, the condensate is a certain size, h bar divided by the size is the momentum spread if you do Heisenberg's uncertainty relation. And now I plug in this momentum spread into a formula for the Doppler broadening. And you would get the same result.

But especially when you think in terms of a coherent wave function, this picture of losing overlap between the two parts of the wave packets is very intuitive, very useful. And it actually guided a lot of our intuition when we looked at the limitations of super radians and optical spectroscopy with Bose-Einstein condensates. Any questions? OK.

So now we are ready to take it to the next level. When I told you that the spectral widths is the inverse of the coherence time. And one way to think about the coherence time is that the particles spread out over one wavelength. So if you take this thought seriously and say, what happens if I confine atoms in a container or an atom trap to less than the optical wavelengths, then you would say, they can never spread out by an optical wavelength. Does it mean that the coherence time is now infinite and that we can do spectroscopy, which is no longer affected in any way by Doppler broadening.

Well, what I just motivated in words is the so-called Lamb-Dicke limit of tight confinement. And as I want to show you now, yes indeed, you have a very, very sharp line which is not broadened by Doppler broadening, which is not shifted by the recoil shift. It's really the unperturbed line of the atom which can be probed by confining the atoms to less than a wavelength.

So therefore, let's now discuss the line shape of confined particles. So what I want

to present you now is we have particles trapped in a harmonic oscillator. And in one limit, which I want to explain you, we should just find the normal Gaussian Doppler profile which we have obtained for free gas.

This must be the limit when the [INAUDIBLE] confinement is very weak. But for tight confinement, we should actually find, unless we assume other means of line broadening, a delta function spectral feature. And I will explain to you that this is actually the same as the Mossbauer effect. It's a Mossbauer line due to the confinement.

So therefore, to have trapped particles allows us to go to the ultimate limit in precision spectroscopy. What happens when you have trapped particles-- the Mossbauer effect, which I mentioned, or simply the effect of confinement-- in other words, the trapping potential is completely eliminating the Doppler effect.

But I want to be specific. It only eliminates the first-order Doppler effect. Everything I just did with the correlation function assumed first-order Doppler effect. If you want to get rid of the second-order Doppler effect, then you need some form of proving. But usually, when you do experiments with trapped particles, you do confinement and cooling at the same time. OK.

Let me start out with very basic things. So let's talk about the spectrum of an oscillating emitter. If we have an atom, it undergoes a transition from excited state b to excited state a, this is an internal state. But now we want to include motion. And we have to include the external degree of freedom. And for our discussion right now, the external degree of freedom is [INAUDIBLE] trapping potential. So now we look at the combined system, combined-- we can say Hubert space, which combines external and internal motion.

And of course, the external motion is now quantized. I can't assume, but it doesn't really add anything to it at this point-- that the trap frequency and the ground and excited state are different. I simply assume that the trap frequency omega trap is the same. This, of course, is excellently fulfilled in ion traps. If you reduce spectroscopy of neutral atoms in a dipole trap, of course, the ground and excited

state may experience a different AC stark shift. And then you have to account for two different frequencies. But let me just make this simplifying assumption.

So if you assume that an atom emits radiation, it will, for energy conservation, emit at the electronic energy. But then there is an extra term, which, in general, is the energy of the external motion, or the trapping potential for the initial state minus the final state.

And if we now make our simplifying assumption that everything is harmonic, that hyper-potential is harmonic, and the trapping frequency is the same in the excited state and the ground state, we simply have the electronic energy plus n quanta-- n is the change of the number of quanta of the harmonic motion.

I want to point out, it looks so trivial. But you should at least think for a second about this statement, that this formula includes the Doppler shift and the recoil shift.

And of course, this is trivial, because we are talking here about the total energy of the external state, the total energy of the final state after photon emission. And the energy and the trapping potential includes all the kinetic energy of the particle, which includes whatever comes from [INAUDIBLE] velocity or from the motion of the atom.

So therefore, we obtain what is called the sideband spectrum. I'll just show you a stick diagram. Here is the electronic transition. And then we have sidebands. And the spacing of those sidebands is nothing else than the harmonic oscillator frequency. Any questions? At that level, I want you to appreciate that this is radically different from Doppler broadening. There is no Doppler broadening. By fully quantizing the motion in the harmonic oscillator, we obtain a discrete spectrum.

And what I want to show you is when we calculate the intensity in the peak for strong confinement, almost all of the intensity's in the central peak. And therefore, there is no Doppler broadening. But I want to later show you how we can go from the discrete spectrum back to the Doppler broadening which we just described in free space.

You should say, well, but the motion, the recall, it must come in. Where is it? Well, it's not in this stick diagram. But the sticks are the only possibilities for the possible photon frequencies or photon energies. But the question is, what is the amplitude? What is the probability that this will happen? And you already see where I'm aiming to. In the limit that we have many, many sticks and we are not resolving the sticks, we will get back to them in the standard Doppler broadening.

So the big question is, how many of those sticks do we have? Are we in the limit where things are heavily discrete? This is our new limit? Or do we have many, many of them?

So therefore, the recoil and the velocity, they really enter when we calculate the intensity. And whatever our formulation is with Fermi's golden rule, the rate is proportional to the relevant matrix element squared. And now I want to show you how we calculate those matrix element. You had a question?

AUDIENCE: But is it true that each of these sticks has the intrinsic line width of the atom?

PROFESSOR: Yes. We come to that in a few minutes. I ignore here the spontaneous broadening just for pedagogical reasons. But a little bit later, I will-- I first want to sort of discuss the number of sticks. Do we have a few? This is sort of new, then we have only few sidebands, and we have the Mossbauer effect. If we have many, that's sort of more the continuum, which we described with the classical velocity distribution. That's my message number one.

But then the next message is, do we resolve the sticks? Do we have resolved sidebands or not? And for that, the criterion is, is the natural line widths large or smaller than the sideband spacing? It's not just one parameter; there are two parameters.

One is which will be the Lamb-Dicke parameter-- how many sticks do we have? And the second question is, do we resolve the sticks? And you can say there are four different regimes, you know-- yes or no for question one, or yes or no for questions two. Other questions? OK.

So the rate is proportional to the matrix element squared. And yes, we have all kind of the matrix elements involving the internal degree. But the new thing is the matrix element for the center of mass wave function of the atom, which is-- you can see just the emission where the polynomials of the harmonic oscillator. So we have the eigen functions for the harmonic oscillator between initial and final state. And what acts on the only part of the electromagnetic field operator, which acts on the position of the atom, is this term, e to the minus ikr. OK.

I mentioned already that the new regime is that the confinement is tight. So let's just look at this situation. When kr is much smaller than 1, then we can expand this exponential into 1 minus ikr. And now I want to remind you that the position operator, when we treat the harmonic oscillator, is nothing else than a plus a dagger.

So therefore, if kr is small and therefore we can do the first-order expansion of e to the ikr, our spectrum, our operator is here-- 1 minus e [? vector ?] a plus a dagger. And therefore, the only possible sidebands are the one where the change in harmonic oscillator quantum number is plus or minus 1.

So we are already obtaining the result, which is called the Lamb-Dicke limit when kr is much smaller than 1, that we have a strong carrier. We have only sticks-- the delta n equals plus, minus 1. And the intensity in each of those sticks is actually proportional to k squared r squared, which is nothing else than the square of the extension of the atomic wave function. And k is 1 over lambda bar squared, divided by the optical wavelengths we get.

So we see already-- I mean, without any major formalism or mathematical tools-- what happens in the limit of tight confinement. The spectrum of confined particles is eclectic, the spontaneous line widths. A delta function without any Doppler shift, without any recoil shift, right at the resonance frequency, electronic frequency of the atom.

The only thing which is reminiscent of the motional degree of freedom are those

small satellites. But their intensity goes to 0 with the extension squared over lambda squared. So therefore, if you confine the particle to less than an optical wavelength, the older picture I told you, it can never spread out over wavelengths, can never get out of coherence with the drive field. And here, we have a quantitative description that at that moment, we can obtain spectroscopic information about the resonance completely unperturbed by motional effects. Questions? OK.

So that was maybe the most fun part, the extreme limit and you realize what happens. But now we want to sort of fill in the gaps. I first want to sort of contrast what I just described to you with a semi-classical picture. The semi-classical picture-- if you have an emitting oscillator or absorber and we have electromagnetic plane wave, we can now ask, what is the phase of the plane wave experienced by the atom?

And of course, the phase is affected by the motion of the atom if we assume the atom moves [INAUDIBLE] with harmonic frequency omega t and an amplitude x0. Then this here is the phase seen by the atom in its own reference frame.

And if I then define-- with another quotation mark, because it's sort of something which needs explanation. If I define an instantaneous frequency, which is nothing else than the derivative of the phase, then I'll retrieve the normal Doppler broadening. So now you see where sort of normal Doppler broadening would come in.

But the question is, you cannot measure an instantaneous frequency. It would violate the Fourier theorem. But if we can apply-- if the atom oscillates slowly enough-- the motion is slow enough that we can apply the concept that we can look at the frequency the atom experiences, you would at least say, before the atom changes its velocity, it should see a few cycles. If it oscillates fairly fast, this concept, of course, cannot be applied. But at least you see where your normal Doppler shift comes in. It comes in in the concept of an instantaneous frequency.

Of course, what we should do now is we should not take the concept of the instantaneous frequency. We should do rather do it correctly. What we have is we

see we have the phase of the atom. So therefore, we should-- by using the motion of the atom for x, we should have formed Fourier transform. The Fourier transform tells us what is the tonal spectrum which the atom experiences.

So therefore, we take this plane wave, we put in the oscillatory behavior of the atom, and then we take away transform. OK.

So the electromagnetic field seen by the atom is an amplitude which is the cosine of the phase. And the phase has a temporal dependence that's just the frequency of the plane wave. But now the precision involves the oscillation of the atom at the [INAUDIBLE] frequency omega t.

And what I've introduced here is the data [? vector ?] is called the modulation index. This is the relevant quantity. It is k, which comes from the plane wave, e to the ikr, times the amplitude of the atom. And so the modulation index is nothing else than the amplitude of the atomic motion divided by the reduced wavelengths of the light.

Just if you remember for a second, this extreme stick diagram where we had one big stick and two smaller sticks, remember, the intensity in the smaller sticks was the extension of the atom divided by the wavelength squared. So we had exactly the same parameter, but the previous description assumed a quantized picture for the harmonic oscillator. I really used a and a daggers in the atomic wave function.

This is a semi-classical picture where I treat the oscillation of the atom in a classical description. But in both cases, of course, the relevant parameter is the ratio over which the atom moves-- the amplitude of the atomic motion or the size of the atomic wave function in relation to the wavelengths of the light. OK.

So we want to Fourier transform this function. And this gives us the spectrum. I can show you that the result is that we can have a Fourier expansion of this amplitude, and it involves basal functions. So it is the basal function Jm which tells us what the intensity is in the n's sideband. And the argument of the basal function, whether we take the basal function at the origin or at a finite argument, is given by beta modulation index.

You can't immediately [INAUDIBLE] in just two lines. We write the Fourier transform. All you have to use is those identities which involve the basal function. Whenever you have the cosine of a sine or the sine of a sine and you Fourier transform, you get, naturally, basal functions. OK.

So let's just look at the result. We talked about the stick diagram. The stick diagram was actually motivated quantum mechanically. But now we also find a stick diagram purely classically because of the periodic motion in the harmonic trap.

But now what we obtain in the semi-classical limit, we obtain what is the height of each stick. Well, the height of the stick, which is n sidebands away, is given by the square of the n's basal function. And beta is the modulation index. And if beta goes to 0, that means the atom is not moving at all. The amplitude x0 is 0. This is the limit of tight confinement. Then all basal functions are 0 except the 0's order. And that means we are back to a single delta function, a single stick in our stick diagram.

So let's now take the result and discuss it. So how does the spectrum look like? Well, I want to give you two limits where the atom is extended much larger than the wavelengths. This means large modulation index beta and small modulation index beta.

But first, in the limit where beta is large-- for large beta, the argument of the basal function becomes just cosine beta minus phase. So if we use that-- if we assume a thermal distribution where beta over kt d beta squared.

So we take our spectrum, and we just convolute it with a thermal distribution of amplitudes x0, which is, of course, just the Boltzmann factor. And in that limit, we actually obtain a spectrum where the envelope of the sticks looks like the Doppler width. So if those two conditions are fulfilled, then we obtain the normal Doppler widths.

If, in addition, we assume that the sidebands are not resolved, because the natural line width is larger than the spacing. So if we assume that in addition, then we just find the normal Doppler broadening how we have derived it as in free space.

So normal Doppler broadening is the limit of the large modulation index, a thermal distribution of modulation index in the case of not resolved sidebands.

The opposite limit, of course, is when beta is much smaller than 1, it means the amplitude of the oscillating particle is smaller than the wavelengths. I mentioned it a few times, so I should write it down. This is the Lamb-Dicke regime. And then the relevant approximation for the basal function is in the limit. And this limit is 1 over n factorial times beta over 2 n.

And for the case that n is 1, this is just beta. And the amplitude squared is beta squared. It's x0 over lambda squared, as we had discussed before in the quantum mechanic unit. So you'll see that to use matrix element for the harmonic oscillator or use a semi-classical Fourier transformed with a basal function, both lead to the same result, that deep in the Lamb-Dicke limit, we have essentially three peaks. And the satellites are quadratically small in the modulation index squared.

So this regime is particularly interesting for atomic locks and for meteorological applications if the sidebands are resolved. If the sidebands are not resolved, you have sort of a line shape which depends, let's say, on temperature, because x0 squared, if it's the thermal excitation, is proportional to temperature. So as you cool it down, you will actually see that your line shape changes.

But the good thing is that once you resolve the sidebands, the line shape of the central carrier does not change anymore. And once you can resolve the sidebands, you can just observe the central carrier, and you obtain spectroscopic information which is no longer blurred, which is no longer affected by motion or by temperature. And that's, of course, a regime where the atomic frequency standards want to operate. Questions? OK. Yes.

Let me just write that down, because this is important. For resolved sidebands, you have sharp lines, no motional broadening. And the physics I described to you is actually analogous to the Mossbauer effect.

In the Mossbauer effect, the intensity of the recoil-less emission of this recoil-less

line is described by the [INAUDIBLE] [? vector. ?] So in this case, the [INAUDIBLE] [? vector ?] is 1 minus the probability for the two sidebands. So the same concept of Mossbauer line and [INAUDIBLE] [? vector ?] describes the physics for tightly confined particles.

So if I use the analogy to the Mossbauer effect, the Mossbauer effect is called the recoil-less emission and absorption of x-rays. So what we have here is we have a recoil-less absorption and emission of photons. Of course, the photon which is emitted and absorbed has momentum. There should be momentum recoil.

So the question is, where does the momentum recoil go when the confined particle emits or absorbs on the carrier of the central line. How do we reconcile the result I derived for you with momentum conservation?

A trapped particle, tightly confined. And photon comes, has momentum, the atom absorbs it. But the spectrum does not show any evidence for recoil shifts and such. Colin.

AUDIENCE: Must be absorbed by the trap somehow.

PROFESSOR: Must be absorbed by the trap, yes. Your trap is anchored in the laboratory. And you transfer the momentum. The atom is attached to the trap. The trap is attached to the laboratory. So the object which takes over the momentum is your whole apparatus or, in the extreme case, a whole building. And there is, of course, a kinetic energy associated-- momentum squared over 2 m. But the mass is now the mass of the building. So therefore, there is no energy associated with absorbing the recoil.

So it is as if-- and this comes back to some earlier remarks I've made-- is that when you have this absorption in the Lamb-Dicke regime is it is as if your two-level system has an infinite mass behind it. And that's sort of the situation how I told you you should often think about it, you should separate effects of the internal degree of freedom and the external degree of freedom by just assuming I can assume first that the atom has infinite mass. And once the atom has infinite mass, the motion

degree of freedom doesn't matter.

What I just explained to you is a practical way to endow your particle with infinite mass. Just connect it with type confinement to your apparatus. And then for the momentum exchange, there is a photon field, it is actually the mass of the whole apparatus [INAUDIBLE]. Questions?

AUDIENCE: Is there a more direct way to think about this for maybe the example of a magnetic trap, the mechanism by which this momentum is transferred? Or absorbed.

PROFESSOR: OK. Magnetic traps for neutral particles are usually not in the Lamb-Dicke regime, so you have to be a little bit careful. But the explanation would be the same. The magnetic fields are like tight springs which connect the neutral atom to your coins. And therefore, you should just think about the magnetic trap in a mechanical model.

Your neutral atom is connected to your coils with strings. And if you now hit the atom with a photon, because of the quantization of the discreteness of this specturm, the photon, in most cases, does not have enough recoil to create a mechanical oscillation of your particle. The momentum goes through the sphinx to the support structure.

But occasionally, with a probability which is a modulation index squared, you will actually promote the particle to the first state of harmonic motion. And then the atom has acquired some kinetic energy. But this probability can be made as small as you want by going to a smaller and smaller modulation index. That's the way how I would think about it. Nancy?

AUDIENCE: About the multiple lines that we get, I was wondering if the levels of the harmonic traps itself are blurred, which could happen in the lab if the trap depth is moving, for example-- the levels of the trap would get blurred. Would that result in additional broadening of this? Or how would that affect?

PROFESSOR: It depends. I think if you have some temporal broadening, you know, you just plug it into your correlation function, whatever shakes your system. If that means the atom sees some shaking in the phase of the electromagnetic field, it affects it.

If you have an ensemble of atoms or if you do the experiment repeatedly, and your measurements are the ensemble, and every time you do the measurement, your magnetic trap has a slightly different [INAUDIBLE], and therefore, a slightly different confinement, well, what you would see is that those sidebands fluctuate, that the carrier is independent of the trap frequency.

So therefore, the carrier is actually the central peak-- would not be broadened by fluctuations in the harmonic confinement.

Let me just make one comment. I derived the result for you-- at least that one-- by taking the amplitude of the atomic motion, the amplitude of the phase, and doing the Fourier analysis. I mean, this is exactly what we learned from the formalism of correlation function. You should take the amplitude of the perturbing field and fully analyze it. I didn't phrase it here in the language of correlation functions. But what I did was exactly what we learned from the correlation function formalism. OK.

Let me maybe summarize in words what we learned. So what we learned from this discussion is that what matters for line broadening and obtaining spectroscopic information is the accumulated phase, the phase which the atom accumulates. And if different atoms in the ensemble accumulate a phase which is different by 2 pi, at that moment, we have reached what we call the coherence time of the correlation function. And the inverse of this time is a line broadening.

But now we also discussed the case of tight confinement. The atom can be in very, very rapid motion. And the phi dot, the change of phase can be very rapid due to the instantaneous velocity. But if the atom turns around because it's in a singulatory motion, positive and negative Doppler shifts completely cancel, because you never allow the atom, in this periodic motion, to acquire a net phase. And therefore, the motional broadening is absent.

So one way to think about this carrier is that the atom rapidly oscillates through plus kv and minus kv Doppler shifts. And the two cancel. So this is the reason for that. OK.

I think we are now very well prepared for Dicke narrowing. Actually, I have to say, it's the first class this semester that I was teaching a little bit faster than I assumed. So I'm now right at the end what I prepared for today. But I know my notes sufficiently well that I can go on for 10 minutes.

So the Dicke narrowing-- the last time I looked at it was two years ago. But let's get the physical picture. So I want to now apply what we have learned not to a trapped atom, but to an atom which is embedded in buffer gas. So just think one rubidium, or one sodium, or one lithium atom. And it is surrounded by a buffer gas of argon or neon.

And I know sometimes when we do saturation spectroscopy to stabilize our lasers, we have a little glass cell, which has sodium or rubidium in it. But we also put an argon buffer gas into it. So that's the situation I want to describe now. But Colin, you had a question.

AUDIENCE: Yeah. You wrote down the condition for resolving the sidebands as your trapped frequency being larger than your natural line width. An alkalizer-- if you had, say it were 10 megahertz line width. How do you actually resolve this without a ridiculously-- because people do sideband [INAUDIBLE], and they don't have 10 megahertz trap frequencies, do they?

PROFESSOR: OK. So good question. The question is, the resolved sideband limit, how can we reach it? Well, it can be reached in ion traps. In ion traps, because you can put kilovolts on electrodes, you can really create harmonic oscillator frequencies which are many, many megahertz. And then you are at the resolved sideband limit, assuming that the natural lifetime, if that's the case, of many ions is in the megahertz [INAUDIBLE].

For neutral atoms, it looks like, you know, mission impossible. However, there's a way out, and this is the following. One is you can maybe [INAUDIBLE] one-dimensional optical latice. So in the latices, you at least have tight confinement of many, many kilohertz. But now you want to use a very narrow transition. If you use a very, very narrow transition, then even for 10 kilohertz external harmonic oscillator

potential, [INAUDIBLE] resolved sideband [INAUDIBLE].

Now, for alkalis, you won't find an excited state which has a natural line width of 10 kilohertz. And this will be our discussion on Wednesday. If you use a Raman transition, we go from one count state with an off-resonant laser to another ground state, [INAUDIBLE] will be two photon transitions. But those two photons, since there's no intermediate state, can be regarded as, click-click, you absorb two photons. And in some picture-- that's a message I will give you next week-- the equivalent to a single photon.

So now you have a two-photon transition, which transfers recall to the atom. So the effective wavelengths of the two-photon transition is because you have twice the photon energy. You have two photons involved which both have a recoil. So the effective k vector is two times the k vector of an atom.

But the spontaneous line width is close to 0, literally 0, because you have a Raman transition between two common states.

If you do Raman sideband cooling of neutral atoms, then you reach the Lamb-Dicke limit, you reach the limit of strong confinement. But you need a better [INAUDIBLE] transition. And there are, of course, some atoms which have a very narrow clock transition. But for many, many atoms which have hyperfine structure, you can resolve to Raman transition. OK.

Let's now talk about Dicke narrowing. So we have a situation that we have an atom in buffer gas. And in most situations, when you put an atom into buffer gas, you get what is called collisional broadening. I will talk about collisional broadening on Friday. Just a reminder, we have class on Friday but not on the following Monday.

So on Friday, we'll talk about collisional broadening. And I will discuss, for instance, the model-- an atom in the excited state, when it collides, it gets de-excited. And then you have pretty much a situation where you have, in effect, a shortened lifetime of the excited state. And what you get is a Lorentian which is broader, which has a width not of the natural line width, but a width which is [INAUDIBLE].

But there are situations-- and that's what I want to discuss here-- that we have atoms in a more benign buffer gas. Where we can assume that this is actually fulfilled, that collisions do not change. Well, they're not de-exciting the excited state. But they're not even changing the coherence between counter-excited state.

So the phase evolution, the internal state, grounded excited state just-- if you can assume you have a Bloch vector which oscillates, and the Bloch vector superposition [INAUDIBLE] excited state oscillates at the natural frequency, and this Bloch vector just rotates, it doesn't have a hiccup, it doesn't change its phase when the atom collides with the buffer gas atom.

So we assume that we have such a buffer gas where collisions don't change the internal coherence. And by internal coherence, I mean the phase between the ground state and the excited state.

So in this situation-- but it's actually a very important situation which has been reached, in many cases. In this situation, we have-- thus, the buffer gas acts only on the external motion of the atom. And now you can say, in some way, the buffer gas acts like a trap. The particle wants to fly away, but it collides with the buffer gas atom. And with a certain probability or after a few collisions, it returns back to the origin.

However, it's a lousy trap, because there is some randomness and effusive motion. So if you want to describe it as a trap, it would be a trap with a wide spread of trap frequencies. OK.

So if we use this picture now, what we have learned from ion traps-- remember, we had an ion trap with a sharp carrier. And then we had sidebands at the trap frequency. But if we have sort of now a lousy trap which each realization, each moment has a different trap frequency when waving all my arms, I would say for the other part of the ensemble, we get a carrier, and we have something else. And for another realization, we have another trap frequency.

So if I use a little bit of artistic intuition here, I would expect, based on what we

learned from the previous discussion, that in such a buffer gas, I would have a sharp carrier, and then I would have sort of a pedestal, which is the envelope of many trap frequencies.

And we know sort of that the envelope of all our sticks-- this was actually given by the Doppler effect. So what we may expect now is that in this situation with buffer gas, we get a sharp line. And then we have this broad pedestal, which you can think [INAUDIBLE] intuitive picture as smeared outside bands. And I may call that the Doppler pedestal. And we would expect-- and I discussed that with the basal function, that there is one limit where the envelope of all those sticks eventually looks like a Doppler-broadened line.

Anyway, time is over. But let me just give an outlook. On Friday, what I want to do is I want to calculate the width of this line with you. And remember, all we have to do is we have to calculate the correlation function. Previously, when I derived for you Doppler broadening, the correlation function was the kx becoming kvt. kvt, where x became vt how much the atom moves with the velocity v.

By simply replacing the linear motion, V times t, with a diffusive model, we can calculate the line shape. And we will actually find that the central line is not infinitely sharp, but it has a width which is given by the diffusion constant. And if the diffusion constant is very, very small, we approach a very sharp line.

And the final comment is, and this is called Dicke narrowing. It is this counterintuitive result that collisions, if they have those properties, are not broadening the line. They actually narrow the line from the [INAUDIBLE] Doppler width to something which is much sharper. And this has been useful for high precision spectroscopy.

But I think with the concept which we discussed today of confinement, you realize why collisions can actually reduce the line widths, namely by preventing the atoms from acquiring random phases with respect to the drive field [INAUDIBLE]. Any questions? So see you on Friday in this other building, this other lecture hall that's just been announced on the website.