Massachusetts Institute of Technology
Department of Physics

Physics 8.942 Fall 2001

Problem Set #6
Due in class Tuesday, October 30, 2001.

1. **Sachs-Wolfe Angular Power Spectrum**
The 8.942 notes *Cosmic Microwave Background Anisotropy* give give a formula for the angular power spectrum C_l as an integral over the three-dimensional power spectrum of gravitational potential fluctuations. In the Sachs-Wolfe approximation, the large-scale (low l) anisotropy in observed direction $\underline{n}$ is $\Delta_0(\underline{n}) = \frac{1}{3}\phi(-\chi_{\rm e}\underline{n}, \tau_{\rm e})$.

a) Complete the steps of the Sachs-Wolfe derivation in the notes to derive the angular power spectrum for a primordial scale-invariant spectrum $P_\phi = A(c/H_0)^3(ck/H_0)^{n-4}$,

$$C_l \propto \frac{\Gamma(3-n)\,\Gamma\left(\frac{2l+n-1}{2}\right)}{\Gamma^2\left(\frac{4-n}{2}\right)\,\Gamma\left(\frac{2l+5-n}{2}\right)} \tag{1}$$

and find the constant of proportionality in terms of A. (Hint: be sure to include the factor 0.9 relating the potential between $y \ll 1$ and $y \gg 1$ for a matter plus radiation universe as discussed in the notes.)

b) Verify Peacock equation (18.31) and find the relation between $Q_{\rm rms-PS}$ (Peacock eq. 18.33) and A.

c) Suppose that the primeval spectrum has a tilt with $n = 0.8$ over the range of scales probed by the CMB. What is the predicted ratio between $l(l+1)C_l$ at $l = 20$ and $l = 2$?

Hint: for part a) you will need

$$\int_0^\infty x^\nu j_l(x) j_{l'}(x)\, dx = \frac{\pi 2^{\nu-2}\Gamma(1-\nu)\Gamma\left(\frac{l+l'+\nu+1}{2}\right)}{\Gamma\left(\frac{l-l'-\nu+2}{2}\right)\Gamma\left(\frac{l'-l-\nu+2}{2}\right)\Gamma\left(\frac{l+l'-\nu+3}{2}\right)} \tag{2}$$

which is valid for $-(l+l'+1) < \nu < 1$.

2. **Cosmic Variance of the CMB Quadrupole**
As Peacock discusses on page 595, the $Q^2_{\rm rms}$ derived in Problem 1 is the *mean* of a statistically fluctuating quantity. Because each spherical harmonic component of the

anisotropy has a Gaussian distribution with zero mean and with variance $C_l \equiv \langle |a_{lm}|^2 \rangle$, the low degree harmonic coefficients are expected to show large statistical fluctuations. (We can only measure one CMB sky; no matter how precise the measurement, we can never get more than $2l + 1$ measurements of degree l.) For the quadrupole the situation is even worse because emission from the plane of our own galaxy practically eliminates one component, so that the observational estimate of $Q_{\rm rms} = 10.7\ \mu$K (Kogut et al 1996, ApJ, 464, L5) is drawn from a distribution that is effectively chi-squared with four degrees of freedom. Given this fact, determine 95%-confidence upper and lower limits for the ensemble mean $Q_{\rm rms-PS}$. Does the low observed quadrupole rule out $Q_{\rm rms-PS} = 18\ \mu$K, the best-fit value based on fitting the entire angular power spectrum? (Hint: The chi-square distribution with four degrees of freedom has one parameter, namely $Q_{\rm rms-PS}$. Tune this parameter so that the measured value lies at the 5% and 95% values of the cumulative distribution, which you can find in any elementary statistics book. See Hinshaw et al 1996, ApJ, 464, L17 for discussion.)

3. **CMBFAST**
Download and build the numerical code CMBFAST (available at the 8.942 links page — note you must have access to a f77 compiler, e.g. on MIT server machines). Before running cmbfast you will need to run jlgen and/or ujlgen. Read the online documentation carefully.

a) Run CMBFAST on the three standard models (SCDM, ΛCDM, OCDM) of Problem Set 2. (Use the default parameters given by cmbfast in square brackets whenever you are unsure.) Compare with Figure 2 of Miller et al (1999, ApJ, 524, L1). This will require you to change the C_l to temperature units.

b) Holding fixed $\Omega_\Lambda = 0$ and other parameters (e.g. $H_0 = 72$, $\Omega_B h^2 = .02$, and $n = 1$), vary Ω over the range $(0.1, 1)$. Make a graph of $l_{\rm max}(\Omega)$, the location of the first acoustic peak. Then hold fixed $\Omega = 1$ (i.e. spatially flat models) and vary Ω_Λ over the same range. Superpose the $l_{\rm max}(\Omega_m = 1 - \Omega_\Lambda)$. Are the results consistent with Problem 2 of Problem Set 3?

c) Run CMBFAST with a tilt $n = 0.8$ and compare the ratio of $l(l + 1)C_l$ at $l = 2$ and $l = 20$ with the Sachs-Wolfe result from Problem 1.

4. **Gamma Ray Burst Luminosity**
Gamma ray burst 990123 occurred in a galaxy of redshift $z = 1.60$. The measured fluence was 3.0×10^{-4} erg cm^{-2} for photon energies above 20 keV (Briggs et al 1999, ApJ, 524, 82). Convert this to a total emitted burst energy in units of $M_\odot c^2$ assuming isotropic emission. You will need to calculate the luminosity distance to $z = 1.60$. Evaluate it for the three models of Problem Set 2 with $h = 0.72$. You will have to perform a numerical integral for the ΛCDM model. You may use whatever is convenient, e.g. Matlab, Maple, or Mathematica on MIT servers— see the 8.942 links for documentation.