Chapter 3: Duality Toolbox

MIT OpenCourseWare Lecture Notes

Hong Liu, Fall 2014

Lecture 24

3.3: HOLOGRAPHIC ENTANGLEMENT ENTROPY

3.3.1: ENTANGLEMENT ENTROPY

Consider a quantum system that is divided into A and B two parts such that the full Hilbert space can be represented as the tensor product of the Hilbert spaces of A and B:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{1}$$

A typical wave function of this system is $\Psi = \sum_n \psi_n(A) \otimes \chi_n(B)$, in which A and B in the state $|\Psi\rangle$ are entangled since $|\Psi\rangle$ cannot be written as a simple product of those of A and B. Entanglement entropy (EE) is a measure to quantify how much A and B are entangled. Define the reduced density matrix of A by tracing over all B state in $|\Psi\rangle$:

$$\rho_A = Tr_B |\Psi\rangle\langle\Psi| \tag{2}$$

Then EE is defined as

$$S_A = -Tr \rho_A \log \rho_A \tag{3}$$

From the definition, an obvious corollary is

$$S_A = 0 \iff \rho_A \text{ denotes a pure state} \iff |\Psi\rangle \text{ can be written as a simple product} \tag{4}$$

Indeed, for any pure state $|\Psi\rangle$, we can always write it as

$$|\Psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle \qquad \lambda_i \geq 0 \tag{5}$$

by Schmidt decomposition of $|\Psi\rangle$ into some complete set of A and B. It follows that

$$S_A = S_B \tag{6}$$

For AB composite system in a mixed state, in general, we have $S_A \neq S_B$. In such a case, EE typically contains classical statistical correlation of the mixed state in addition to quantum correlations.

Here are some examples. A two spin system, we can have a state

$$\begin{aligned}|\Psi\rangle &= \frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \\ &= \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)\end{aligned} \tag{7}$$

that is not entangled. However the state

$$|\Psi\rangle = \cos\theta|\downarrow\uparrow\rangle + \sin\theta|\uparrow\downarrow\rangle \tag{8}$$

is entangled. For this state, the reduced density matrix for A is

$$\begin{aligned}\rho_A &= Tr_B \cos^2\theta|\downarrow\uparrow\rangle\langle\downarrow\uparrow| + \sin^2\theta|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + \sin\theta\cos\theta(|\downarrow\uparrow\rangle\langle\uparrow\downarrow| + |\uparrow\downarrow\rangle\langle\downarrow\uparrow|) \\ &= \cos^2\theta|\downarrow\rangle\langle\downarrow| + \sin^2\theta|\uparrow\rangle\langle\uparrow|\end{aligned} \tag{9}$$

thus EE is

$$S_A = -\cos^2\theta \log\cos^2\theta - \sin^2\theta\log\sin^2\theta \tag{10}$$

We see S_A has period of $\pi/2$ w.r.t. θ and the minima are $S_A = 0$ when $\theta = n\pi/2$ and maxima are $S_A = \log 2$ when $\theta = n\pi/2 + \pi/4$, where $n \in \mathbb{Z}$. At maxima, the state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$ is most entangled.

We list some important properties of EE as follows. Suppose A, B, C are three parts of the system without any intersection between any two of them. We have

1. Subadditivity:

$$|S(A) - S(B)| \le S(AB) \le S(A) + S(B) \tag{11}$$

2. Strong subadditivity (hard to prove):

$$S(AC) + S(BC) \ge S(ABC) + S(C) \tag{12}$$

$$S(AC) + S(BC) \ge S(A) + S(B) \tag{13}$$

EE plays an important role in quantum information and quantum computation as it provides quantitative measure of quantum correlations that are not present classically.

3.3.2: ENTANGLEMENT ENTROPY IN MANY-BODY SYSTEMS

Consider a system AB. If Hamiltonian is $H = H_A + H_B$, then the ground state is unentangled. Moreover, starting with a general unentangled state $|\Psi(t = 0)\rangle$, the system will remain unentangled. If we add coupling between A and B, $H = H_A + H_B + H_{AB}$, then the ground state is generally entangled. Certainly, starting with an initial unentangled state, entanglement will be generated during time evolution. In all realistic condensed matter (CM) systems and quantum field theories, H_{AB} is local, *e.g.* Heisenberg model,

$$H = \sum_{\text{nearest neighbors}} J_{ij} \vec{S}_i \cdot \vec{S}_j \tag{14}$$

and ϕ^4 QFT,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 \tag{15}$$

Locality implies (as shown in the picture below) that the interaction only happens near the interface of A and B, so H_{AB} only involves degrees of freedom near ∂A and ∂B. This has important implications.

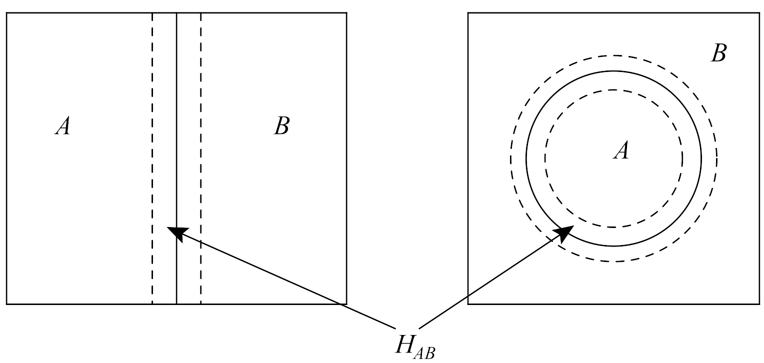


Figure 1: Locality of interaction

One finds, in general, in the ground state for a local Hamiltonian, EE is given by

$$S_A = \# \frac{Area(\partial A)}{\epsilon^{d-2}} + \cdots \tag{16}$$

where ϵ is the lattice spacing for CM system or short-distance cutoff for QFTs, which characterizes the geometric width of interface. This formula shows that EE between A and B is dominated by short-range entanglement near ∂A, where H_{AB} is supported. While (16) appears to be universal, # is not, depending on short-distance physics of specific systems. For the subleading order terms, the excitement of last decade on EE shows that they come from large-range entanglement and can provide important characterization of a system:

1. Characterize topological order ($2+1$ dimension). In typical gapped systems, ground state contains only short-range correlations. In topological ordered systems, ground state contains long-range correlations not accessible via standard observables such as correlation functions of local operators etc. The EE for such a system is

$$S_A = \# \frac{L(\partial A)}{\epsilon} - \gamma \tag{17}$$

where $\gamma \neq 0$ is topological order, a constant independent of shape and size of A.

2. Characterize the number of degree of freedom of a QFT. For a $(1+1)-$dimensional CFT, there is no leading contribution (area of ∂A is zero), and one finds

$$S_A = \frac{c}{3}\log\frac{l}{\epsilon} \tag{18}$$

where c is central charge and l is the length of A. We know c contains the information of number of degree of freedom of a 2d CFT, *i.e.* via Cardy formula. For $(2+1)-$dimensional CFT, one finds

$$S_A = \#\frac{L(\partial A)}{\epsilon} - \gamma \tag{19}$$

where γ depending on the shape of A can be used to characterize the number of degree of freedom. Similar results can be found in other dimensions.

3.3.3: HOLOGRAPHIC ENTANGLEMENT ENTROPY

Suppose we have a CFT with a gravity dual, how can we calculate EE on gravity side? As shown in the picture below, we propose to find the minimal area surface γ_A which extends into the bulk with ∂A as boundary and EE is given by

$$S_A = \frac{Area(\gamma_A)}{4G_N} \tag{20}$$

This formula was first guessed by S. Ryu and T. Takayanagi (R-T) who were motivated by black hole entropy.

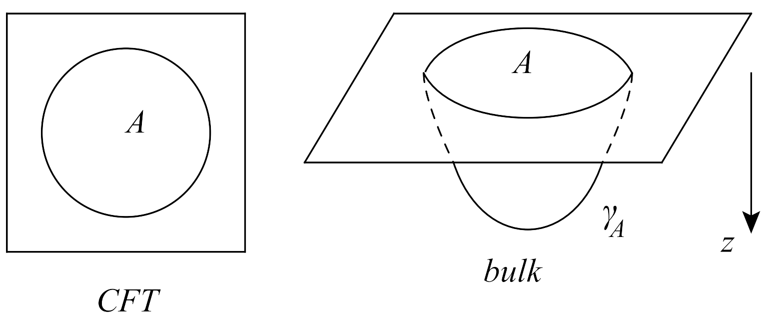


Figure 2: Minimal area

There are some supports for this formula: (1) satisfy the strong subadditivity conditions; (2) reproduce known results about EE; (3) give many new results that are all sensible. While easy to define, EE is very complicated to compute for a general many-body system. Even for a free QFT, the computation is highly non-trivial, and often numerical calculation are needed. R-T provides a very simple way to compute EE in a class of strongly interacting QFTs.

Here are proofs of strong subadditivity.

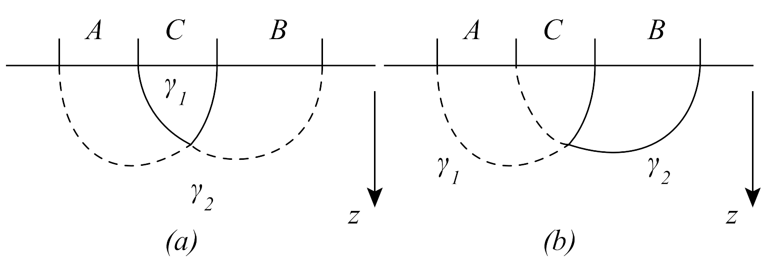


Figure 3: Proof of strong subadditivity

In Figure 3(a), we can easily find

$$\gamma_{AC} + \gamma_{BC} = \gamma_1 + \gamma_2 \tag{21}$$

and by definition of minimal surface

$$\gamma_1 \geq \gamma_C \qquad \gamma_2 \geq \gamma_{ABC} \tag{22}$$

which implies

$$S(AC) + S(BC) \geq S(ABC) + S(C) \tag{23}$$

In Figure 3(b), we also have

$$\gamma_{AC} + \gamma_{BC} = \gamma_1 + \gamma_2 \tag{24}$$

and by definition of minimal surface

$$\gamma_1 \geq \gamma_A \qquad \gamma_2 \geq \gamma_B \tag{25}$$

which implies

$$S(AC) + S(BC) \geq S(A) + S(C) \tag{26}$$

Now we can apply this formula to $(1+1)-$dimensional CFT, whose dual is AdS_3 where the metric is

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + dx^2 + dz^2) \tag{27}$$

We know each CFT is characterized by a central charge c, *e.g.* density of state:

$$D(E) = \exp\left(2\pi\sqrt{\frac{cn_L}{6}} + 2\pi\sqrt{\frac{cn_R}{6}}\right) \qquad E \propto n_L + n_R \tag{28}$$

and the trace anomaly gives

$$\langle T^\mu_\mu \rangle = -\frac{c}{12}\mathcal{R} \tag{29}$$

where $\mathcal{R}$ is Ricci scalar. For holographic CFT, the central charge is given by

$$c = \frac{3R}{2G_N} \tag{30}$$

On the gravity side, like Wilson loop, we consider constant time slice where the metric is

$$ds^2 = \frac{R^2}{z^2}(dx^2 + dz^2) \tag{31}$$

Assume the minimal surface is a function of $x(z)$. The differential length of the surface is

$$dl^2 = \frac{R^2}{z^2}(1 + x'^2)dz^2 \tag{32}$$

Consider right half, $x(0) = \frac{L}{2}$,

$$S_A = \frac{1}{4G_N} \times 2\int_0^{z_0} dz \frac{R}{z}\sqrt{1 + x'^2} \tag{33}$$

Extreming it leads to the well known answer, $x = \sqrt{L^2/4 - z^2}$, the half circle with $z_0 = L/2$.

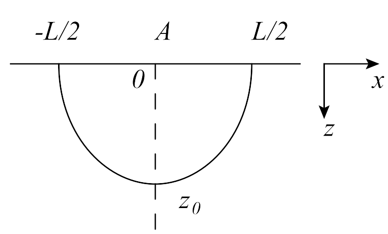


Figure 4: Minimal surface in bulk

Finally, we get EE

$$S_A = \frac{1}{4G_N} \times 2R \times \frac{L}{2}\int_0^{L/2} \frac{dz}{z}\frac{1}{\sqrt{L^2/4 - z^2}}$$

$$\begin{aligned} &\simeq \frac{RL}{4G_N}\int_\epsilon^{L/2}\frac{dz}{z}\frac{1}{\sqrt{L^2/4-z^2}} \\ &= \frac{2R}{4G_N}\log\frac{L}{\epsilon} \\ &= \frac{c}{3}\log\frac{L}{\epsilon} \end{aligned} \tag{34}$$

where ϵ is the cutoff to regularize the divergence. Clearly this result agrees with the calculation from CFT.

In finite temperature, we may discuss the connection to black hole entropy. Consider the CFT on a circle (Figure 5), we should recover black hole entropy if we take A to be the whole boundary space because by definition

$$S = -Tr\rho\log\rho \tag{35}$$

is the thermal entropy and in this case the reduced density matrix is the same as the real density matrix of A. Indeed, graphically the minimal surface is just black hole horizon and black hole entropy is recovered. Generally, if the minimal surface is always perpendicular to boundary, area law can be proved in AdS in arbitrary dimensions.

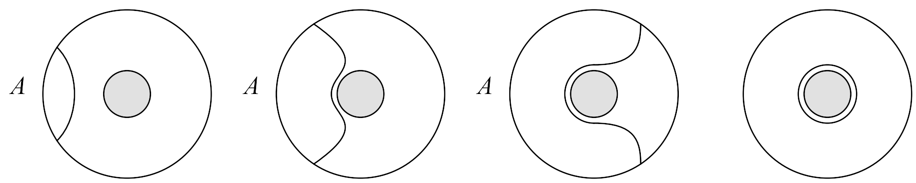
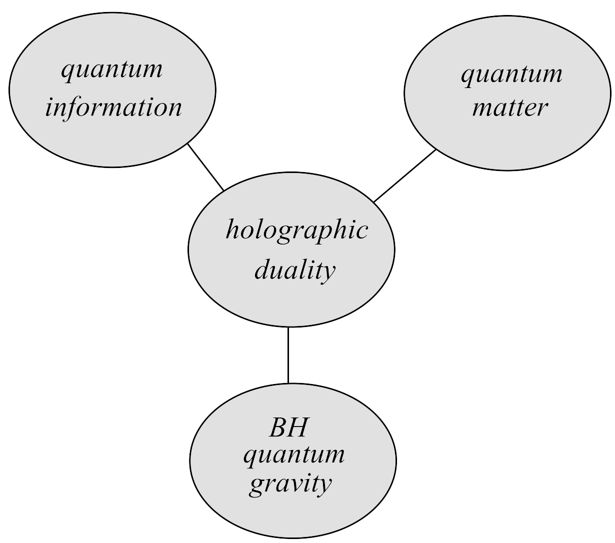


Figure 5: Entanglement entropy and black hole entropy

R-T formula not only provides a simple way to calculate EE, but more importantly, it implies some connection between spacetime and geometrization of quantum entanglement, geometry and quantum information. In the end, we may expect a unified paradigm:



8.821 / 8.871 String Theory and Holographic Duality
Fall 2014